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STRONGLY SINGULAR CALDERÓN-ZYGMUND OPERATORS AND THEIR COMMUTATORS

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Abstract In this paper, the authors obtain two kinds of endpoint estimates for strongly singular Calderón-Zygmund operators. Moreover, the pointwise estimate for the sharp maximal function of commutators generated by strongly singular Calderón-Zygmund operators and BMO functions is also established. As its applications, the boundedness of the commutators on Morrey type spaces will be obtained.

1. Introduction

The introduction of strongly singular integral operators is motivated by a class of multiplier operators whose symbol is given by $e^{i|\xi|^a}/|\xi|^\beta$ away from the origin, where $0 < a < 1$ and $\beta > 0$. Fefferman and Stein [8] have enlarged the multiplier operators onto a class of convolution operators. Coifman [6] has also considered a related class of operators for $n = 1$.

The strongly singular non-convolution operator was introduced by Alvarez and Milman in [3], whose properties are similar to those of Calderón-Zygmund operators, but the kernel is more singular than that of the standard case near the diagonal. Furthermore, following a suggestion of Stein, the authors in [3] showed that the pseudo-differential operators with symbols in the class $S^{-\beta}_{\alpha,\delta}$, where $0 < \delta \le \alpha < 1$ and $n(1-\alpha)/2 \le \beta < n/2$, are included in the strongly singular Calderón-Zygmund operator defined as follows.

Definition 1.1. *Let $T : \mathcal{S} \to \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied:*

(1) T can be extended into a continuous operator from L^2 into itself.

(2) There exists a continuous function $K(x, y)$ away from the diagonal $\{(x, y) : x = y\}$ such that

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \le C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}},$$

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if

$$2|y-z|^{\alpha} \leq |x-z| \quad \textit{for some} \quad 0<\delta\leq 1, 0<\alpha<1,$$

and $\langle Tf, g\rangle = \int K(x,y)f(y)g(x)dydx$, *for* $f, g \in \mathcal{S}$ *with disjoint supports.*

(3) For some $n(1-\alpha)/2 \leq \beta < n/2$, *both* T *and its conjugate operator* T^* *can be extended into continuous operators from* L^q *to* L^2, *where* $1/q = 1/2 + \beta/n$.

Alvarez and Milman [3, 4] discussed the boundedness of strongly singular Calderón-Zygmund operators on Lebesgue spaces.

Theorem A [3] *If* T *is a strongly singular Calderón-Zygmund operator, then* T *can be defined to be a continuous operator from* L^∞ *to BMO.*

Theorem B [4] *If* T *is a strongly singular Calderón-Zygmund operator, then* T *is of weak* (L^1, L^1) *type.*

Moreover, the authors in [10] obtained that the strongly singular Calderón-Zygmund operator T is bounded from H^1 to L^1. Obviously, T is bounded on L^p, $1<p<\infty$, by the interpolation theory.

Now, let us return to the classical singular integral defined by

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x-y)f(y)dy,$$

where the kernel $K \in C(\mathbb{R}^n \setminus \{0\})$ satisfies the following three conditions:

(a) $\int_{\varepsilon<|x|<N} K(x)dx = 0$, for any $0<\varepsilon<N<\infty$;

(b) $|K(x)| \leq C|x|^{-n}$, $x \neq 0$;

(c) $|K(x-y) - K(x)| \leq C|y||x|^{-n-1}$, for all $|x| > 2|y|$.

A well-known result in [14] showed that the above operator T is bounded on the BMO space.

Definition 1.2. *LMO is a subspace of BMO, equipped with the semi-norm*

$$[f]_{LMO} = \sup_{0<r<1} \frac{1+|\ln r|}{|B_r|}\int_{B_r} |f(x) - f_{B_r}|dx + \sup_{r\geq 1}\frac{1}{|B_r|}\int_{B_r}|f(x)-f_{B_r}|dx,$$

where B_r *denotes a ball in* $\mathbb{R}^n$ *with radius* r.

The authors in [12, 13] also obtained the LMO-boundedness of classical singular integral operators.

These results mentioned above essentially depend on the cancellation condition of the kernel in (a). A natural question is: whether the strongly singular Calderón-Zygmund operator T is bounded on the BMO and LMO spaces if we add a condition similar to (a) to it. In Section 2, we will give an affirmative answer.

On the other hand, a pointwise estimate for the sharp maximal function of strongly singular Calderón-Zygmund operators was obtained in [4]:

$$(Tf)^\sharp(x) \le CM_2(f)(x).$$

Here and in what follows, for $1 < p < \infty$, $M_p(f)(x) = M(|f|^p)^{1/p}(x)$, where M stands for the Hardy-Littlewood maximal function. As a matter of fact, this estimate was generalized in [9] as follows:

$$(Tf)^\sharp(x) \le CM_s(f)(x), \quad \text{for any} \quad \frac{n(1-\alpha)+2\beta}{2\beta} \le s < \infty,$$

where α, β are given as in Definition 1.1. Then a weighted norm inequality can be established immediately. T is bounded on $L^p_\omega(\mathbb{R}^n)$ for $\frac{n(1-\alpha)+2\beta}{2\beta} \le s < p < \infty$ and $\omega \in A_{p/s}$. By the well known result of Alvarez-Bagby-Kurtz-Pérez in [2], the commutator $[b, T]$ generated by a strongly singular Calderón-Zygmund operator T and a BMO function b, which is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for suitable functions f, is also bounded on $L^p_\omega(\mathbb{R}^n)$ for $\frac{n(1-\alpha)+2\beta}{2\beta} \le s < p < \infty$ and $\omega \in A_{p/s}$. In particular, $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$, $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$. A standard discussion about duality and interpolation yields the boundedness of $[b, T]$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Besides the above method, there is another way to obtain the weighted norm estimate of $[b, T]$. In Section 3, we will establish a pointwise estimate for the sharp maximal function of $[b, T]$ directly. Furthermore, this estimate can be applied to get boundedness properties of $[b, T]$ on other function spaces. In Section 4, we will state the boundedness of $[b, T]$ on Morrey type spaces, which can be regarded as applications of the result in Section 3.

In what follows, for $1 < p < \infty$, p' is the conjugate index of p, that is, $1/p + 1/p' = 1$. χ_E is the characteristic function of a set E. $E^c = \mathbb{R}^n \setminus E$ is the complementary set of E. C's will be constants which may vary from line to line. We will always denote by $B(x, R)$ the ball centered at x with radius $R > 0$, $CB(x, R) = B(x, CR)$ for $C > 0$, $|B(x, R)|$ the Lebesgue measure of $B(x, R)$ and $f_{B(x,R)} = \frac{1}{|B(x,R)|}\int_{B(x,R)} f(y)dy$.

2. Endpoint estimates

The most useful property of a BMO function is the classical John-Nirenberg inequality, which shows that functions in BMO are locally exponentially integrable. This implies that for any $1 \le q < \infty$, the functions in BMO can be described by means of the condition:

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^q dx \right)^{1/q} < \infty.$$

Based on the above property, the following estimate can be established, which is very convenient in applications.

Lemma 2.1. *Let f be a function in BMO. Suppose $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p dy\right)^{1/p} \le C\left(1 + \left|\ln \frac{r_1}{r_2}\right|\right)\|f\|_{\mathrm{BMO}},$$

where $C > 0$ is independent of f, x, r_1 and r_2.

Proof We only consider the case $0 < r_1 \le r_2$. Actually, the similar procedure also works for another case $0 < r_2 < r_1$.

For $0 < r_1 \le r_2$, there are $k_1, k_2 \in \mathbb{Z}$ such that $2^{k_1 - 1} < r_1 \le 2^{k_1}$ and $2^{k_2-1} < r_2 \le 2^{k_2}$. Then $k_1 \le k_2$ and

$$(k_2 - k_1 - 1)\ln 2 < \ln \frac{r_2}{r_1} < (k_2 - k_1 + 1)\ln 2.$$

Thus, we have

$$\begin{aligned}
&\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p dy\right)^{1/p} \\
&\le \left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,2^{k_1})}|^p dy\right)^{1/p} + |f_{B(x,2^{k_1})} - f_{B(x,r_2)}| \\
&\le \left(\frac{2^n}{|B(x,2^{k_1})|}\int_{B(x,2^{k_1})} |f(y) - f_{B(x,2^{k_1})}|^p dy\right)^{1/p} + |f_{B(x,r_2)} - f_{B(x,2^{k_2})}| \\
&\qquad + \sum_{j=k_1}^{k_2-1} |f_{B(x,2^{j+1})} - f_{B(x,2^j)}| \\
&\le C\|f\|_{\mathrm{BMO}} + \frac{1}{|B(x,r_2)|}\int_{B(x,r_2)} |f(y) - f_{B(x,2^{k_2})}| dy \\
&\qquad + \sum_{j=k_1}^{k_2-1} \frac{1}{|B(x,2^j)|}\int_{B(x,2^j)} |f(y) - f_{B(x,2^{j+1})}| dy \\
&\le C\|f\|_{\mathrm{BMO}} + \frac{2^n}{|B(x,2^{k_2})|}\int_{B(x,2^{k_2})} |f(y) - f_{B(x,2^{k_2})}| dy \\
&\qquad + \sum_{j=k_1}^{k_2-1} \frac{2^n}{|B(x,2^{j+1})|}\int_{B(x,2^{j+1})} |f(y) - f_{B(x,2^{j+1})}| dy \\
&\le \|f\|_{\mathrm{BMO}}\big(C + 2^n + 2^n(k_2 - k_1)\big)
\end{aligned}$$

$$\le C\left(1+\ln\frac{r_2}{r_1}\right)\|f\|_{\mathrm{BMO}}.$$

This completes the proof of the lemma. □

LMO is essentially a special case of a kind of function spaces introduced by Spanne in [12]. For a LMO function, there are some properties similar to those of a BMO function. One can refer to [1] for the details.

For $1\le p<\infty$, define

$$[f]_{\mathrm{LMO}^p}=\sup_{0<r<\frac{1}{2}}(1+|\ln r|)\left(\frac{1}{|B_r|}\int_{B_r}|f(x)-f_{B_r}|^p dx\right)^{1/p}.$$

Lemma 2.2. [1] *If $f\in$ LMO, then for any $1\le p<\infty$, there exists a constant $C>0$ depending only on n and p such that*

$$[f]_{\mathrm{LMO}^p}\le C[f]_{\mathrm{LMO}}.$$

Lemma 2.3. *Let $\varepsilon>0$ and $f\in$ LMO. Then for any ball $B=B(x,r)$ with $0<r<\frac{1}{2}$,*

$$\int_{B^c}\frac{|f(y)-f_B|}{|x-y|^{n+\varepsilon}}dy\le Cr^{-\varepsilon}(1+|\ln r|)^{-1}[f]_{\mathrm{LMO}},$$

where $C>0$ is independent of f, x and r.

Proof Since $r^{-1}>2$, there exists a $k\in\mathbb{N}^+$ such that $2^k<r^{-1}\le 2^{k+1}$. Then $k\sim|\ln r|$.

$$\begin{aligned}
&\int_{B^c}\frac{|f(y)-f_B|}{|x-y|^{n+\varepsilon}}dy\le\sum_{j=0}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{|f(y)-f_B|}{|x-y|^{n+\varepsilon}}dy\\
&\le C\sum_{j=0}^{\infty}(2^jr)^{-\varepsilon}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_B|dy\\
&\le C\sum_{j=0}^{\infty}(2^jr)^{-\varepsilon}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|dy+\sum_{i=0}^{j}|f_{2^{i+1}B}-f_{2^iB}|\right)\\
&=C\sum_{j=0}^{k-1}(2^jr)^{-\varepsilon}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|dy+\sum_{i=0}^{j}|f_{2^{i+1}B}-f_{2^iB}|\right)\\
&\quad+C\sum_{j=k}^{\infty}(2^jr)^{-\varepsilon}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|dy+\sum_{i=0}^{j}|f_{2^{i+1}B}-f_{2^iB}|\right)\\
&:=I+II.
\end{aligned}$$

Let us estimate II first.

$$\begin{aligned} II &\leq C\sum_{j=k}^{\infty}(2^j r)^{-\varepsilon}\Bigg(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|dy \\ &\qquad + \sum_{i=0}^{j}\frac{1}{|2^i B|}\int_{2^i B}|f(y)-f_{2^{i+1}B}|dy\Bigg) \\ &\leq C\|f\|_{\mathrm{BMO}} r^{-\varepsilon}\sum_{j=k}^{\infty} j\, 2^{-\varepsilon j} \\ &\leq Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{j=k}^{\infty}\frac{j^2}{k+1}2^{-\varepsilon j} \\ &\leq Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{j=1}^{\infty}\frac{j^2}{1+|\ln r|}2^{-\varepsilon j} \\ &= Cr^{-\varepsilon}(1+|\ln r|)^{-1}[f]_{\mathrm{LMO}}. \end{aligned}$$

To estimate I, we use the fact that $0 < 2^{j+1}r < 1$ for $0 \leq j \leq k-1$.

$$\begin{aligned} I &\leq C\sum_{j=0}^{k-1}(2^j r)^{-\varepsilon}\Bigg(\frac{1+|\ln 2^{j+1}r|}{(1+|\ln 2^{j+1}r|)|2^{j+1}B|}\int_{2^{j+1}B}|f(y)-f_{2^{j+1}B}|dy \\ &\qquad + \sum_{i=0}^{j}\frac{1+|\ln 2^{i+1}r|}{(1+|\ln 2^{i+1}r|)|2^{i+1}B|}\int_{2^{i+1}B}|f(y)-f_{2^{i+1}B}|dy\Bigg) \\ &\leq Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{j=0}^{k-1}2^{-\varepsilon j}\Bigg(\frac{1}{1+|\ln 2^{j+1}r|}+\sum_{i=0}^{j}\frac{1}{1+|\ln 2^{i+1}r|}\Bigg) \\ &\leq Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{2^{-\varepsilon j}}{1+|\ln 2^{i+1}r|} \\ &= Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{i=0}^{\infty}\frac{1}{1+|\ln r+(i+1)\ln 2|}\sum_{j=i}^{\infty}2^{-\varepsilon j} \\ &= Cr^{-\varepsilon}[f]_{\mathrm{LMO}}\sum_{i=0}^{\infty}\frac{2^{-\varepsilon i}}{1+|\ln r+(i+1)\ln 2|} \\ &\leq Cr^{-\varepsilon}(1+|\ln r|)^{-1}[f]_{\mathrm{LMO}}. \end{aligned}$$

The following basic inequality was applied to get the last inequality above.

$$1+|a+b| \geq (1+|a|)^{-1}(1+|b|), \quad \text{for any } a,b\in\mathbb{R}. \tag{2.1}$$

Combining the above two estimates, we can obtain the desired result. □

Now, let us state the main results in this section.

Theorem 2.1. *Let T be a strongly singular Calderón-Zygmund operator and $T1 = 0$. Suppose $f \in \mathrm{BMO}$ such that $Tf(x)$ exists a.e. in $\mathbb{R}^n$. Then $Tf \in \mathrm{BMO}$ and*

$$\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}},$$

where $C > 0$ is independent of f.

Proof Let α, β and δ be given as in Definition 1.1. For any ball $B = B(x_0, r) \subset \mathbb{R}^n$, there are two cases.

(i) $r > 1$.

Write

$$\begin{aligned} f(x) &= f_{2B} + (f(x) - f_{2B})\chi_{8B}(x) + (f(x) - f_{2B})\chi_{(8B)^c}(x) \\ &:= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

It follows from the hypothesis $T1 = 0$ that $Tf_1 = 0$.

By Hölder's inequality, the L^2-boundedness of T and Lemma 2.1, we have

$$\begin{aligned} \frac{1}{|B|}\int_B |Tf_2(x)|dx &\leq \left(\frac{1}{|B|}\int_B |Tf_2(x)|^2 dx\right)^{1/2} \\ &\leq C\left(\frac{1}{|B|}\int_{\mathbb{R}^n} |f_2(y)|^2 dy\right)^{1/2} \\ &= C\left(\frac{1}{|8B|}\int_{8B} |f(y) - f_{2B}|^2 dy\right)^{1/2} \\ &\leq C\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Since $Tf(x)$ and $Tf_2(x)$ exist a.e. in $\mathbb{R}^n$, there is a point $z_1 \in B$ such that $|Tf_3(z_1)| < \infty$. For any $x \in B$ and $y \in (8B)^c$, $2|x - z_1|^\alpha \leq 2(2r)^\alpha < 4r < |y - z_1|$ since $r > 1$. It follows from (2) of Definition 1.1, Lemma 2.1 and $r > 1$ that

$$\begin{aligned} &\frac{1}{|B|}\int_B |Tf_3(x) - Tf_3(z_1)|dx \\ &\leq \frac{1}{|B|}\int_B\int_{(8B)^c} |K(x,y) - K(z_1,y)||f(y) - f_{2B}|dydx \\ &\leq C\frac{1}{|B|}\int_B\int_{(8B)^c} \frac{|x - z_1|^\delta}{|y - z_1|^{n+\delta/\alpha}}|f(y) - f_{2B}|dydx \\ &\leq Cr^\delta \sum_{k=3}^{\infty}\int_{2^{k+1}B\setminus 2^kB} \frac{|f(y) - f_{2B}|}{|y - z_1|^{n+\delta/\alpha}}dy \end{aligned}$$

$$
\begin{aligned}
&\leq Cr^{\delta}\sum_{k=3}^{\infty}(2^k r)^{-\delta/\alpha}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y)-f_{2B}|dy\\
&\leq Cr^{\delta-\delta/\alpha}\|f\|_{\mathrm{BMO}}\sum_{k=3}^{\infty}k2^{-k\delta/\alpha}\\
&\leq C\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

The last inequality is due to $\delta-\delta/\alpha<0$.

Thus,

$$
\begin{aligned}
&\frac{1}{|B|}\int_B|Tf(x)-(Tf)_B|dx\\
&\leq\frac{2}{|B|}\int_B|Tf(x)-Tf_3(z_1)|dx\\
&\leq\frac{2}{|B|}\int_B|Tf_2(x)|dx+\frac{2}{|B|}\int_B|Tf_3(x)-Tf_3(z_1)|dx\\
&\leq C\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

(ii) $0<r\leq 1$.

Let $\widetilde{B}=B(x_0,r^{\alpha})$. Write

$$
\begin{aligned}
f(x) &= f_{2\widetilde{B}}+(f(x)-f_{2\widetilde{B}})\chi_{8\widetilde{B}}(x)+(f(x)-f_{2\widetilde{B}})\chi_{(8\widetilde{B})^c}(x)\\
&:= f_4(x)+f_5(x)+f_6(x).
\end{aligned}
$$

It follows from the hypothesis $T1=0$ that $Tf_4=0$.

By Hölder's inequality, the $(L^2,L^{q'})$-boundedness of T in Definition 1.1, where $1/q'=1/2-\beta/n$, Lemma 2.1 and $0<r\leq 1$, we have

$$
\begin{aligned}
\frac{1}{|B|}\int_B|Tf_5(x)|dx &\leq \left(\frac{1}{|B|}\int_B|Tf_5(x)|^{q'}dx\right)^{1/q'}\\
&\leq C|B|^{-1/q'}\left(\int_{8\widetilde{B}}|f(y)-f_{2\widetilde{B}}|^2dy\right)^{1/2}\\
&\leq C\|f\|_{\mathrm{BMO}}|B|^{-1/q'}|\widetilde{B}|^{1/2}\\
&= C\|f\|_{\mathrm{BMO}}r^{n(\alpha/2-1/q')}\\
&\leq C\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

The last inequality is due to $\alpha/2-1/q'\geq 0$ which follows from $\beta\geq n(1-\alpha)/2$ in Definition 1.1.

Since $Tf(x)$ and $Tf_5(x)$ exist a.e. in $\mathbb{R}^n$, there is a point $z_2\in B$ such that $|Tf_6(z_2)|<\infty$. For any $x\in B$ and $y\in(8\widetilde{B})^c$, $2|x-z_2|^{\alpha}\leq 2(2r)^{\alpha}<4r^{\alpha}<|y-z_2|$

since $0 < r \leq 1$. It follows from (2) of Definition 1.1 and Lemma 2.1 that

$$\begin{aligned}
&\frac{1}{|B|}\int_B |Tf_6(x) - Tf_6(z_2)|dx \\
&\leq \frac{1}{|B|}\int_B\int_{(8\widetilde{B})^c} |K(x,y) - K(z_2,y)||f(y) - f_{2\widetilde{B}}|dydx \\
&\leq C\frac{1}{|B|}\int_B\int_{(8\widetilde{B})^c} \frac{|x-z_2|^\delta}{|y-z_2|^{n+\delta/\alpha}}|f(y) - f_{2\widetilde{B}}|dydx \\
&\leq Cr^\delta \sum_{k=3}^\infty \int_{2^{k+1}\widetilde{B}\setminus 2^k\widetilde{B}} \frac{|f(y) - f_{2\widetilde{B}}|}{|y-z_2|^{n+\delta/\alpha}}dy \\
&\leq Cr^\delta \sum_{k=3}^\infty (2^k r^\alpha)^{-\delta/\alpha}\frac{1}{|2^{k+1}\widetilde{B}|}\int_{2^{k+1}\widetilde{B}} |f(y) - f_{2\widetilde{B}}|dy \\
&\leq C\|f\|_{\mathrm{BMO}}\sum_{k=3}^\infty k2^{-k\delta/\alpha} \\
&= C\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{|B|}\int_B |Tf(x) - (Tf)_B|dx \\
&\leq \frac{2}{|B|}\int_B |Tf(x) - Tf_6(z_2)|dx \\
&\leq \frac{2}{|B|}\int_B |Tf_5(x)|dx + \frac{2}{|B|}\int_B |Tf_6(x) - Tf_6(z_2)|dx \\
&\leq C\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

Therefore, in both cases, we have

$$\frac{1}{|B|}\int_B |Tf(x) - (Tf)_B|dx \leq C\|f\|_{\mathrm{BMO}},$$

which completes the proof of the theorem. □

Remark 2.1. *If we assume that* $T^*1 = 0$, *then by a discussion similar to that in Theorem 2.1, it follows that*

$$\|T^*f\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}$$

for $f \in \mathrm{BMO}$ *such that* $T^*f(x)$ *exists a.e. in* $\mathbb{R}^n$.

Given $f \in H^1$, for any $g \in \mathrm{VMO}$ with compact support, the duality relation $(H^1)' = \mathrm{BMO}$ and $T^*1 = 0$ imply that

$$|\langle Tf, g\rangle| = |\langle f, T^*g\rangle| \leq \|T^*g\|_{\mathrm{BMO}}\|f\|_{H^1} \leq C\|g\|_{\mathrm{BMO}}\|f\|_{H^1}.$$

Because the set of VMO functions with compact support is dense in VMO, we can get that $Tf \in (\mathrm{VMO})' = H^1$. Moreover

$$\|Tf\|_{H^1} \leq C\|f\|_{H^1}.$$

Actually, this conclusion has been obtained by Alvarez and Milman in [3].

From a contrasting point of view, the BMO-boundedness in Theorem 2.1 can be also formulated by a duality discussion based on the (H^1, H^1)-boundedness in [3]. As a matter of fact, we give a straightforward proof for it in this paper.

On the other hand, the boundedness of strongly singular Calderón-Zygmund operators on the LMO space can be established as follows.

Theorem 2.2. *Let T be a strongly singular Calderón-Zygmund operator and $T1 = 0$. Suppose $f \in \mathrm{LMO}$ such that $Tf(x)$ exists a.e. in $\mathbb{R}^n$. Then $Tf \in \mathrm{LMO}$ and*

$$[Tf]_{\mathrm{LMO}} \leq C[f]_{\mathrm{LMO}},$$

where $C > 0$ is independent of f.

Proof Let α, β and δ be given as in Definition 1.1. For any ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $r \geq 1$, by the BMO-boundedness of T in Theorem 2.1, we have

$$\frac{1}{|B|}\int_B |Tf(x) - (Tf)_B|dx \leq \|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}} \leq C[f]_{\mathrm{LMO}}.$$

It suffices to prove that, for any ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $0 < r < 1$, the following inequality holds.

$$\frac{1 + |\ln r|}{|B|}\int_B |Tf(x) - (Tf)_B|dx \leq C[f]_{\mathrm{LMO}}.$$

We consider two cases respectively.

(i) $16^{-1/\alpha} \leq r < 1$.

The BMO-boundedness of T also implies that

$$\begin{aligned}
&\frac{1 + |\ln r|}{|B|}\int_B |Tf(x) - (Tf)_B|dx \\
&= \frac{1 + \ln \frac{1}{r}}{|B|}\int_B |Tf(x) - (Tf)_B|dx \\
&\leq C\frac{1}{|B|}\int_B |Tf(x) - (Tf)_B|dx \\
&\leq C\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}} \leq C[f]_{\mathrm{LMO}}.
\end{aligned}$$

(ii) $0 < r < 16^{-1/\alpha}$.

Let $\widetilde{B} = B(x_0, r^\alpha)$. Write

$$\begin{aligned} f(x) &= f_{8\widetilde{B}} + (f(x) - f_{8\widetilde{B}})\chi_{8\widetilde{B}}(x) + (f(x) - f_{8\widetilde{B}})\chi_{(8\widetilde{B})^c}(x) \\ &:= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

It follows from the hypothesis $T1 = 0$ that $Tf_1 = 0$.

Notice that $0 < 8r^\alpha < 1/2$. By Hölder's inequality, the $(L^2, L^{q'})$-boundedness of T, Lemma 2.2 and (2.1), we have

$$\begin{aligned} \frac{1}{|B|}\int_B |Tf_2(x)|dx &\leq \left(\frac{1}{|B|}\int_B |Tf_2(x)|^{q'}dx\right)^{1/q'} \\ &\leq C|B|^{-1/q'}\left(\int_{8\widetilde{B}} |f(y) - f_{8\widetilde{B}}|^2 dy\right)^{1/2} \\ &\leq C[f]_{\mathrm{LMO}^2}|B|^{-1/q'}|\widetilde{B}|^{1/2}(1 + |\ln 8r^\alpha|)^{-1} \\ &\leq C[f]_{\mathrm{LMO}} r^{n(\alpha/2-1/q')}(1 + |\ln 8 + \alpha \ln r|)^{-1} \\ &\leq C[f]_{\mathrm{LMO}}(1 + \ln 8)(1 + \alpha|\ln r|)^{-1} \\ &\leq C[f]_{\mathrm{LMO}}(1 + |\ln r|)^{-1}. \end{aligned}$$

Since $Tf(x)$ and $Tf_2(x)$ exist a.e. in $\mathbb{R}^n$, there is a point $x^* \in B$ such that $|Tf_3(x^*)| < \infty$. For any $x \in B$ and $y \in (8\widetilde{B})^c$, $2|x - x^*|^\alpha \leq 2(2r)^\alpha < 4r^\alpha < |y - x^*|$ since $0 < r < 1$. It follows from (2) of Definition 1.1, Lemma 2.3 and (2.1) that

$$\begin{aligned} &\frac{1}{|B|}\int_B |Tf_3(x) - Tf_3(x^*)|dx \\ &\leq \frac{1}{|B|}\int_B\int_{(8\widetilde{B})^c} |K(x,y) - K(x^*,y)||f(y) - f_{8\widetilde{B}}|dydx \\ &\leq C\frac{1}{|B|}\int_B\int_{(8\widetilde{B})^c} \frac{|x - x^*|^\delta}{|y - x^*|^{n+\delta/\alpha}}|f(y) - f_{8\widetilde{B}}|dydx \\ &\leq Cr^\delta \int_{(8\widetilde{B})^c} \frac{|f(y) - f_{8\widetilde{B}}|}{|y - x_0|^{n+\delta/\alpha}}dy \\ &\leq Cr^\delta (8r^\alpha)^{-\delta/\alpha}(1 + |\ln 8r^\alpha|)^{-1}[f]_{\mathrm{LMO}} \\ &\leq C[f]_{\mathrm{LMO}}(1 + |\ln r|)^{-1}. \end{aligned}$$

Thus,

$$\frac{1 + |\ln r|}{|B|}\int_B |Tf(x) - (Tf)_B|dx$$

$$\leq 2\frac{1+|\ln r|}{|B|}\int_B |Tf(x)-Tf_3(x^*)|dx$$
$$\leq 2\frac{1+|\ln r|}{|B|}\int_B |Tf_2(x)|dx + 2\frac{1+|\ln r|}{|B|}\int_B |Tf_3(x)-Tf_3(x^*)|dx$$
$$\leq C[f]_{\mathrm{LMO}}.$$

This gives the desired result. □

Remark 2.2. *It should be pointed out that there is a counterpart of the above result for T^* under the hypothesis $T^*1 = 0$, but we omit the details for their similarity.*

3. A pointwise estimate for the sharp maximal function

The definition and properties of BMO functions lead us naturally to study the sharp maximal function $f^\sharp$, associated to any locally integrable function f. It is defined by

$$f^\sharp(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)-f_B|dy$$
$$\sim \sup_{B\ni x}\inf_{a\in\mathbb{C}}\frac{1}{|B|}\int_B |f(y)-a|dx,$$

where the supremum is taken over all balls B containing x. In fact, the above definition is equivalent to the one by taking the supremum over all balls B centered at x.

A function f is in the BMO exactly when $f^\sharp$ is a bounded function. This observation illustrates that sometimes significant aspects of f are most directly expressed in terms of $f^\sharp$.

In this section, we will state a pointwise estimate for the sharp maximal function of commutators generated by strongly singular Calderón-Zygmund operators and BMO functions. First, the following elementary inequality is necessary.

Lemma 3.1. *Given $\varepsilon > 0$, there is*

$$\ln x \leq \frac{1}{\varepsilon}x^\varepsilon, \quad \text{for all } x \geq 1.$$

Let $\varphi(x) = \ln x - \frac{1}{\varepsilon}x^\varepsilon$, $x \geq 1$. The above result comes from the monotone property of the function φ.

Besides the (L^p, L^p)-boundedness, $1 < p < \infty$, the strongly singular Calderón-Zygmund operator T still has other boundedness properties on Lebesgue spaces. By interpolating between $(L^2, L^{q'})$ and (L^∞, BMO), where q is given as in Definition 1.1 and $1/q + 1/q' = 1$, T is bounded from L^u to L^v, $2 \leq u < \infty$ and $v = \frac{uq'}{2}$. It is easy to see that $0 < \frac{u}{v} \leq \alpha$. Then we interpolate between $(L^2, L^{q'})$ and weak (L^1, L^1)

to obtain the boundedness of T from L^u to L^v, $1 < u \le 2$ and $v = \frac{uq'}{2q'-uq'+2u-2}$. In this situation, $0 < \frac{u}{v} \le \alpha$ if and only if $\frac{n(1-\alpha)+2\beta}{2\beta} \le u \le 2$. In a word, T is bounded from L^u to L^v, $\frac{n(1-\alpha)+2\beta}{2\beta} \le u < \infty$ and $0 < \frac{u}{v} \le \alpha$. In particular, if we restrict $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$ in (3) of Definition 1.1, then T is bounded from L^u to L^v, $\frac{n(1-\alpha)+2\beta}{2\beta} < u < \infty$ and $0 < \frac{u}{v} < \alpha$.

Theorem 3.1. *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given as in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$. If $b \in \mathrm{BMO}$, then for any s satisfying $\frac{n(1-\alpha)+2\beta}{2\beta} < s < \infty$, there exists a constant $C > 0$ such that for all smooth functions f with compact support,*

$$([b,T]f)^\sharp(x) \le C\|b\|_{\mathrm{BMO}}(M_s(Tf)(x) + M_s(f)(x)).$$

Proof For any ball $B = B(x_0, r) \subset \mathbb{R}^n$, there are two cases.

(i) $r > 1$.

Write

$$\begin{aligned}[b,T]f(x) &= [b - b_{2B}, T]f(x) \\ &= (b - b_{2B})Tf(x) - T((b - b_{2B})f\chi_{2B})(x) - T((b - b_{2B})f\chi_{(2B)^c})(x).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{|B|}\int_B |[b,T]f(x) - T((b_{2B} - b)f\chi_{(2B)^c})(x_0)|dx \\ &\le \frac{1}{|B|}\int_B |b(x) - b_{2B}||Tf(x)|dx + \frac{1}{|B|}\int_B |T((b - b_{2B})f\chi_{2B})(x)|dx \\ &\quad + \frac{1}{|B|}\int_B |T((b - b_{2B})f\chi_{(2B)^c})(x) - T((b - b_{2B})f\chi_{(2B)^c})(x_0)|dx \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

For I_1, Hölder's inequality yields that

$$\begin{aligned}I_1 &\le \left(\frac{1}{|B|}\int_B |b(x) - b_{2B}|^{s'}dx\right)^{1/s'}\left(\frac{1}{|B|}\int_B |Tf(x)|^s dx\right)^{1/s} \\ &\le C\|b\|_{\mathrm{BMO}}M_s(Tf)(x_0).\end{aligned}$$

Since $\frac{n(1-\alpha)+2\beta}{2\beta} < s < \infty$, there exists an s_0 such that $\frac{n(1-\alpha)+2\beta}{2\beta} < s_0 < s < \infty$. Denote by $1/s_1 = 1/s_0 - 1/s$. To estimate I_2, we use Hölder's inequality and the (L^{s_0}, L^{s_0})-boundedness of T.

$$I_2 \le \left(\frac{1}{|B|}\int_B |T((b - b_{2B})f\chi_{2B})(x)|^{s_0}dx\right)^{1/s_0}$$

$$
\begin{aligned}
&\leq C\left(\frac{1}{|B|}\int_{2B}|b(y)-b_{2B}|^{s_0}|f(y)|^{s_0}dy\right)^{1/s_0}\\
&\leq C\left(\frac{1}{|2B|}\int_{2B}|b(y)-b_{2B}|^{s_1}dy\right)^{1/s_1}\left(\frac{1}{|2B|}\int_{2B}|f(y)|^{s}dy\right)^{1/s}\\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0).
\end{aligned}
$$

For any $x \in B$ and $y \in (2B)^c$, $2|x-x_0|^\alpha \leq 2r^\alpha < 2r \leq |y-x_0|$ since $r > 1$. It follows from (2) of Definition 1.1 and Lemma 2.1 that

$$
\begin{aligned}
I_3 &\leq \frac{1}{|B|}\int_B\int_{(2B)^c}|K(x,y)-K(x_0,y)||b(y)-b_{2B}||f(y)|dydx\\
&\leq C\frac{1}{|B|}\int_B\int_{(2B)^c}\frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta/\alpha}}|b(y)-b_{2B}||f(y)|dydx\\
&\leq Cr^\delta\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{1}{|y-x_0|^{n+\delta/\alpha}}|b(y)-b_{2B}||f(y)|dy\\
&\leq Cr^\delta\sum_{k=1}^{\infty}(2^kr)^{-\delta/\alpha}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy\\
&\leq Cr^{\delta-\delta/\alpha}\sum_{k=1}^{\infty}2^{-k\delta/\alpha}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(y)-b_{2B}|^{s'}dy\right)^{1/s'}\\
&\qquad\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f(y)|^{s}dy\right)^{1/s}\\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0)r^{\delta-\delta/\alpha}\sum_{k=1}^{\infty}k2^{-k\delta/\alpha}\\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0).
\end{aligned}
$$

(ii) $0 < r \leq 1$.

For the index s_0 which we chose above, there exists an l_0 such that T is bounded from L^{s_0} to L^{l_0} and $0 < \frac{s_0}{l_0} < \alpha$. Then we can take a θ satisfying $0 < \frac{s_0}{l_0} < \theta < \alpha$. Let $\widetilde{B} = B(x_0, r^\theta)$.

Write

$$
\begin{aligned}
[b,T]f(x) &= [b-b_{2B},T]f(x)\\
&= (b-b_{2B})Tf(x) - T((b-b_{2B})f\chi_{2\widetilde{B}})(x) - T((b-b_{2B})f\chi_{(2\widetilde{B})^c})(x).
\end{aligned}
$$

Then

$$
\frac{1}{|B|}\int_B|[b,T]f(x) - T((b_{2B}-b)f\chi_{(2\widetilde{B})^c})(x_0)|dx
$$

$$\leq \frac{1}{|B|}\int_B |b(x)-b_{2B}||Tf(x)|dx + \frac{1}{|B|}\int_B |T((b-b_{2B})f\chi_{2\widetilde{B}})(x)|dx$$
$$+\frac{1}{|B|}\int_B |T((b-b_{2B})f\chi_{(2\widetilde{B})^c})(x) - T((b-b_{2B})f\chi_{(2\widetilde{B})^c})(x_0)|dx$$
$$:= II_1 + II_2 + II_3.$$

The estimate of II_1 is the same as that of I_1.

$$II_1 \leq C\|b\|_{\mathrm{BMO}} M_s(Tf)(x_0).$$

The inequality $0 < \frac{s_0}{l_0} < \theta$ implies that $\varepsilon_1 := n(\frac{\theta}{s_0} - \frac{1}{l_0}) > 0$. By Hölder's inequality, the (L^{s_0}, L^{l_0})-boundedness of T, Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}
II_2 &\leq \left(\frac{1}{|B|}\int_B |T((b-b_{2B})f\chi_{2\widetilde{B}})(x)|^{l_0}dx\right)^{1/l_0} \\
&\leq C|B|^{-1/l_0}\left(\int_{2\widetilde{B}} |b(y)-b_{2B}|^{s_0}|f(y)|^{s_0}dy\right)^{1/s_0} \\
&\leq C|B|^{-1/l_0}|\widetilde{B}|^{1/s_0}\left(\frac{1}{|2\widetilde{B}|}\int_{2\widetilde{B}} |b(y)-b_{2B}|^{s_1}dy\right)^{1/s_1}\left(\frac{1}{|2\widetilde{B}|}\int_{2\widetilde{B}} |f(y)|^{s}dy\right)^{1/s} \\
&\leq C\|b\|_{\mathrm{BMO}} M_s(f)(x_0)|B|^{-1/l_0}|\widetilde{B}|^{1/s_0}\left(1+(1-\theta)\ln\frac{1}{r}\right) \\
&\leq C\|b\|_{\mathrm{BMO}} M_s(f)(x_0)|B|^{-1/l_0}|\widetilde{B}|^{1/s_0}\left(1+\frac{1}{\varepsilon_1}r^{-\varepsilon_1}\right) \\
&\leq C\|b\|_{\mathrm{BMO}} M_s(f)(x_0)r^{n(\frac{\theta}{s_0}-\frac{1}{l_0})-\varepsilon_1} \\
&= C\|b\|_{\mathrm{BMO}} M_s(f)(x_0).
\end{aligned}$$

The fact $\theta < \alpha$ implies that $\varepsilon_2 := \frac{\delta}{\alpha}(\alpha-\theta) > 0$. For any $x \in B$ and $y \in (2\widetilde{B})^c$, we have $2|x-x_0|^\alpha \leq 2r^\alpha \leq 2r^\theta \leq |y-x_0|$ since $0 < r \leq 1$. It follows from (2) of Definition 1.1, Lemma 2.1 and Lemma 3.1 that

$$\begin{aligned}
II_3 &\leq \frac{1}{|B|}\int_B\int_{(2\widetilde{B})^c} |K(x,y)-K(x_0,y)||b(y)-b_{2B}||f(y)|dydx \\
&\leq C\frac{1}{|B|}\int_B\int_{(2\widetilde{B})^c} \frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta/\alpha}}|b(y)-b_{2B}||f(y)|dydx \\
&\leq Cr^\delta\sum_{k=1}^\infty \int_{2^{k+1}\widetilde{B}\setminus 2^k\widetilde{B}} \frac{1}{|y-x_0|^{n+\delta/\alpha}}|b(y)-b_{2B}||f(y)|dy \\
&\leq Cr^\delta\sum_{k=1}^\infty (2^k r^\theta)^{-\delta/\alpha}\frac{1}{|2^{k+1}\widetilde{B}|}\int_{2^{k+1}\widetilde{B}} |b(y)-b_{2B}||f(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&\leq Cr^{\delta-\theta\delta/\alpha}\sum_{k=1}^{\infty}2^{-k\delta/\alpha}\bigg(\frac{1}{|2^{k+1}\widetilde{B}|}\int_{2^{k+1}\widetilde{B}}|b(y)-b_{2B}|^{s'}dy\bigg)^{1/s'} \\
&\qquad\times\bigg(\frac{1}{|2^{k+1}\widetilde{B}|}\int_{2^{k+1}\widetilde{B}}|f(y)|^{s}dy\bigg)^{1/s} \\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0)r^{\frac{\delta}{\alpha}(\alpha-\theta)}\sum_{k=1}^{\infty}2^{-k\delta/\alpha}\bigg(1+k+(1-\theta)\ln\frac{1}{r}\bigg) \\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0)r^{\frac{\delta}{\alpha}(\alpha-\theta)}\sum_{k=1}^{\infty}2^{-k\delta/\alpha}\bigg(1+k+\frac{1}{\varepsilon_2}r^{-\varepsilon_2}\bigg) \\
&\leq C\|b\|_{\mathrm{BMO}}M_s(f)(x_0)r^{\frac{\delta}{\alpha}(\alpha-\theta)-\varepsilon_2}\sum_{k=1}^{\infty}k2^{-k\delta/\alpha} \\
&= C\|b\|_{\mathrm{BMO}}M_s(f)(x_0).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
([b,T]f)^{\sharp}(x_0) &\sim \sup_{B(x_0,r)\subset\mathbb{R}^n}\inf_{a\in\mathbb{C}}\frac{1}{|B|}\int_B|[b,T]f(x)-a|dx \\
&\leq C\|b\|_{\mathrm{BMO}}(M_s(Tf)(x_0)+M_s(f)(x_0)),
\end{aligned}
$$

which completes the proof of the theorem. □

4. Applications

The estimate for the sharp maximal function of $[b,T]$ can be applied to obtain not only the weighted norm estimate of the commutator, but also the boundedness properties of it on Morrey type spaces.

Morrey spaces have been of great value through the years in studying the local behavior of solutions to second elliptic partial differential equations.

Definition 4.1. *A function $f\in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the classical Morrey space $M^q_p(\mathbb{R}^n)$, $1\leq p\leq q<\infty$, if*

$$
\|f\|_{M^q_p(\mathbb{R}^n)}=\sup_{B\subset\mathbb{R}^n}|B|^{\frac{1}{q}-\frac{1}{p}}\bigg(\int_B|f(x)|^pdx\bigg)^{\frac{1}{p}}<\infty.
$$

Remark 4.1. *It can be seen from the special case $M^p_p(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ with $1\leq p<\infty$ that Morrey spaces are the generalization of Lebesgue spaces.*

Definition 4.2. *For a general positive function φ on $\mathbb{R}^n\times\mathbb{R}^+$, the generalized Morrey space $L^{p,\varphi}$ with $1\leq p<\infty$ is defined as follows.*

$$
L^{p,\varphi}=\{f\in L^p_{loc}(\mathbb{R}^n),\ \|f\|_{L^{p,\varphi}}<+\infty\},
$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{x\in\mathbb{R}^n,\, r>0}\left(\frac{1}{\varphi(x,\, r)}\int_{B(x,r)}|f(y)|^p dy\right)^{1/p}.$$

Remark 4.2. *For the case $\varphi(x,\, r) = r^{n(1-p/q)}$, we have $L^{p,\varphi} = M^q_p(\mathbb{R}^n)$, $1 \leq p \leq q < \infty$. Thus, generalized Morrey spaces are the generalization of classical Morrey spaces.*

Lemma 4.1. [11] *Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and there exists a C_0 satisfying $0 < C_0 < 2^n$ such that*

$$\varphi(x,\, 2r) \leq C_0\varphi(x,\, r) \quad \textit{for all} \quad x \in \mathbb{R}^n, r > 0. \tag{4.1}$$

If $1 < p < \infty$, then

$$\|Mf\|_{L^{p,\varphi}} \leq C\|f\|_{L^{p,\varphi}} \quad \textit{and} \quad \|Mf\|_{L^{p,\varphi}} \leq C\|f^\sharp\|_{L^{p,\varphi}},$$

where C is independent of f.

Remark 4.3. *As a matter of fact, the conditions of φ are stronger in* [11] *than here. However, just for the result of Lemma 4.1, the hypothesis here is sufficient.*

The boundedness of classical Calderón-Zygmund operators on Morrey spaces was established by Chiarenza and Frasca in [5]. More generally, the authors in [7] obtained that a sublinear operator T is bounded on Morrey spaces if T is bounded on $L^p(\mathbb{R}^n)$ and satisfies the following size condition:

$$|Tf(x)| \leq C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy,$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \mathrm{supp} f$.

For the case when T is a strongly singular Calderón-Zygmund operator, the corresponding conclusion has been obtained in [9].

Lemma 4.2. [9] *Let T be a strongly singular Calderón-Zygmund operator, and α, β, δ be given as in Definition 1.1. Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ such that (4.1) holds. If $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$, then T is bounded on $L^{p,\varphi}$.*

Now, let us proceed with the boundedness of commutators generated by strongly singular Calderón-Zygmund operators and BMO functions on Morrey spaces.

Theorem 4.1. *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given as in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$. Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ such that (4.1) holds. If $b \in \mathrm{BMO}$, then $[b, T]$ is bounded on $L^{p,\varphi}$, where $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$.*

Proof Noticing that $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$, there exists an s such that $\frac{n(1-\alpha)+2\beta}{2\beta} < s < p < \infty$. By Lemma 4.1, Theorem 3.1 and Lemma 4.2, we have

$$\begin{aligned}
\|[b,T]f\|_{L^{p,\varphi}} &\le \|M([b,T]f)\|_{L^{p,\varphi}} \le C\|([b,T]f)^{\sharp}\|_{L^{p,\varphi}} \\
&\le C\|b\|_{\mathrm{BMO}}\big(\|M_s(Tf)\|_{L^{p,\varphi}} + \|M_s(f)\|_{L^{p,\varphi}}\big) \\
&= C\|b\|_{\mathrm{BMO}}\big(\|M(|Tf|^s)\|^{1/s}_{L^{p/s,\varphi}} + \|M(|f|^s)\|^{1/s}_{L^{p/s,\varphi}}\big) \\
&\le C\|b\|_{\mathrm{BMO}}\big(\||Tf|^s\|^{1/s}_{L^{p/s,\varphi}} + \||f|^s\|^{1/s}_{L^{p/s,\varphi}}\big) \\
&= C\|b\|_{\mathrm{BMO}}\big(\|Tf\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}\big) \\
&\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of the theorem. □

In particular, if we take $\varphi(x, r) = r^{n(1-p/q)}$, $1 \le p \le q < \infty$, then Remark 4.2 implies the following conclusion.

Corollary 4.1. *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given as in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$. If $b \in \mathrm{BMO}$, then $[b,T]$ is bounded on $M_p^q(\mathbb{R}^n)$, where $\frac{n(1-\alpha)+2\beta}{2\beta} < p \le q < \infty$.*

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