Jordan Journal of Mathematics and Statistics (JJMS) 2008, 1(2), pp143-162

SOME INEQUALITIES OF HILBERT'S TYPE AND APPLICATIONS

LAITH EMIL AZAR

ABSTRACT. By introducing some parameters we establish an extension of Hardy-Hilbert's integral inequality and the corresponding inequality for series. As an application the reverses, some particular results and their equivalent forms are given.

1. INTRODUCTION

If $f(x), g(x) \geq 0$, $0 < \int\limits_0^\infty f^2(x)dx < \infty$, and $0 < \int\limits_0^\infty g^2(x)dx < \infty$, then (see[5])

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int\limits_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int\limits_0^\infty g^2(x)dx \right\}^{\frac{1}{2}} \tag{1.1}$$

where the constant factor π is the best possible in (1.1). Inequality (1.1) is called Hilbert's integral inequality which has been extended by Hardy (see[6]) as: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) > 0$, $0 < \int\limits_0^\infty f^p(x)dx < \infty$, and $0 < \int\limits_0^\infty g^q(x)dx < \infty$, then

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int\limits_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int\limits_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factors $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible in (1.2). Hardy-Hilbert's inequality is important in analysis and it's applications (see[7]). Recently Yang [2] gave some generalizations and the reverse form of (1.2) as: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

2000 *Mathematics Subject Classification.* : 26D15.

Key words and phrases. : Hardy-Hilbert's integral inequality, Hölder inequality, Weight function, Equivalent form.

Received: April 15, 2008 Accepted: Oct.30, 2008.

$r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $f(x), g(x) \geq 0$, $0 < \int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)dx < \infty$, and $0 < \int_0^\infty x^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)} \left\{ \int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty x^{q\left(1-\frac{\lambda}{s}\right)-1} g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $\frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)}$ is the best possible.

The corresponding inequality for series of (1.2) is:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=0}^\infty a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^\infty b_n^q \right\}^{\frac{1}{q}}, \tag{1.4}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are such that $0 < \sum_{n=0}^\infty a_n^p < \infty$, $0 < \sum_{n=0}^\infty a_n^q < \infty$, and the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible. By introducing a parameter $0 < \lambda \leq 2$ an extension of (1.4) $(p = q = 2)$ was given by Yang [2,3] as:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda} \left\{ \sum_{n=0}^\infty n^{1-\lambda} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=0}^\infty n^{1-\lambda} b_n^2 \right\}^{\frac{1}{2}}.$$

Very recently, in [8] the following extensions were given

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{A \min\{x^\lambda, y^\lambda\} + B \max\{x^\lambda, y^\lambda\}} dxdy \\ &< C_\lambda(A,B) \left\{ \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx \right\}^{\frac{1}{q}} \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{A \min\{m^\lambda, n^\lambda\} + B \max\{m^\lambda, n^\lambda\}} \\ &< C_\lambda(A,B) \left\{ \sum_{n=1}^\infty n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q \right\}^{\frac{1}{q}}, \end{aligned} \tag{1.6}$$

where the constant factor $C_\lambda(A,B)$ (see[8, Lemma 2.1]) is the best possible in both inequalities. Inequalities (1.5) and (1.6) are called Hilbert's type inequalities. For more information related to this subject see for example [4].

In this paper by introducing some parameters we give some new inequalities of Hilbert's type for both integral and discrete forms and we obtain the reverse form for each of them. Some particular results and the equivalent form is also considered.

2. Main Results

Lemma 2.1. *Suppose that* $\lambda > 0,\ A \geq 0,\ B > 0, C > 0$. *Define the weight coefficients* $\omega_\lambda(A,B,C,x)$ *by*

$$\omega_\lambda(A,B,C,x) := \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dy, \tag{2.1}$$

then we have $\omega_\lambda(A,B,C,x) = K_\lambda(A,B,C)$ *is a constant defined by*

(2.2)

$$K_\lambda(A,B,C) = \begin{cases} \frac{1}{\lambda}\left[\frac{2}{\sqrt{B(A+C)}}\arctan\sqrt{\frac{A+C}{B}} + \frac{2}{\sqrt{C(A+B)}}\arctan\sqrt{\frac{A+B}{C}}\right], & \text{for } A,B,C>0 \\ \frac{\pi}{\lambda\sqrt{BC}}, & \text{for } A=0, \text{ and } B,C>0. \end{cases}$$

Proof. For a fixed x, setting $t = \left(\frac{y}{x}\right)^\lambda$, we have

$$\begin{aligned} \omega_\lambda(A,B,C,x) &= \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dy, \\ &= \frac{1}{\lambda}\int_0^\infty \frac{t^{-\frac{1}{2}}}{A\min\{1,t\} + B + Ct} dt := I \end{aligned} \tag{2.3}$$

(i) for $A,B,C > 0$, we obtain

$$
\begin{aligned}
I &= \frac{1}{\lambda}\left\{\int_0^1 \frac{t^{-\frac{1}{2}}}{B+(A+C)t}dt + \int_1^\infty \frac{t^{-\frac{1}{2}}}{A+B+Ct}dt\right\} \\
&= \frac{1}{\lambda}\left\{\frac{2}{\sqrt{B(A+C)}}\int_0^{\sqrt{\frac{A+C}{B}}} \frac{dt}{t^2+1} + \frac{2}{\sqrt{C(A+B)}}\int_0^{\sqrt{\frac{A+B}{C}}} \frac{dt}{t^2+1}\right\} \\
&= \frac{1}{\lambda}\left[\frac{2}{\sqrt{B(A+C)}}\arctan\sqrt{\frac{A+C}{B}} + \frac{2}{\sqrt{C(A+B)}}\arctan\sqrt{\frac{A+B}{C}}\right]
\end{aligned}
$$

(ii) for $A = 0$, and $B, C > 0$, we find

$$
\begin{aligned}
I &= \frac{1}{\lambda}\int_0^\infty \frac{t^{-\frac{1}{2}}}{B+Ct}dt \\
&= \frac{\pi}{\lambda\sqrt{BC}}.
\end{aligned}
$$

Hence, $\omega_\lambda(A,B,C,x) = K_\lambda(A,B,C)$. The Lemma is proved.

Lemma 2.2 *For* $p > 1$(*or* $0 < p < 1$), $\frac{1}{p}+\frac{1}{q} = 1, \lambda > 0, A \geq 0, B > 0, C > 0$ *and* $0 < \varepsilon < \frac{p\lambda}{2}$, *setting*

$$
J(\varepsilon) = \int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda}dxdy, \tag{2.4}
$$

then for $\varepsilon \to 0^+$, *we get*

$$
\frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right] - O(1) < J(\varepsilon) < \frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right]. \tag{2.5}
$$

Proof. Setting $t = \left(\frac{x}{y}\right)^\lambda$, we find

$$
\begin{aligned}
J(\varepsilon) &= \int_1^\infty \int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dxdy \\
&= \frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon} \int_{y^{-\lambda}}^\infty \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A\min\{t,1\} + Bt + C} dtdy \\
&= \frac{1}{\lambda\varepsilon}\int_0^\infty \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A\min\{t,1\} + Bt + C} dtdy \\
&\quad -\frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon} \int_0^{y^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A\min\{t,1\} + Bt + C} dtdy \\
&= \frac{1}{\varepsilon}\left[K_\lambda(A,B,C) + o(1)\right] - \frac{1}{\lambda}\int_1^\infty y^{-1-\varepsilon} \int_0^{y^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{At + Bt + C} dtdy \\
&\geq \frac{1}{\varepsilon}\left[K_\lambda(A,B,C) + o(1)\right] - \frac{1}{\lambda}\int_1^\infty y^{-1} \int_0^{y^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{C} dtdy \\
&= \frac{1}{\varepsilon}\left[K_\lambda(A,B,C) + o(1)\right] - O(1).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
J(\varepsilon) &= \int_1^\infty \int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dxdy \\
&< \int_1^\infty \left[\int_0^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dx\right] y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}} dy \\
&= \frac{1}{\varepsilon}\left[K_\lambda(A,B,C) + o(1)\right].
\end{aligned}
$$

Hence (2.5) is valid. The Lemma is proved.

Theorem 2.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, A \geq 0, B > 0, C > 0, f(x), g(x) \geq 0$ *such that* $0 < \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx < \infty, 0 < \int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx < \infty$*, then*

$$S \quad : \quad = \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{A\min\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dxdy$$

$$(2.6) \quad < \quad K_\lambda(A,B,C) \left\{ \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(x)dx \right\}^{\frac{1}{q}},$$

where the constant factor $K_\lambda(A,B,C)$ defined in (2.2) is the best possible. In particular:

(i) for $A = 0$, $\lambda = B = C = 1$ we get $K_1(0,1,1) = \pi$, and inequality (2.6) reduces to Hardy-Hilbert's inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty x^{\frac{p}{2}-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{q}{2}-1} g^q(x)dx \right\}^{\frac{1}{q}},$$

(ii) for $\lambda = A = B = C = 1$, we get $K_1(1,1,1) = 2\sqrt{2}\arctan\sqrt{2}$ and (2.6) reduces to the Hardy-Hilbert's type inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{\min\{x,y\} + x + y} dxdy < 2\sqrt{2}\arctan\sqrt{2} \left\{ \int_0^\infty x^{\frac{p}{2}-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{q}{2}-1} g^q(x)dx \right\}^{\frac{1}{q}}.$$

Proof. By the Hölder inequality, taking into account (2.1), we get

$$\begin{aligned}
S &= \int_0^\infty\int_0^\infty \left[\frac{1}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right]^{\frac{1}{p}} \frac{x^{\left(1-\frac{\lambda}{2}\right)/q}}{y^{\left(1-\frac{\lambda}{2}\right)/p}} f(x) \\
&\quad \times \left[\frac{1}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right]^{\frac{1}{q}} \left[\frac{y^{\left(1-\frac{\lambda}{2}\right)/p}}{x^{\left(1-\frac{\lambda}{2}\right)/q}} g(y)\right] dxdy \\
&\le \left\{\int_0^\infty\int_0^\infty \frac{x^{\left(1-\frac{\lambda}{2}\right)(p-1)} y^{\frac{\lambda}{2}-1}}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} f^p(x)dx\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_0^\infty\int_0^\infty \frac{y^{\left(1-\frac{\lambda}{2}\right)(q-1)} x^{\frac{\lambda}{2}-1}}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} g^q(y)dy\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty \omega_\lambda(A,B,C,x) x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \omega_\lambda(A,B,C,y) y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy\right\}^{\frac{1}{q}} \\
&\le K_\lambda(A,B,C) \left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy\right\}^{\frac{1}{q}}. \tag{2.7}
\end{aligned}$$

If (2.7) takes the form of equality, then there exists constants M and N which are not all zero such that

$$\begin{aligned}
M\frac{x^{\left(1-\frac{\lambda}{2}\right)(p-1)} y^{\frac{\lambda}{2}-1}}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} f^p(x) &= N\frac{y^{\left(1-\frac{\lambda}{2}\right)(q-1)} x^{\frac{\lambda}{2}-1}}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} g^q(y) \\
Mx^{p\left(1-\frac{\lambda}{2}\right)} f^p(x) &= Ny^{q\left(1-\frac{\lambda}{2}\right)} g^q(y), \text{ a.e. in } (0,\infty)\times(0,\infty).
\end{aligned}$$

Hence, there exists a constant c such that

$$Mx^{p\left(1-\frac{\lambda}{2}\right)} f^p(x) = Ny^{q\left(1-\frac{\lambda}{2}\right)} g^q(y) = c \text{ a. e in } (0,\infty).$$

We claim that $M=0$. In fact if $M\neq 0$, then

$$x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x) = \frac{c}{Mx} \text{ a.e. in } (0,\infty),$$

which contradicts the fact that $0<\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx<\infty$. Hence by (2.7) we get (2.6).

If the constant factor $K_\lambda(A,B,C)$ is not the best possible, then there exists a positive constant L (with $L < K_\lambda(A,B,C)$), thus (2.6) is still valid if we replace $K_\lambda(A,B,C)$ by L. For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{f}$ and $\widetilde{g}$ as $\widetilde{f}(x) = \widetilde{g}(x) = 0$ for $x \in (0,1)$, $\widetilde{f}(x) = x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}$; $\widetilde{g}(x) = x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$ for $x \in [1,\infty)$, then we have

$$\begin{aligned} & L\left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{g}^q(x)dx\right\}^{\frac{1}{q}} \\ = \; & L\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{\frac{1}{q}} = \frac{L}{\varepsilon}. \end{aligned}$$

By using (2.5), we find

$$\begin{aligned} \int_0^\infty\int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(x)dxdy}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} &= \int_1^\infty\left[\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}dx}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right] y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}\,dy \\ &> \frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right]-O(1). \end{aligned}$$

Therefore, we get

$$\frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right]-O(1) < \frac{L}{\varepsilon}$$

or

$$\left[K_\lambda(A,B,C)+o(1)\right]-\varepsilon O(1) < L,$$

For $\varepsilon \to 0^+$, it follows that $K_\lambda(A,B,C) \le L$ which contradicts the fact that $L < K_\lambda(A,B,C)$. Hence the constant factor $K_\lambda(A,B,C)$ in (2.6) is the best possible. The Theorem is proved.

Theorem 2.2. *If* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda > 0, A \ge 0, B > 0, C > 0, f(x), g(x) \ge 0$ *such that* $0 < \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx < \infty, 0 < \int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx < \infty$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dxdy$$

$$(2.8)\quad > \quad K_\lambda(A,B,C)\left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}g^q(x)dx\right\}^{\frac{1}{q}},$$

where the constant factor $K_\lambda(A,B,C)$ defined in (2.2) is the best possible. In particular:

(i) for $A=0,\lambda=B=C=1$ we get $K_1(0,1,1)=\pi$, and inequality (2.8) reduces to Hardy-Hilbert's inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x+y}dxdy>\pi\left\{\int_0^\infty x^{\frac{p}{2}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{q}{2}-1}g^q(x)dx\right\}^{\frac{1}{q}},$$

(ii) for $\lambda=A=B=C=1$, we get $K_1(1,1,1)=2\sqrt{2}\arctan\sqrt{2}$ and (2.8) reduces to the Hardy-Hilbert's type inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{\min\{x,y\}+Bx+Cy}dxdy>2\sqrt{2}\arctan\sqrt{2}\left\{\int_0^\infty x^{\frac{p}{2}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{q}{2}-1}g^q(x)dx\right\}^{\frac{1}{q}}.$$

Proof. By the reverse Hölder's inequality, and the same way, we have (2.8). If the constant factor $K_\lambda(A,B,C)$ in (2.8) is not the best possible, then there exists a positive constant H (with $H>K_\lambda(A,B,C)$) such that (2.8) is still valid if we replace $K_\lambda(A,B,C)$ by H. For $0<\varepsilon<\frac{p\lambda}{2}$, setting $\widetilde{f}$ and $\widetilde{g}$ as in Theorem 2.1, then we have

$$\begin{aligned}&H\left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{f}^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{g}^q(x)dx\right\}^{\frac{1}{q}}\\ =\;&H\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{-1-\varepsilon}dx\right\}^{\frac{1}{q}}=\frac{H}{\varepsilon}.\end{aligned}$$

By using (2.5), we find

$$\int_0^\infty\int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(x)dxdy}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} = \int_1^\infty\left[\int_1^\infty \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}dx}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right]y^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}\,dy$$
$$< \frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right].$$

Therefore, we get

$$\frac{1}{\varepsilon}\left[K_\lambda(A,B,C)+o(1)\right] > \frac{H}{\varepsilon}$$

or

$$K_\lambda(A,B,C)+o(1) > H.$$

For $\varepsilon \to 0^+$, it follows that $K_\lambda(A,B,C) \geq H$ which contradicts the fact that $H > K_\lambda(A,B,C)$. Hence the constant factor $K_\lambda(A,B,C)$ in (2.8) is the best possible. The Theorem is proved.

Theorem 2.3. *Under the assumption of Theorem 2.1, we have*

(2.9)
$$\int_0^\infty y^{\frac{\lambda p}{2}-1}\left[\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right]^p dy < [K_\lambda(A,B,C)]^p\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1}f^p(x)dx,$$

where the constant factor $[K_\lambda(A,B,C)]^p$ *is the best possible. Inequalities (2.6) and (2.9) are equivalent.*

Proof. Setting

$$g(y) = y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right\}^{p-1},$$

then by (2.6) we have

$$\begin{aligned}
\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy &= \int_0^\infty y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right\}^p dy \\
&= \int_0^\infty \left\{\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right\} \\
&\quad \times \left\{y^{\frac{\lambda p}{2}-1}\left\{\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right\}^{p-1}\right\} dy \\
&= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dxdy \\
&\le K_\lambda(A,B,C)\left\{\int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx\right\}^{\frac{1}{p}} \\
&\quad \times\left\{\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy\right\}^{\frac{1}{q}}. \qquad (2.10)
\end{aligned}$$

Hence, we obtain

$$\int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy \le [K_\lambda(A,B,C)]^p \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx. \qquad (2.11)$$

Thus, by (2.6), both (2.10) and (2.11) keep the form of strict inequalities, then we have (2.9).

By Hölder's inequality, we find

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dxdy$$
$$= \int_0^\infty \left\{ y^{\frac{\lambda}{2}-\frac{1}{p}} \int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx \right\} \left\{ y^{\frac{1}{p}-\frac{\lambda}{2}} g(y)\right\} dy$$
$$\le \left\{ \int_0^\infty y^{\frac{p\lambda}{2}-1} \left\{ \int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx \right\}^p \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_0^\infty y^{q\left(1-\frac{\lambda}{2}\right)-1} g^q(y)dy \right\}^{\frac{1}{q}}. \tag{2.12}$$

Therefore, by (2.9) we have (2.6), and inequalities (2.9) and (2.6) are equivalent. If the constant factor in (2.9) is not the best possible, then by (2.12) we can get a contradiction that the constant factor in (2.6) is not the best possible. The Theorem is proved.

Theorem 2.4. *Under the assumption of Theorem 2.2, we have*

(2.13)
$$\int_0^\infty y^{\frac{\lambda p}{2}-1} \left[\int_0^\infty \frac{f(x)}{A\min\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx \right]^p dy > [K_\lambda(A,B,C)]^p \int_0^\infty x^{p\left(1-\frac{\lambda}{2}\right)-1} f^p(x)dx.$$

where the constant factor $[K_\lambda(A,B,C)]^p$ *is the best possible. Inequalities (2.8) and (2.13) are equivalent.*

The proof of Theorem 2.4 is similar to that of Theorem 2.3, so we omit it.

3. Discrete Analogous

Lemma 3.1. *Suppose that* $0<\lambda\le 2,\ A\ge 0,\ B>0, C>0$. *Then the weight coefficients* $\varpi_\lambda(A,B,C,m)$ *is defined by*

$$\varpi_\lambda(A,B,C,m) := \sum_{n=1}^\infty \frac{m^{\frac{\lambda}{2}} n^{-1+\frac{\lambda}{2}}}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda} \qquad (m\in N) \tag{3.1}$$

satisfies the following inequality

$$K_\lambda(A,B,C)\left[1-\theta_\lambda(A,B,C,m)\right] < \varpi_\lambda(A,B,C,m) < K_\lambda(A,B,C), \tag{3.2}$$

where $\theta_\lambda(A,B,C,r) := \frac{1}{K_1(A,B,C)} \int\limits_0^{r^{-\lambda}} \frac{t^{-\frac{1}{2}}}{(A+C)t+B}dt = O(\frac{1}{r^{\frac{\lambda}{2}}}) \in (0,1)$ $(r \in N)$ $(r \to \infty)$, *and* $K_\lambda(A,B,C)$ *is defined by (2.2).*

Proof. Since $0 < \lambda \leq 2$, $A \geq 0$, $B > 0, C > 0$, by Lemma 2.1 we get

$$\begin{aligned} \varpi_\lambda(A,B,C,m) &< \int\limits_0^\infty \frac{m^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A\min\{m^\lambda, y^\lambda\} + Bm^\lambda + Cy^\lambda} dy \\ &= \omega_\lambda(A,B,C,m) = K_\lambda(A,B,C). \qquad (3.3) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \varpi_\lambda(A,B,C,m) &> \int\limits_1^\infty \frac{m^{\frac{\lambda}{2}} y^{-1+\frac{\lambda}{2}}}{A\min\{m^\lambda, y^\lambda\} + Bm^\lambda + Cy^\lambda} dy \\ &= \frac{1}{\lambda} \int\limits_{m^{-\lambda}}^\infty \frac{t^{-\frac{1}{2}}}{A\min\{1,t\} + B + Ct} dt \\ &= I - \frac{1}{\lambda} \int\limits_0^{m^{-\lambda}} \frac{t^{-\frac{1}{2}}}{(A+C)t+B} dt \\ &= I(1 - \theta_\lambda(A,B,C,m)), \end{aligned}$$

where $I = \frac{1}{\lambda} K_1(A,B,C)$ and

$$0 < \theta_\lambda(A,B,C,m) = \frac{1}{K_1(A,B,C)} \int\limits_0^{m^{-\lambda}} \frac{t^{-\frac{1}{2}}}{(A+C)t+B} dt < 1.$$

Since

$$\int\limits_0^{m^{-\lambda}} \frac{t^{-\frac{1}{2}}}{(A+C)t+B} dt \leq \int\limits_0^{m^{-\lambda}} \frac{t^{-\frac{1}{2}}}{B} dt = \frac{2}{Bm^{\frac{\lambda}{2}}},$$

then $\theta_\lambda(A,B,C,m) = O\left(\frac{1}{m^{\frac{\lambda}{2}}}\right)$. Therefore, (3.2) is valid. The lemma is proved.

Lemma 3.2 *If* $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq 2, A \geq 0, B > 0, C > 0$ *and* $0 < \varepsilon < \frac{p\lambda}{2}$, *setting*

$$L(\varepsilon) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A \min\{m^\lambda, n^\lambda\} + Bm^\lambda + Cn^\lambda}, \tag{3.4}$$

then for $\varepsilon \to 0^+$, *we get*

$$[K_\lambda(A,B,C) - o(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < L(\varepsilon) < [K_\lambda(A,B,C) + \tilde{o}(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}. \tag{3.5}$$

Proof. Setting $t = \left(\frac{x}{n}\right)^\lambda$ in the following, by (3.2) we have

$$\begin{aligned} L(\varepsilon) &< \sum_{n=1}^{\infty} \int_0^{\infty} \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A \min\{x^\lambda, n^\lambda\} + Bx^\lambda + Cn^\lambda} dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[\frac{1}{\lambda} \int_0^{\infty} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A \min\{t, 1\} + Bt + C} dt \right] \\ &= [K_\lambda(A,B,C) + \tilde{o}(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\varepsilon \to 0^+\right). \end{aligned}$$

$$
\begin{aligned}
L(\varepsilon) &> \sum_{n=1}^{\infty} \int_{1}^{\infty} \frac{x^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}} n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}}{A\min\{x^{\lambda}, n^{\lambda}\} + Bx^{\lambda} + Cn^{\lambda}} dx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[\frac{1}{\lambda} \int_{n^{-\lambda}}^{\infty} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{A\min\{t,1\} + Bt + C} dt \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[K_{\lambda}(A,B,C) + \tilde{o}(1) - \frac{1}{\lambda} \int_{0}^{n^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{(A+B)t + C} dt \right] \\
&> \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[K_{\lambda}(A,B,C) + \tilde{o}(1) \right] - \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{n^{-\lambda}} \frac{t^{-\frac{1}{2}-\frac{\varepsilon}{\lambda p}}}{C} dt \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[K_{\lambda}(A,B,C) + \tilde{o}(1) \right] - \frac{1}{\left(\frac{\lambda C}{2} - \frac{\varepsilon C}{p}\right)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[K_{\lambda}(A,B,C) + \tilde{o}(1) - \frac{1}{\left(\frac{\lambda C}{2} - \frac{\varepsilon C}{p}\right)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right)^{-1} \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[K_{\lambda}(A,B,C) - o(1) \right] \left(\varepsilon \to 0^{+}\right).
\end{aligned}
$$

Thus, inequality (3.5) holds. The Lemma is proved.

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \le 2, A \ge 0, B > 0, C > 0, a_n, b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q < \infty$, *then*

$$
\begin{aligned}
D \quad : &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{A\min\{m^{\lambda}, n^{\lambda}\} + Bm^{\lambda} + Cn^{\lambda}} \\
&< K_{\lambda}(A,B,C) \left\{ \sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1} b_n^q \right\}^{\frac{1}{q}}, \qquad (3.6)
\end{aligned}
$$

where the constant factor $K_{\lambda}(A,B,C)$ *defined in (2.2) is the best possible. In particular:*

(i) for $A = 0, \lambda = B = C = 1$ *we get* $K_1(0,1,1) = \pi$, *and inequality (3.6) reduces to Hardy-Hilbert's inequality*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}},$$

(ii) for $\lambda=A=B=C=1$, *we get* $K_1(1,1,1)=2\sqrt{2}\arctan\sqrt{2}$ *and (3.6) reduces to the Hardy-Hilbert's type inequality*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n}<2\sqrt{2}\arctan\sqrt{2}\left\{\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

Proof. By the Hölder inequality, taking into account (3.1), we get

$$\begin{aligned}
D &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right\}^{\frac{1}{p}}\left[\frac{m^{\left(1-\frac{\lambda}{2}\right)/q}}{n^{\left(1-\frac{\lambda}{2}\right)/p}}a_m\right]\\
&\quad\times\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right\}^{\frac{1}{q}}\left[\frac{n^{\left(1-\frac{\lambda}{2}\right)/p}}{m^{\left(1-\frac{\lambda}{2}\right)/q}}b_n\right]\\
&\le \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\frac{m^{\left(1-\frac{\lambda}{2}\right)(p-1)}}{n^{1-\frac{\lambda}{2}}}a_m^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\frac{n^{\left(1-\frac{\lambda}{2}\right)(q-1)}}{m^{1-\frac{\lambda}{2}}}b_n^q\right\}^{\frac{1}{q}}\\
&= \left\{\sum_{m=1}^{\infty}\varpi_\lambda(A,B,m)\,m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(A,B,n)\,n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then, by (3.2) we obtain (3.6).

It remains to show that the constant factor $K_\lambda(A,B,C)$ is the best possible, to do that we set for $0<\varepsilon<\frac{p\lambda}{2}$, $\widetilde{a}_m=m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}};\widetilde{b}_n=n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$, by (3.4) we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}=L(\varepsilon).$$

If there exists a constant $0<L\le K_\lambda(A,B,C)$ such that (3.6) is still valid if we replace $K_\lambda(A,B,C)$ by L, then in particular by (3.5) we find

$$\begin{aligned}[K_\lambda(A,B,C)-o(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &< L(\varepsilon) \\ &< L\left\{\sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}} \\ &= L\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},\end{aligned}$$

it follows that $K_\lambda(A,B,C) - o(1) < L$ and then $K_\lambda(A,B,C) \le L\,(\varepsilon \to 0^+)$. Therefore, $L = K_\lambda(A,B,C)$ is the best constant factor in (3.6). The Theorem is proved.

Theorem 3.2. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \le 2, A \ge 0, B > 0, C > 0, a_n, b_n \ge 0$ *such that* $0 < \sum\limits_{n=1}^{\infty} n^{p\left(1-\frac{\lambda}{2}\right)-1}a_n^p < \infty, 0 < \sum\limits_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}$$

$$K_\lambda(A,B,C)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(A,B,C,n)]\,n^{p\left(1-\frac{\lambda}{2}\right)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3.7}$$

where the constant factor $K_\lambda(A,B,C)$ *defined in (2.2) is the best possible. In particular:*

(i) for $A = 0$, $\lambda = B = C = 1$ *we get* $K(0,1,1) = \pi$, *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} > \pi\left\{\sum_{n=1}^{\infty}\left[1-\frac{2}{\pi}\arctan\frac{1}{n^{\frac{1}{2}}}\right]n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}},$$

(ii) for $\lambda = A = B = C = 1$, *we get* $K(1,1,1) = 2\sqrt{2}\arctan\sqrt{2}$ *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n} > 2\sqrt{2}\arctan\sqrt{2}\left\{\sum_{n=1}^{\infty}\left[1-\frac{\arctan\sqrt{2}n^{-\frac{1}{2}}}{2\arctan\sqrt{2}}\right]n^{\frac{p}{2}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right\}^{\frac{1}{q}}.$$

Proof. By the reverse Hölder's inequality, we get

$$\begin{aligned} D &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda} \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right\}^{\frac{1}{p}}\left[\frac{m^{\left(1-\frac{\lambda}{2}\right)/q}}{n^{\left(1-\frac{\lambda}{2}\right)/p}}a_m\right] \\ &\quad\times\left\{\frac{1}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right\}^{\frac{1}{q}}\left[\frac{n^{\left(1-\frac{\lambda}{2}\right)/p}}{m^{\left(1-\frac{\lambda}{2}\right)/q}}b_n\right] \\ &\geq \left\{\sum_{m=1}^{\infty}\varpi_\lambda(A,B,C,m)\,m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_\lambda(A,B,C,n)\,n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (3.2) , in view of $q<0$, we have (3.7). For $0<\varepsilon<\frac{p\lambda}{2}$, setting $\widetilde{a}_m = m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}; \widetilde{b}_n = n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}\,(m,n\in N)$, If there exists a constant $L\geq K_\lambda(A,B,C)$ such that (3.7) is still valid if we replace $K_\lambda(A,B,C)$ by L, then in particular by (3.4) and (3.5) we find

$$\begin{aligned} [C_\lambda(A,B,C)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &> L(\varepsilon) \\ &> L\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(A,B,C,n)]\,n^{p\left(1-\frac{\lambda}{2}\right)-1}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{\lambda}{2}\right)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}} \\ &= L\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{\frac{1}{q}} \\ &= L\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}}, \end{aligned}$$

it follows that

$$K_\lambda(A,B,C)+\widetilde{o}(1) > L\left\{1-\left[\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right]^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\frac{\lambda}{2}}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{\frac{1}{p}}.$$

Hence, if $\varepsilon\to 0^+$, we get $K_\lambda(A,B,C)\geq L$. Thus, $L=K_\lambda(A,B,C)$ is the best constant factor in (3.7).

Theorem 3.3. *Under the assumption of Theorem 3.1, we have*

(3.8)
$$\sum_{n=1}^{\infty} n^{\frac{\lambda p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right]^p < [K_\lambda(A,B,C)]^p\sum_{m=1}^{\infty} m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p,$$

where the constant factor $[K_\lambda(A,B,C)]^p$ *is the best possible. Inequalities (3.6) and (3.8) are equivalent.*

Proof. Setting

$$b_n = n^{\frac{\lambda p}{2}-1}\left\{\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right\}^{p-1},$$

we get

$$\sum_{n=1}^{\infty} n^{q\left(1-\frac{\lambda}{2}\right)-1}b_n^q = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}.$$

By (3.6) and using the same method of Theorem (2.3), we obtain (3.8). We may show that the constant factor in (3.8) is the best possible and inequality (3.6) is equivalent to (3.8).

Theorem 3.4. *Under the assumption of Theorem 3.2, we have*

(3.9)
$$\sum_{n=1}^{\infty} n^{\frac{\lambda p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m^\lambda,n^\lambda\}+Bm^\lambda+Cn^\lambda}\right]^p > [K_\lambda(A,B,C)]^p\sum_{m=1}^{\infty} m^{p\left(1-\frac{\lambda}{2}\right)-1}a_m^p.$$

where the constant factor $[K_\lambda(A,B,C)]^p$ *is the best possible. Inequalities (3.7) and (3.9) are equivalent.*

Acknowledgement

I would like to thank the anonymous referee for his or her suggestions and corrections.

References

[1] Bicheng Yang, On an Extension of Hilbert's Integral Inequality with Some Parameters, The Australian J. Math. Anal. Appl., 2004, 5:1-8.

[2] Bicheng Yang, On the Extended Hilbert's Integral Inequality, Journal of Inequalities in Pure and Applied Mathematics, 2004,(1):1-8.

[3] Bicheng Yang, On a Generalization of the Hilbert's Type Inequality and it's Applications, Chinese Journal of Engineering Mathematics, 2004,21 (5):821-824.

[4] Bicheng Yang,On a relation between Hilberts inequality and a Hilbert-type inequality. Applied Mathematics Letters, Volume 21, Issue 5, May 2008, Pages 483-488.

[5] Hardy G.H., Littlewood J.E., and Polya G.,"*Inequalities*", Cambridge Univ. Press, London, 1952.

[6] Hardy G. H., Note on a Theorem of Hilbert concerning series of positive terms,Proceedings London Math. Soc.,1925,23(2).

[7] Mitrinović, D.S., Pecarić, J.E., Fink, A.M., *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers,1991.

[8] Laith Emil Azar, On Some Extensions of Hardy-Hilbert's Inequality and Applications . Journal of Inequalities and Applications, volume 2008(2008), Article ID 546828, 14 pages. doi:10.1155/2008/546829.

Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq, Jordan

E-mail address: azar_laith@yahoo.com