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MATRIX TRANSFORMATIONS BETWEEN SETS OF THE FORM W_ξ AND OPERATOR GENERATORS OF ANALYTIC SEMIGROUPS

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Abstract. In this paper we establish a relation between the notion of operators of analytic semigroups and matrix transformations from a set of sequences into w_∞. We get extensions of some results given by Labbas and de Malafosse concerning applications of the sum of operators in the nondifferential case.

1. Introduction

In this paper we consider spaces that generalize the well-known sets w_0 and w_∞ introduced and studied by Maddox [5]. Recall that w_0 and w_∞ are the sets of sequences that are *strongly summable to zero and bounded* by *the Cesàro method of order* 1. In [14], Malkowsky and Rakočević gave the characterizations of matrix maps between w_0, w, or w_∞ and w_∞^p and between w_0, w, or w_∞ and ℓ_1.

More recently it was shown by de Malafosse and Malkowsky in [10] that if λ is an *exponentially bounded sequence* then $(w_\infty(\lambda), w_\infty(\lambda))$ is a *Banach algebra*. Here we give some properties of operators on the sets $W_\tau = D_\tau w_\infty$ and apply these results to particular matrix transformations between W_τ and w_∞. In this way we are led to explictly represent two unbounded operators that are given by infinite matrices and are *operator generators of an analytic semigroup (OGASG)*. Recall that this notion is a part of the theory of the *sum of operators* which was studied by many authors such as Da Prato and Grisvard [1], R. Labbas and B. Terreni [3]. In Labbas and de Malafosse [4] and de Malafosse [7] there are some applications of the sum of operators in the theory of summability in the noncommutative case. In this paper we extend some results given in [4, 7] using the same infinite matrices

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A and B defined here in sets $W_\xi \subset w_\infty$. The *relative boundedness with respect to A or B is not satisfied*, so we are not within the framework of the classical perturbation theory given by Kato [2].

In this paper we establish a relation between results in summability and the basic notions used in the theory of the sum of operators.

2. Preliminaries and well known results

For a given infinite matrix $M = (a_{nm})_{n,m\geq 1}$ we define the operators M_n for every integer $n \geq 1$ by $M_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m$, where $X = (x_n)_{n\geq 1}$ and the series are convergent. So we are led to the study of the infinite linear system $M_n(X) = y_n$ with $n = 1, 2, ...$ where $Y = (y_n)_{n\geq 1}$ is a one-column matrix and X is the unknown, see [6, 9]. The equations $M_n(X) = y_n$ for $n = 1, 2, ...$ can be written in the form $MX = Y$, where $MX = (M_n(X))_{n\geq 1}$. We write s for the set of all complex sequences and ℓ_∞, c_0 for the set of all *bounded and null sequences*. Recall that ℓ_∞ and c_0 are Banach spaces with norm $\|X\|_{\ell_\infty} = \sup_{n\geq 1}(|x_n|)$.

For subsets E and F of s we denote by (E, F) the set of all matrices that map E to F. For any subset E of s, we write ME for the set of all $Y \in s$ such that $Y = MX$ for some $X \in E$. If F is a subset of s, we will denote

$$F(M) = F_M = \{X \in s \ : \ Y = MX \in F\}. \tag{1}$$

A Banach space E of complex sequences with the norm $\|\cdot\|_E$ *is a BK space if each projection $P_n : X \mapsto P_nX = x_n$ is continuous*, (cf. [15]). A *BK space $E \subset s$ is said to have AK if every sequence $X = (x_n)_{n\geq 1} \in E$ has a unique representation $X = \sum_{n=1}^{\infty} x_n e^{(n)}$* where $e^{(n)}$ is the sequence with 1 in the n-th position and 0 otherwise. The set $B(E)$ of all bounded linear operators L mapping E to E normed by

$$\|L\|^*_{B(E)} = \sup_{X\neq 0} \frac{\|L(X)\|_E}{\|X\|_E}$$

is a Banach algebra and it is well known that if E is a BK space with AK, then $B(E) = (E, E)$.

Throughout we write U^+ for the set of all sequences $(u_n)_{n\geq 1}$ with $u_n > 0$ for all n, and $e = (1, ..., 1, ...)$. For $\lambda = (\lambda_n)_{n\geq 1} \in U^+$ we define the operator $C(\lambda) = (c_{nm})_{n,m\geq 1}$ by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases} \qquad (n = 1, 2, \dots).$$

It can be proved that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise} \end{cases}$$

is the inverse of $C(\lambda)$, see [8]. In the following we use the spaces of *strongly bounded and summable sequences* defined by

$$\begin{aligned} w_\infty(\lambda) &= \{X = (x_n)_{n\geq 1} \in s \ : C(\lambda)|X| \in \ell_\infty \}, \\ w_0(\lambda) &= \{X \in s : C(\lambda)|X| \in c_0 \} \end{aligned}$$

and

$$w(\lambda) = \{X \in s : \ X - le \in w_0(\lambda) \text{ for some } l \in \mathbb{C}\}.$$

These spaces were studied by Malkowsky with the concept of *exponentially bounded sequences*, see [12]. Recall that Maddox [5] defined and studied the special case $\lambda_n = n$ for all n of these sets, and denoted them by w_∞, w^0 and w.

3. The set W_τ and matrix transformations between sets of the form W_ξ

In this section we state some results on $W_\tau = D_\tau w_\infty$ and deal with the triangles Δ_ρ and Δ_ρ^T that map W_τ to itself.

3.1. **Some properties of the set W_τ.** For a given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we define the infinite diagonal matrix $D_\tau = (\tau_n \delta_{nm})_{n,m\geq 1}$. For any subset E of s, $D_\tau E$ *is the set of sequences with* $(x_n/\tau_n)_{n\geq 1} \in E$. We put $W_\tau = D_\tau w_\infty$, $W_\tau^0 = D_\tau w^0$ *for* $\tau \in U^+$. So $W_e = w_\infty$ *and* $W_e^0 = w^0$. It is well known [5] that w_∞ and w^0 are BK spaces with the norm

$$\|X\|_{w_\infty} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} |x_m| \right)$$

and w^0 has AK. It was shown in [10] that the class (w_∞, w_∞) is a *Banach algebra* normed by

$$\|M\|^*_{(w_\infty, w_\infty)} = \sup_{X\neq 0} \left(\frac{\|MX\|_{w_\infty}}{\|X\|_{w_\infty}} \right). \tag{2}$$

To study operator generators of analytic semigroups (OGASG) mapping into w_∞ we need the following results.

Proposition 3.1. *Let τ, $\nu \in U^+$. Then*

i) The sets W_τ and W_τ^0 are BK spaces normed by

$$\|X\|_{W_\tau} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{\tau_m} \right) \tag{3}$$

and W_τ^0 has AK.

ii) We have $\tau/\nu \in \ell_\infty$ if and only $W_\tau \subset W_\nu$.

iii) We have $W_\tau = W_\nu$ if and only if there are C_1, $C_2 > 0$ with

$$C_1 \leq \frac{\tau_n}{\nu_n} \leq C_2 \text{ for all } n. \tag{4}$$

Proof. i) Since w_∞ and w^0 are BK spaces with the norm $\|\cdot\|_{w_\infty}$ by [14, Theorem 3.3, pp. 179], the sets $W_\tau = w_\infty(D_{1/\tau})$ and $W_\tau^0 = w^0(D_{1/\tau})$ are BK spaces normed by $\|X\|_{W_\tau} = \|D_{1/\tau}X\|_{w_\infty}$ and W_τ^0 has AK.

ii) The necessity is a direct consequence of the inequality

$$\|X\|_{W_\nu} \leq \left\| \frac{\tau}{\nu} \right\|_{\ell_\infty} \|X\|_{W_\tau} \text{ for all } X \in W_\tau.$$

Conversely put $\lambda_n = n$ for all n. The inclusion $W_\tau \subset W_\nu$ means that $y = C(\lambda)D_{1/\tau}|X| \in \ell_\infty$ implies $C(\lambda)D_{1/\nu}|X| \in \ell_\infty$ for all $X \in s$. Since $(C(\lambda)D_{1/\tau})^{-1} = D_\tau\Delta(\lambda)$, we have $W_\tau \subset W_\nu$ if and only if

$$Y \in \ell_\infty \text{ implies } C(\lambda)D_{1/\nu}D_\tau\Delta(\lambda)Y \in \ell_\infty \text{ for all } Y,$$

that is

$$C(\lambda)D_{\tau/\nu}\Delta(\lambda) \in (\ell_\infty, \ell_\infty). \tag{5}$$

An elementary calculation gives

$$\left[C(\lambda)D_{\tau/\nu}\Delta(\lambda)\right]_{nm} = \begin{cases} \left(\dfrac{\tau_m}{\nu_m} - \dfrac{\tau_{m+1}}{\nu_{m+1}}\right)\dfrac{m}{n} & \text{for } m \leq n-1, \\ \dfrac{\tau_n}{\nu_n} & \text{for } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then using the characterization of $(\ell_\infty, \ell_\infty)$ and condition (5), we obtain

$$\frac{\tau_n}{\nu_n} \leq \sup_{n \geq 2} \left(\sum_{m=1}^{n-1} \left| \frac{\tau_m}{\nu_m} - \frac{\tau_{m+1}}{\nu_{m+1}} \right| \frac{m}{n} + \frac{\tau_n}{\nu_n} \right) < \infty \text{ for all } n$$

and we conclude $\tau/\nu \in \ell_\infty$.

iii) $W_\tau = W_\nu$ is equivalent to τ/ν, $\nu/\tau \in \ell_\infty$ that is (4). $\square$

3.2. **On the operators Δ_ρ^+ and Δ_ρ^- considered as maps in W_τ.** For given a given sequence $\rho = (\rho_n)_{n\geq 1}$ we consider the operator Δ_ρ^+ defined by

$$\left[\Delta_\rho^+ X\right]_n = x_n - \rho_n x_{n+1} \text{ for all } n \geq 1.$$

Then we get, putting $(\Delta_\rho^+)^T = \Delta_\rho^-$,

$$\left[\Delta_\rho^- X\right]_n = x_n - \rho_{n-1} x_{n-1} \text{ for all } n \geq 1$$

with the convention $x_0 = 0$.

To state the next Lemma we write for all $\tau \in U^+$ and all integers k

$$\rho^+(\tau) = (\rho_n \tau_{n+1}/\tau_n)_{n\geq 1},\ \ \rho^-(\tau) = (\rho_n \tau_{n-1}/\tau_n)_{n\geq 2},$$

$$\theta_k^+(\tau) = (1 + \frac{1}{k}) \sup_{n\geq k}(|\rho_n^+(\tau)|) \text{ and } \theta_k^-(\tau) = (1 + \frac{1}{k}) \sup_{n\geq k+1}\left(|\rho_n^-(\tau)|\right).$$

We also use the infinite matrices $\Sigma_\rho^{+(N)}$ and $\Sigma_\rho^{-(N)}$ defined by

$$\Sigma_\rho^{+(N)} = \begin{pmatrix} \left[\Delta_\rho^{+(N)}\right]^{-1} & & 0 \\ & 1 & \\ 0 & & . \end{pmatrix} \text{ and } \Sigma_\rho^{-(N)} = \begin{pmatrix} \left[\Delta_\rho^{-(N)}\right]^{-1} & & 0 \\ & 1 & \\ 0 & & . \end{pmatrix}$$

where $\Delta_\rho^{+(N)}$ and $\Delta_\rho^{-(N)}$ are the finite matrices whose elements are those of the N first rows and columns of Δ_ρ^+ and Δ_ρ^-. Now we state the next lemma.

Lemma 3.2. *Let $\rho, \tau \in U^+$.*

i) a) For every $N \geq 1$ we have $\|I - \Delta_\rho^+ \Sigma_\rho^{+(N)}\|^*_{(W_\tau, W_\tau)} \leq \theta_N^+(\tau)$;

b) if $\overline{\lim}_{n\to\infty} |\rho_n^+(\tau)| = 0$ then

$$\lim_{n\to\infty} \left\|I - \Delta_\rho^+ \Sigma_\rho^{+(n)}\right\|^*_{(W_\tau, W_\tau)} = 0.$$

ii) a) For every $N \geq 1$ we have $\|I - \Delta_\rho^- \Sigma_\rho^{-(N)}\|^*_{(W_\tau, W_\tau)} \leq \theta_N^-(\tau)$;

b) if $\overline{\lim}_{n\to\infty} |\rho_n^-(\tau)| = 0$ then

$$\lim_{n\to\infty} \left\|I - \Delta_\rho^- \Sigma_\rho^{-(n)}\right\|^*_{(W_\tau, W_\tau)} = 0.$$

Proof. (i) a) First we note that the finite matrix $\Delta_\rho^{+(N)}$ is invertible, since it is an upper triangle. We get $\Delta_\rho^+ \Sigma_\rho^{+(N)} = (a_{nm})_{n,m\geq 1}$ with $a_{nn} = 1$ for all n; $a_{n,n+1} = -\rho_n$ for all $n \geq N$; and $a_{nm} = 0$ otherwise. For every $X \in W_\tau$, we have $(I - \Delta_\rho^+ \Sigma_\rho^{+(N)})X = (\xi_n(X))_{n\geq 1}$ with $\xi_n(X) = 0$ for all $n \leq N - 1$ and $\xi_n(X) = \rho_n x_{n+1}$ for all $n \geq N$. Then we obtain for every $X \in W_\tau$

$$\begin{aligned}
\left\|(I-\Delta_\rho^+\Sigma_\rho^{+(N)})X\right\|_{W_\tau} &= \sup_{n\geq N}\left(\frac{1}{n}\sum_{k=N}^{n}\frac{|\rho_k x_{k+1}|}{\tau_k}\right)\\
= \sup_{n\geq N}\left(\frac{1}{n}\sum_{k=N}^{n}\frac{|\rho_k|}{\tau_k}\tau_{k+1}\frac{|x_{k+1}|}{\tau_{k+1}}\right) &\leq \sup_{n\geq N}\left[\left(\sup_{k\geq N}|\rho_k^+(\tau)|\right)\frac{1}{n}\sum_{k=N+1}^{n+1}\frac{|x_k|}{\tau_k}\right]\\
&\leq \sup_{n\geq N}\left(\frac{n+1}{n}\right)\sup_{k\geq N}\left(|\rho_k^+(\tau)|\right)\sup_{n\geq N}\left(\frac{1}{n+1}\sum_{k=N+1}^{n+1}\frac{|x_k|}{\tau_k}\right)\\
&\leq \left[\left(1+\frac{1}{N}\right)\sup_{k\geq N}(|\rho_k^+(\tau)|)\right]\|X\|_{W_\tau} = \theta_N^+(\tau)\|X\|_{W_\tau}.
\end{aligned}$$

We conclude

$$\left\|I-\Delta_\rho^+\Sigma_\rho^{+(N)}\right\|^*_{(W_\tau,W_\tau)} \leq \theta_N^+(\tau) < 1.$$

i) b) Part b) is a direct consequence of Part a).
ii) Part ii) can be shown similarly. □

As a direct consequence of the preceding lemma we get the following

Proposition 3.3. *Let $\rho,\tau\in U^+$.*
i) a) If $\rho^+(\tau)\in\ell_\infty$ then $\Delta_\rho^+\in(W_\tau,W_\tau)$ and

$$\|\Delta_\rho^+\|^*_{(W_\tau,W_\tau)} \leq 1+2\|\rho^+(\tau)\|_{\ell_\infty}. \tag{6}$$

b) If

$$\overline{\lim_{n\to\infty}}(|\rho_n^+(\tau)|) < 1 \tag{7}$$

then the operator Δ_ρ^+ is a bijection from W_τ to itself.
ii) a) If $\rho^-(\tau)\in\ell_\infty$ then $\Delta_\rho^-\in(W_\tau,W_\tau)$ and

$$\|\Delta_\rho^-\|^*_{(W_\tau,W_\tau)} \leq 1+\|\rho^-(\tau)\|_{\ell_\infty};$$

b) if

$$\overline{\lim_{n\to\infty}}(|\rho_n^-(\tau)|) < 1 \tag{8}$$

then the operator Δ_ρ^- is a bijection from W_τ to itself and

$$W_\tau(\Delta_\rho^-) = W_\tau.$$

Proof. i) a) We have for each $X\in W_\tau$

$$\|\Delta_\rho^+X\|_{W_\tau} = \sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\frac{|x_m+\rho_m x_{m+1}|}{\tau_m}\right)$$

and, since $\rho_m x_{m+1}/\tau_m = \rho_m^+(\tau) x_{m+1}/\tau_{m+1}$, we deduce

$$\begin{aligned}\|\Delta_\rho^+ X\|_{W_\tau} &\le \|X\|_{W_\tau} + \sup_m (|\rho_m^+(\tau)|) \sup_n \left(\frac{n+1}{n} \frac{1}{n+1} \sum_{m=2}^{n+1} \left(\frac{|x_m|}{\tau_m} \right) \right) \\ &\le \left(1 + 2\|\rho^+(\tau)\|_{\ell_\infty}\right) \|X\|_{W_\tau},\end{aligned}$$

and conclude that (6) holds.

i) b) By (7), for given l with $0 < l < 1$ and for every $\varepsilon > 0$, there is an integer n_0 such that

$$\sup_{n \ge n_0} (|\rho_n^+(\tau)|) < l + \varepsilon.$$

Then there is an integer n_1 with $\sup_{n \ge n_1}(1 + 1/n) < 1 + \varepsilon$. So there is $N \ge \max\{n_0, n_1\}$ and a sufficiently small $\varepsilon > 0$ such that

$$\theta_N^+(\tau) \le (1+\varepsilon)(l+\varepsilon) < 1.$$

We obtain

$$\left\| I - \Delta_\rho^+ \Sigma_\rho^{+(N)} \right\|_{(W_\tau, W_\tau)}^* \le \theta_N^+(\tau) < 1,$$

and $\Delta_\rho^+ \Sigma_\rho^{+(N)}$ has a unique inverse in the Banach algebra (W_τ, W_τ). Since $\Sigma_\rho^{+(N)}$ obviously is bijective from W_τ into itself, the operators defined by $\Delta_\rho^+ \Sigma_\rho^{+(N)}$ and $\Delta_\rho^+ = (\Delta_\rho^+ \Sigma_\rho^{+(N)})(\Sigma_\rho^{+(N)})^{-1}$ are bijective from W_τ into itself. So for every given $Y \in W_\tau$ the equation $\Delta_\rho^+ X = Y$ has a unique solution in W_τ.

ii) a) Here we have

$$\begin{aligned}\|\Delta_\rho^- X\|_{W_\tau} &= \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m + \rho_m x_{m-1}|}{\tau_m} \right) \\ &\le \|X\|_{W_\tau} + \|\rho^-(\tau)\|_{\ell_\infty} \sup_n \left(\frac{n-1}{n} \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{|x_m|}{\tau_m} \right) \\ &\le \left(1 + \|\rho^-(\tau)\|_{\ell_\infty}\right) \|X\|_{W_\tau} \text{ for all } X \in W_\tau.\end{aligned}$$

This concludes the proof of ii) a).

Reasoning as in the proof of i) b), we get ii) b). □

4. Spectral properties of an unbounded operator mapping into w_∞

In this section we apply the results obtained in the previous section to special matrix transformations A from W_ξ to w_∞, and give some spectral properties of A.

4.1. **Definition and first properties of an upper triangle mapping into w_∞.** Let $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ be two sequences and define the matrix A by $[A]_{nn} = a_n$, $[A]_{n,n+1} = b_n$ for all n and $[A]_{nm} = 0$ otherwise. Then we have

$$A_n(X) = a_n x_n + b_n x_{n+1} \text{ for all } n \text{ and all } X \in s. \tag{9}$$

We assume that A satisfies the following properties

$$\left\{\begin{array}{ll} \text{i)} & a \in U^+, \text{ and there is } \alpha_A > 0 \text{ with } a_n \geq \alpha_A n \text{ for all } n, \\ \text{ii)} & \text{there is } M_A > 0 \text{ such that } |b_n| \leq M_A \text{ for all } n. \end{array}\right. \tag{10}$$

We immediatly get the following properties.

Lemma 4.1. *Let a be as in (10). Then we have*
i) $W_{1/a} \subset w_\infty$,
ii) $\overline{W_{1/a}} \neq w_\infty$,
iii) $A \in (W_{1/a}, w_\infty)$, *and for every* $X \in W_{1/a}$

$$\|AX\|_{w_\infty} \leq \|X\|_{W_{1/a}} + 2M_A\|X\|_{w_\infty}. \tag{11}$$

Proof. i) Part i) is a consequences of Proposition 3.1.

ii) To show Part (ii), let $e \in w_\infty$ and assume $X'_p = (x_{np})_{n\geq 1}$ tends to e in $W_{1/a}$, that is

$$\|X'_p - e\|_{W_{1/a}} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} a_m |x_{mp} - 1| \right) \to 0 \ (p \to \infty).$$

We have for every n by (10) i)

$$\|X'_p - e\|_{W_{1/a}} \geq \alpha_A |x_{np} - 1|$$

and $x_{np} \to 1$ $(p \to \infty)$. There is p_0 such that for every $p \geq p_0$ and every n, $|x_{np}| \geq 1/2$ and

$$\|X'_p\|_{W_{1/a}} \geq \frac{1}{n} \sum_{m=1}^{n} a_m |x_{mp}| \geq \frac{1}{2n} \sum_{m=1}^{n} a_m.$$

Again we have for every k by(10) i)

$$\|a\|_{w_\infty} \geq \frac{1}{2k} \sum_{m=k}^{2k} a_m \geq \alpha_A \frac{1}{2k} \sum_{m=k}^{2k} m \geq \frac{3}{4} \alpha_A (k-1).$$

Finally we obtain $\|X'_p\|_{W_{1/a}} \geq \|a\|_{w_\infty}/2 = \infty$ for $p \geq p_0$. This contradicts the fact that $X'_p \in W_{1/a}$ for all p.

iii) Part iii comes from the inequality

$$\frac{1}{n} \sum_{m=1}^{n} |b_{m+1} x_{m+1}| = \frac{n+1}{n} \frac{1}{n+1} \sum_{k=2}^{n+1} |b_k x_k| \leq 2M_A \|X\|_{w_\infty} \text{ for all } n.$$

□

4.2. **Spectral properties of** A**.** We state some elementary lemmas which can be found in [4].

Lemma 4.2. *Let* $\varepsilon \in]0, \pi/2[$ *and* $x > 0$*. Then we have*

$$|x - \lambda| \geq x \sin \varepsilon \text{ for all } \lambda \in \mathbb{C} \text{ with } |Arg(\lambda)| \geq \varepsilon.$$

Lemma 4.3. *Let* $x > 0$*. Then we have*

$$\begin{cases} |x - \lambda| \geq |\lambda| \sin \theta & \text{for all } \lambda = |\lambda| e^{i\theta} \notin \mathbb{R}^-, \\ |x - \lambda| \geq |\lambda| & \text{for all } \lambda \in \mathbb{R}^-. \end{cases}$$

Now we can state the next results on the inverse of $A - \lambda I$*.*

Proposition 4.4. *Let* $\varepsilon_A \in]0, \pi/2[$*. Then the infinite matrix* $A - \lambda I$ *considered as operator in* $W_{1/a}$ *is invertible for every* $\lambda \in \mathbb{C}$ *with* $|Arg(\lambda)| \geq \varepsilon_A$*, that is*

$$(A - \lambda I)^{-1} \in (W_{1/a}, w_\infty)$$

and

$$\|(A - \lambda I)^{-1}\|^*_{(w_\infty, w_\infty)} \leq \frac{M}{|\lambda|} \text{ for all } \lambda \neq 0 \text{ with } |Arg(\lambda)| \geq \varepsilon_A. \tag{12}$$

Proof. i) We fix $\varepsilon_A \in]0, \pi/2[$ and consider the sector

$$\Pi_{\varepsilon_A} = \{\lambda \in \mathbb{C} : |\text{Arg}(\lambda)| < \varepsilon_A\}.$$

We put for every $\lambda \notin \Pi_{\varepsilon_A}$

$$\chi_n = \frac{b_n}{a_n - \lambda}$$

and $D'_\lambda = D_{(1/((a_n - \lambda)_n)}$. Then we have $[D'_\lambda(A - \lambda I)]_{nn} = 1$, $[D'_\lambda(A - \lambda I)]_{n,n+1} = \chi_n$ and $[D'_\lambda(A - \lambda I)]_{nm} = 0$ otherwise. We apply Lemma 4.2 with $\rho = \chi$ and obtain

$$|\chi_n| \leq \frac{M_A}{a_n \sin \varepsilon_A} \text{ for all } n \text{ and all } \lambda \notin \Pi_{\varepsilon_A}$$

and, since $a_n \to \infty$ $(n \to \infty)$, there is n_0 such that

$$\sup_{n \geq n_0} |\chi_n| \frac{a_n}{a_{n+1}} \leq \frac{1}{4} \text{ for all } \lambda \notin \Pi_{\varepsilon_A}.$$

Then we get, putting $\Delta^+_\chi = D'_\lambda(A - \lambda I)$ and applying Lemma 3.2

$$\left\|I - \Delta^+_\chi \Sigma^{+(n_0)}_\chi\right\|^*_{(W_{1/a}, W_{1/a})} \leq \theta^+_{n_0}\left(\frac{1}{a}\right) \leq \frac{1}{4}\left(1 + \frac{1}{n_0}\right) \leq \frac{1}{2} < 1$$
$$\text{for all } \lambda \notin \Pi_{\varepsilon_A}.$$

Therefore $\Delta_\chi^+ \Sigma_\chi^{+(n_0)}$ is bijective from $W_{1/a}$ to itself and $(\Delta_\chi^+ \Sigma_\chi^{+(n_0)})^{-1} \in (W_{1/a}, W_{1/a})$. Now we have for every $Y \in w_\infty$

$$Y' = D'_\lambda Y = (y_n/(a_n - \lambda))_{n \geq 1} \in W_{1/a}.$$

Indeed, we get for every n and $\lambda \notin \Pi_{\varepsilon_A}$

$$\frac{1}{n} \sum_{m=1}^{n} \left| \frac{y_m}{a_m - \lambda} \right| a_m \leq \frac{1}{n} \sum_{m=1}^{n} \frac{|y_m|}{a_m \sin \varepsilon_A} a_m \leq \frac{1}{\sin \varepsilon_A} \|Y\|_{w_\infty}$$

and $\|Y'\|_{W_{1/a}} \leq \|Y\|_{w_\infty} / \sin \varepsilon_A$. We successively obtain

$$\left(\Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right)^{-1} Y' \in W_{1/a},$$

$$(A - \lambda I)^{-1} Y = \Sigma_\chi^{+(n_0)} \left(\Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right)^{-1} Y' \in W_{1/a} \text{ for all } Y \in w_\infty$$

and

$$(A - \lambda I)^{-1} = \Sigma_\chi^{+(n_0)} \left(\Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right)^{-1} D'_\lambda \in (w_\infty, W_{1/a}) \text{ for all } \lambda \notin \Pi_{\varepsilon_A}.$$

Now we show that (12) holds. We have, for all $\lambda \notin \Pi_{\varepsilon_A}$ and for all $Y \in w_\infty$, $\Sigma_\chi^{+(n_0)} \in (w_\infty, w_\infty)$ and

$$\begin{aligned} \left\|(A - \lambda I)^{-1} Y\right\|_{w_\infty} & \\ \leq \left\|\Sigma_\chi^{+(n_0)}\right\|_{(w_\infty, w_\infty)} & \left\|\left(\Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right)^{-1}\right\|_{(w_\infty, w_\infty)} \|D'_\lambda Y\|_{w_\infty}. \end{aligned} \tag{13}$$

It follows that

$$\|D'_\lambda Y\|_{w_\infty} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \left| \frac{y_m}{a_m - \lambda} \right| \right) \leq \sup_{n \geq 1} \frac{1}{|a_n - \lambda|} \|Y\|_{w_\infty} \tag{14}$$

and we get by Lemma 4.3

$$\begin{aligned} \|D'_\lambda Y\|_{w_\infty} &\leq \sup_{n \geq 1} \frac{1}{|a_n - \lambda|} \\ &\leq \begin{cases} 1/|\lambda| \sin \theta & \text{for } \lambda = |\lambda| e^{i\theta} \notin \mathbb{R}^-, \\ 1/|\lambda| & \text{for } \lambda \in \mathbb{R}^-. \end{cases} \end{aligned} \tag{15}$$

Now we have, again by Lemma 3.2

$$\left\|I - \Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right\|_{(w_\infty, w_\infty)} \leq \theta_{n_0}^+(e) = \left(1 + \frac{1}{n_0}\right) \sup_{n \geq n_0} (|\chi_n|) \leq \frac{1}{2}$$

and then we easily get in the Banach algebra (w_∞, w_∞)

$$\left\| \left(\Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right)^{-1} \right\|_{(w_\infty, w_\infty)} \leq \sum_{m=0}^{\infty} \left\| \left(I - \Delta_\chi^+ \Sigma_\chi^{+(n_0)}\right) \right\|_{(w_\infty, w_\infty)}^m \leq \sum_{m=0}^{\infty} 2^{-m} = 2. \tag{16}$$

Finally we have from the expression of $\Sigma_\chi^{+(n_0)}$ in [4, p. 198]

$$\sup_{\lambda \notin \Pi_{\varepsilon_A}} \left\| \Sigma_\chi^{+(n_0)} \right\|_{(w_\infty, w_\infty)} < \infty$$

and we deduce from (13), (14), (15) and (16) that (12) holds. This concludes the proof. □

5. Matrix transformations in w_∞ and (OGASG)

In this section we apply the previous results to explicitly present matrices A and B for which $D(A)$ and $D(B)$ are not embedded in each other and that are (OGASG).

5.1. **Recall of some results in the general case.** Here we recall some results given in Da Prato-Grisvard [1] and Labbas-Terreni [3]. The set E is a Banach space and we consider two *closed linear operators* A *and* B, whose domains are $D(A)$ and $D(B)$ and included in E. Then we define $SX = AX + BX$ for every $X \in D(A) \cap D(B)$. The spectral properties of A and B are

$$(H) \begin{cases} \text{there are } C_A,\ C_B > 0 \text{ and } \varepsilon_A,\ \varepsilon_B \in]0, \pi[\text{ such that} \\ \text{i)} \quad \rho(A) \supset \sum_A = \{z \in \mathbb{C} : |\mathrm{Arg}(z)| < \pi - \varepsilon_A\} \text{ and} \\ \qquad \|(A - zI)^{-1}\|_{\mathcal{L}(E)} \leq \dfrac{C_A}{|z|} \text{ for all } z \in \sum_A - \{0\}; \\ \text{ii)} \quad \rho(B) \supset \sum_B = \{z \in \mathbb{C} : |\mathrm{Arg}(z)| < \pi - \varepsilon_B\} \text{ and} \\ \qquad \|(B - zI)^{-1}\|_{\mathcal{L}(E)} \leq \dfrac{C_B}{|z|} \text{ for all } z \in \sum_B - \{0\}; \\ \text{iii)} \quad \varepsilon_A + \varepsilon_B < \pi \end{cases}$$

If (H) is satisfied then A and B are (OGASG) not strongly continuous at $t = 0$ and we have $\sigma(A) \cap \sigma(-B) = \emptyset$ and $\rho(A) \cup \rho(-B) = \mathbb{C}$. In [4] an application was given for the solvability of the equation $(A + B + \lambda I)X = Y$ where A and B were considered as operators in ℓ_∞ in the noncommutative case.

Note that it is well known (cf. [1]) in the commutative case, that is when

$$(A-\xi I)^{-1}(B-\eta I)^{-1}-(B-\eta I)^{-1}(A-\xi I)^{-1}=0 \quad \text{for all } \xi\in\rho(A),\ \eta\in\rho(B),$$

that if $D(A)$ and $D(B)$ are dense in E, then the bounded operator defined by

$$L_\lambda=-\frac{1}{2i\pi}\int_\Gamma (B+zI)^{-1}(A-\lambda I-zI)^{-1}\,dz \text{ for all } \lambda>0,$$

where Γ is an infinite sectorial curve in $\rho(A-\lambda I)\cap\rho(-B)$, coincides with $(\overline{A+B}-\lambda I)^{-1}$.

5.2. **Matrix transformations as (OGASG).** In our case A is defined in Subsection 4.1 by (9) and (10). Then the matrix B is defined for $\beta=(\beta_n)_{n\geq 1}$, $\gamma=(\gamma_n)_{n\geq 1}\in s$ by

$$B_n(X)=\gamma_n x_{n-1}+\beta_n x_n \text{ for all } n \text{ and all } X\in s$$

with the convention $x_0=0$, where we assume

$$(17)\quad \begin{cases} \text{i)} & \beta\in U^+ \text{ and } \lim_{n\to\infty}\beta_{2k}=L\neq 0,\\ \text{ii)} & \lim_{k\to\infty}\beta_{2k+1}/a_{2k+1}=\infty\\ \text{iii)}\ \alpha) & \text{there is } M_B>0 \text{ such that } |\gamma_{2k}|\leq M_B \text{ for all } n,\\ \phantom{\text{iii)}}\ \beta) & \gamma_{2k+1}=o(1)\ (n\to\infty).\end{cases}$$

We easily see that $B\in(W_{1/\beta},w_\infty)$ and for each $X\in W_{1/\beta}$

$$\|BX\|_{w_\infty}\leq\|X\|_{W_{1/\beta}}+M_B\|X\|_{w_\infty}.$$

These results lead to the next remarks.

Remark 5.1. *We note for the convenience of the reader that, for instance, we can define A and B as follows; $a_n=n$; $b=e$; $\beta_n=1$ if $n=2k$, $\beta_n=k^2$ if $n=2k+1$; $\gamma_n=1$ if $n=2k$, $\gamma_n=1/k$ if $n=2k+1$ for all n and k.*

Remark 5.2. *We will see in Lemma 5.3 and Theorem 5.4 that the condition in (10) ii) implies that A is a closed operator. The conditions in (10) i) and (17) i), ii) imply that $D(A)$ and $D(B)$ are not embedded in each other. We will see in Theorem 5.4 that the conditions in (10) i) and (17) ii), iii) imply that $B+\mu I$ considered as an operator in $W_{1/\beta}$ is invertible and*

$$(B+\mu I)^{-1}\in(w_\infty,W_{1/\beta}) \textit{ for all } \mu \textit{ with } |Arg(\mu)|\leq\pi-\varepsilon_B.$$

Now we state the next result.

Lemma 5.3. *Let A, B be as in (10) and (17). Then we have*

i) $W_{1/a} \subset w_\infty$, $W_{1/\beta} \subset w_\infty$,

ii) $w_\infty(A) = W_{1/a}$ *and* $w_\infty(B) = W_{1/\beta}$,

iii) $W_{1/a}$ *and* $W_{1/\beta}$ *are not embedded in each other,*

iv) $\overline{W_{1/a}} \neq w_\infty$, $\overline{W_{1/\beta}} \neq w_\infty$.

Proof. i) The inclusion $W_{1/a} \subset w_\infty$ follows from Proposition 3.1 where $\tau = 1/a$, $\nu = e$ and $1/a \in c_0 \subset \ell_\infty$. Furthermore we have $1/\beta_{2k+1} = (1/a_{2k+1})(a_{2k+1}/\beta_{2k+1}) = o(1)$ $(k \to \infty)$ and $W_{1/\beta} \subset w_\infty$.

ii) First we have $A \in (W_{1/a}, w_\infty)$ by Lemma 4.1. It remains to show $w_\infty(A) = W_{1/a}$. First we have $W_{1/a} \subset w_\infty(A)$. Indeed it follows from [13, Theorem 1, pp. 260] that $I \in (W_{1/a}, w_\infty(A))$ if and only if $A \in (W_{1/a}, w_\infty)$. Now let $X \in w_\infty(A)$. Then we have $Y = AX \in w_\infty$. In the proof of Proposition 4.4, we can take $\lambda = 0$. Indeed there is n_0 such that

$$\chi_n = \frac{|b_n|}{a_n} \leq \frac{M_A}{n\alpha_A} \text{for all } n \geq n_0.$$

Then, for $Y \in w_\infty$, we successively get $D_{1/a}Y = (y_n/a_n)_{n\geq 1} \in W_{1/a}$, $(\Delta_\chi^+ \Sigma_\chi^{+(n_0)})^{-1} D_{1/a} Y \in W_{1/a}$, and since $A^{-1} = \Sigma_\chi^{+(n_0)} (\Delta_\chi^+ \Sigma_\chi^{+(n_0)})^{-1} D_{1/a}$, we conclude

$$X = A^{-1} Y \in W_{1/a}.$$

This shows $w_\infty(A) \subset W_{1/a}$, and since $W_{1/a} \subset w_\infty(A)$, we conclude $w_\infty(A) = W_{1/a}$. The proof is similar for B.

iii) Part iii) is a direct consequence of Proposition 3.1, since a/β and $\beta/a \notin \ell_\infty$.

iv) The property $\overline{W_{1/a}} \neq w_\infty$ has been shown in Lemma 4.1. To show $\overline{W_{1/\beta}} \neq w_\infty$, we use the notations of Lemma 4.1. Here we have

$$\left\| X'_p - e \right\|_{W_{1/\beta}} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \beta_m |x_{mp} - 1| \right) \to 0 \ (p \to \infty) \tag{18}$$

and

$$\begin{aligned} \left\| X'_p - e \right\|_{W_{1/\beta}} &\geq \sup_n \left(\frac{1}{n} \sum_{k=1}^{\frac{n-1}{2}} \beta_{2k+1} |x_{2k+1,p} - 1| \right) \\ &\geq \sup_n \left(\frac{1}{n} \sum_{k=1}^{\frac{n-1}{2}} \frac{\beta_{2k+1}}{a_{2k+1}} a_{2k+1} |x_{2k+1,p} - 1| \right) \\ &\geq \alpha_A \sup_n \left(\frac{1}{n} \sum_{k=1}^{\frac{n-1}{2}} \frac{\beta_{2k+1}}{a_{2k+1}} (2k+1) |x_{2k+1,p} - 1| \right). \end{aligned}$$

Since $\beta_{2k+1}/a_{2k+1} \to \infty$ $(k \to \infty)$, there is $C_1 > 0$ with $\beta_{2k+1}/a_{2k+1} \geq C_1$ for all k and

$$\left\| X'_p - e \right\|_{W_{1/\beta}} \geq \frac{\alpha_A C_1}{n} n |x_{np} - 1| = \alpha_A C_1 |x_{np} - 1| \text{ for all } n.$$

From (18), we have $x_{np} \to 1$ $(p \to \infty)$ for all n. There is p_0 such that for every n and each $p \geq p_0$, $|x_{np}| \geq 1/2$ and

$$\left\| X'_p \right\|_{W_{1/\beta}} \geq \frac{1}{2n} \sum_{m=1}^{n} \beta_m = \frac{1}{2} \|\beta\|_{w_\infty}. \tag{19}$$

We have for every integer i

$$\begin{aligned} \|\beta\|_{w_\infty} &\geq \frac{1}{4i+3} \sum_{k=1}^{2i+1} \beta_{2k+1} = \frac{1}{4i+3} \sum_{k=1}^{2i+1} \frac{\beta_{2k+1}}{a_{2k+1}} a_{2k+1} \\ &\geq \frac{\alpha_A C_1}{4i+3} \sum_{k=1}^{2i+1} (2k+1) \geq \alpha_A C_1 \frac{(2i+1)(2i+3)}{4i+3}. \end{aligned}$$

Since $(2i+1)(2i+3)/(4i+3) \to \infty$ $(i \to \infty)$, we deduce $\|\beta\|_{w_\infty} = \infty$, and by (19) we get $\|X'_p\|_{W_{1/\beta}} \geq \|\beta\|_{w_\infty}/2 = \infty$ for $p \geq p_0$. This contradicts the fact that $X'_p \in W_{1/\beta}$ for all p. Therefore we conclude $\overline{W_{1/\beta}} \neq w_\infty$. □

We immediatly obtain the next result.

Theorem 5.4. *The two linear operators A and B are closed in the space w_∞ and satisfy*

i) $D(A) = W_{1/a}$,

ii) $D(B) = W_{1/\beta}$,

iii) $\overline{D(A)} \neq w_\infty$, $\overline{D(B)} \neq w_\infty$.

iv) There are ε_A, $\varepsilon_B > 0$ (with $\varepsilon_A + \varepsilon_B < \pi$) such that (12) holds and

$$\left\| (B + \mu I)^{-1} \right\|^*_{(w_\infty, w_\infty)} \leq \frac{M}{|\mu|} \text{ for all } \mu \neq 0 \text{ and } |Arg(\mu)| \leq \pi - \varepsilon_B. \tag{20}$$

Proof. We show that A is a closed operator. For this, we consider a sequence $X'_p = (x_{np})_{n \geq 1}$ tending to $X = (x_n)_{n \geq 1}$ in w_∞ as p tends to infinity, where $X'_p \in W_{1/a}$ for all p. Then we have $AX'_p \to Y$ $(p \to \infty)$ in w_∞ with $Y = (y_n)_{n \geq 1}$. It follows that for every n

$$A_n(X'_p) \to A_n(X) = y_n \ (p \to \infty)$$

with $A_n(X'_p) = a_n x_{np} + b_n x_{n+1,p}$, $y_n = a_n x_n + b_n x_{n+1}$, and since

$$\frac{1}{n}\sum_{m=1}^{n} |a_m x_m| = \frac{1}{n}\sum_{m=1}^{n} |y_m - b_m x_{m+1}|$$
$$\leq \|Y\|_{w_\infty} + 2M_A\|X\|_{w_\infty} \text{ for all } n,$$

we conclude $X \in W_{1/a}$. The proof for B is similar.

i), ii) and iii) These parts are direct consequences of Lemma 5.3.

iv) The first part in iv) has been shown in Proposition 4.4. Let $\varepsilon_B \in]0, \pi/2[$. We show that for every μ with $|\text{Arg}(\mu)| \leq \pi - \varepsilon_B$, the infinite matrix $B + \mu I$, considered as an operator in $W_{1/\beta}$, is invertible and

$$(B + \mu I)^{-1} \in (W_{1/\beta}, w_\infty).$$

We put $\Sigma_B = \{\mu \in \mathbb{C} : |\text{Arg}(\mu)| \leq \pi - \varepsilon_B\}$. To be able to deal with the inverse of $B + \mu I$, we need to study the sequences with $|\gamma_{2k+1}|/\beta_{2k}$ and $|\gamma_{2k}/(\beta_{2k} + \mu)|\beta_{2k}/\beta_{2k-1}$. We have by (17) i), iii)

$$\gamma_{2k+1}/\beta_{2k} \to 0 \ (k \to \infty). \tag{21}$$

On the other hand we get for every $\mu \in \Sigma_B$

$$\left|\frac{\gamma_{2k}}{\beta_{2k} + \mu}\right| \frac{\beta_{2k}}{\beta_{2k-1}} \leq \frac{M_B}{\beta_{2k} \sin \varepsilon_B} \frac{\beta_{2k}}{\beta_{2k-1}} = \frac{M_B}{\sin \varepsilon_B} \frac{1}{\beta_{2k-1}} \tag{22}$$

and

$$\frac{1}{\beta_{2k-1}} = \frac{a_{2k-1}}{\beta_{2k-1}} \frac{1}{a_{2k-1}} = o(1) \ (k \to \infty). \tag{23}$$

We deduce from (21), (22) and (23) that there is n_1 such that

$$|\gamma_{2k+1}| \frac{1}{\beta_{2k}} \leq \frac{1}{4} \sin \varepsilon_B \text{ for } 2k + 1 \geq n_1,$$

and

$$\left|\frac{\gamma_{2k}}{\beta_{2k} + \mu}\right| \frac{\beta_{2k}}{\beta_{2k-1}} \leq \frac{1}{4} \text{ for } 2k \geq n_1 \text{ for all } \mu \in \Sigma_B.$$

We define the matrices $D'_\mu = D_{(1/(\beta_n + \mu)_n)}$, and then $D'_\mu(B + \mu I) = \Delta^-_\kappa$ with $\kappa_n = \gamma_n/(\beta_n + \mu)$. We have

$$\sigma_1 = |\kappa_{2k}| \frac{\beta_{2k}}{\beta_{2k-1}} \leq \frac{M_B}{\beta_{2k} \sin \varepsilon_B} \frac{\beta_{2k}}{\beta_{2k-1}} \leq \frac{1}{4} \text{ for all } k \geq \frac{n_1 - 1}{2}$$

and

$$\sigma_2 = |\kappa_{2k+1}| \frac{\beta_{2k+1}}{\beta_{2k}} \leq \frac{1}{4} \sin \varepsilon_B \frac{1}{\beta_{2k+1}} \sin \varepsilon_B \beta_{2k+1} = \frac{1}{4}$$
$$\text{for all } k \geq \frac{n_1}{2}.$$

It follows that

$$\sup_{n\geq n_1}\left(|\kappa_n|\frac{\beta_n}{\beta_{n-1}}\right)=\max\{\sigma_1,\sigma_2\max\}\leq\frac{1}{4}\text{ for all }\mu\in\Sigma_B.$$

Lemma 3.2 yields

$$\left\|I-\Delta_\kappa^-\Sigma_\kappa^{(n_1-1)}\right\|^*_{(W_{1/\beta},W_{1/\beta})}\leq\theta_{n_1}^-\left(\frac{1}{\beta}\right)\leq\frac{1}{4}\left(1+\frac{1}{n_1-1}\right)\leq\frac{1}{2}<1 \quad \text{for all }\mu\in\Sigma_B$$

So $\Delta_\kappa^-=\Delta_\kappa^-\Sigma_\kappa^{(n_1-1)}(\Sigma_\kappa^{(n_1-1)})^{-1}$ is bijective from $W_{1/\beta}$ to itself. Reasoning as in Proposition 4.4 with $(B+\mu I)^{-1}=(\Delta_\kappa^-)^{-1}D'_\mu$, we conclude that $B+\mu I$, considered as an operator from $W_{1/\beta}$ into w_∞ is invertible, and $(B+\mu I)^{-1}\in(w_\infty,W_{1/\beta})$ for all $\mu\in\Sigma_B$. Condition (20) can be obtained by reasoning as in Proposition 4.4. This concludes the proof of Part iv). □

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