Jordan Journal of Mathematics and Statistics (JJMS) 2008, 1(1), pp.69-80

BEST SIMULTANEOUS APPROXIMATION IN METRIC SPACES

SH. AL-SHARIF

ABSTRACT. For a Banach space X and an increasing subadditive continuous function φ on $[0,\infty)$ with $\varphi(0)=0$, let us denote by $L^{\varphi}(I,X)$, the space of all X-valued φ-integrable functions $f : I \to X$ on a certain positive complete σ-finite measure space $(I,\sum,\mu,)$ with $\int_I \varphi \|f(t)\|\, d\mu(t) < \infty$ and $l^{\varphi}(X) = \left\{(x_k) : \sum_{k=1}^{\infty} \varphi \|x_k\| < \infty,\ x_k \in X\right\}$. The aim of this paper is to prove that for a closed separable subspace G of X, $L^{\varphi}(I,G)$ is simultaneously proximinal in $L^{\varphi}(I,X)$ if and only if G is simultaneously proximinal in X. Other result on simultaneous approximation of $l^{\varphi}(G)$ in $l^{\varphi}(X)$ is presented.

1. Introduction

A function $\varphi : [0,\infty) \to [0,\infty)$ is called a modulus function if it satisfies the following conditions:

(1) φ is continuous and increasing function.
(2) $\varphi(x) = 0$ if and only if $x = 0$.
(3) $\varphi(x+y) \le \varphi(x) + \varphi(y)$.

The functions $\varphi(x) = x^p,\ 0 < p < 1$, and $\varphi(x) = \ln(1+x)$ are modulus functions. In fact if φ is a modulus function, then $\psi(x) = \varphi(x)/(1+\varphi(x))$ is a modulus function. Further the composition of two modulus function is a modulus function.

For a modulus function φ and a Banach space X, let us denote by $L^{\varphi}(I,X)$, the space of all X-valued φ-integrable functions $f : I \to X$ on a certain positive complete σ-finite measure space $(I,\sum,\mu,)$ with $\int_I \varphi \|f(t)\|\, d\mu(t) < \infty$ and

$$l^{\varphi}(X) = \left\{(x_k) : \sum_{k=1}^{\infty} \varphi \|x_k\| < \infty,\ x_k \in X\right\}.$$

For $a = (a_k) \in l^{\varphi}(X)$ and $f \in L^{\varphi}(I,X)$ set

$$\|a\|_{\varphi} = \sum_{k=1}^{\infty} \varphi \|a_k\| \qquad \text{and} \qquad \|f\|_{\varphi} = \int_I \varphi \|f(t)\|\, d\mu(t).$$

If $X = C$, the set of complex numbers, the spaces $l^{\varphi}(X)$ and $L^{\varphi}(I,X)$ is simply denoted by l^{φ} and $L^{\varphi}(I)$ respectively. It is known, [4], that $l^{\varphi} \subseteq l^1$, $L^{\varphi}(I) \supseteq L^1(I)$

1991 *Mathematics Subject Classification.* Primary: 41A65; Secondary: 41A50.
Key words and phrases. Simultaneous, Approximation.

Received Sept. 1, 2007, Accepted March 4, 2008.

and $\left(l^{\varphi}(X), \|.\|_{\varphi}\right)$ and $\left(L^{\varphi}(I,X), \|.\|_{\varphi}\right)$are complete metric linear spaces. For more on l^{φ} and $L^{\varphi}(I)$ we refer to the reader to [3] and [5].

Note that the Banach space X is a metric space with the metric $d(x,y) = \varphi\|x-y\|$.

Definition 1.1. *Let φ be a modulus function and G be a closed subspace of a Banach space X. We say that*

(a) G is simultaneously proximinal in X if for each m-tuple of elements $(x_1, x_2, ..., x_m) \in X^m$ there exists $g \in G$ such that:

$$\sum_{i=1}^{m} \varphi\|x_i - g\| = dist_{\varphi}(x_1, x_2, ..., x_m, G) = \inf_{h\in G} \sum_{i=1}^{m} \varphi\|x_i - h\|.$$

In other words for every $h \in G$

$$\|(x_1, x_2, ..., x_m, 0, ...) - (g, g, ..., g, 0, ...)\|_{\varphi} \leq \left\| \begin{array}{c} (x_1, x_2, ..., x_m, 0, ...) \\ -(h, h, ..., h, 0, ...) \end{array} \right\|_{\varphi}.$$

(b) $L^{\varphi}(I,G)$ is simultaneously proximinal in $L^{\varphi}(I,X)$ if for each m-tuple of elements $f_1, f_2, ..., f_m \in \left(L^{\varphi}(I,X)\right)^m$ there exists $g \in L^{\varphi}(I,G)$ such that

$$\sum_{i=1}^{m} \|f_i - g\|_{\varphi} = dist_{\varphi}(f_1, f_2, ..., f_m, L^{\varphi}(I,G)) = \inf_{h\in L^{\varphi}(I,G)} \sum_{i=1}^{m} \|f_i - h\|_{\varphi}.$$

The problem of best simultaneous approximation has been studied by many authors e.g., [2], [9], [15] and [16]. Most of these works have dealt with the characterization of best simultaneous approximation in spaces of continuous functions with values in a Banach space X. Some existence and uniqueness results were obtained. Results on best simultaneous approximation in general Banach spaces may be found in [11] and [13]. Related results on $L^p(I,X), 1 \leq p < \infty$, are given in [14]. In [14], it is shown that if G is a reflexive subspace of a Banach space X, then $L^p(I,G)$ is simultaneously proximinal in $L^p(I,X)$. If $p = 1$, Abu Sarhan and Khalil [1], proved that if G is a reflexive subspace of the Banach space X or G is a 1-summand subspace of X, then $L^1(I,G)$ is simultaneously proximinal in $L^1(I,X)$.

The aim of this paper is to prove that for a closed separable subspace G of X, $L^{\varphi}(I,G)$ is simultaneously proximinal in $L^{\varphi}(I,X)$ if and only if G is simultaneously proximinal in X. Some results are inspired by the results in [14]. Other result on simultaneous approximation of $l^{\varphi}(G)$ in $l^{\varphi}(X)$ is presented.

Throughout this paper, $(I, \sum, \mu,)$ is a σ-finite measure space, X is a Banach space, G is a closed subspace of X and the norm of $v \in X$ is denoted by $\|v\|$.

2. **Distance Formulae**

Progress in the discussion of simultaneous proximinality when X does not possess pleasant properties is greatly facilitated by the fact that the distance from an m-tuple of elements $f_1, f_2, ..., f_m \in L^{\varphi}(I,X)$ to a subspace $L^{\varphi}(I,G)$ is computed by the following theorem:

Theorem 2.1. *Let φ be a modulus function and $f_1, f_2, ..., f_m \in L^{\varphi}(I, X)$. Then*

$$dist_{\varphi}\left(f_1, f_2, ..., f_m, L^{\varphi}(I, G)\right) = \int_I dist_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s).$$

Proof. Let $f_1, f_2, ..., f_m \in L^{\varphi}(I, X)$. Then for each $i = 1, 2, ..., m$, f_i is the limit almost everywhere of a sequence of simple functions $\{f_{i,n}\}$ in $L^{\varphi}(I, X)$.Since the distance function $\mathrm{dist}_{\varphi}(x, G)$ is continuous in $x \in X$, $\lim_{n\to\infty} \varphi\left(\|f_{i,n}(s) - f_i(s)\|\right) = 0, i = 1, 2, ..., m,$ implies that

$$\lim_{n\to\infty} \left|\mathrm{dist}_{\varphi}\left(f_{1,n}(s), ..., f_{m,n}(s), G)\right) - \mathrm{dist}_{\varphi}\left(f_1(s), ..., f_m(s), G)\right)\right| = 0.$$

Furthermore for each n, the function: $s \to \mathrm{dist}_{\varphi}\left(f_{1,n}(s), f_{2,n}(s), ..., f_{m,n}(s), G\right)$ is a simple function and so we may assume that $\mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G)\right)$ is measurable. Now for any $g \in L^{\varphi}(\mu, G)$

$$\begin{aligned} \int_I \mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s) &\leq \int_I \sum_{i=1}^{m} \varphi\left(\|f_i(s) - g(s)\|\right) d\mu(s) \\ &= \sum_{i=1}^{m} \int_I \varphi\left(\|f_i(s) - g(s)\|\right) d\mu(s). \end{aligned}$$

Therefore

$$\int_I \mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s) \leq \mathrm{dist}_{\varphi}\left(f_1, f_2, ..., f_m, L^{\varphi}(I, G)\right). \tag{1}$$

For the reverse inequality fix $\epsilon > 0$. Since simple functions are dense in $L^{\varphi}(I, X)$, there exist simple functions, $f_j^{'}$ in $L^{\varphi}(I, X)$ such that $\left\|f_j - f_j^{'}\right\|_{\varphi} < \frac{\epsilon}{m}$, $j = 1, 2, ..., m$. Assume that $f_j^{'}(t) = \sum_{i=1}^{n} \varkappa_{A_i}(t) y_i^j$, $j = 1, 2, ..., m$, where $\varkappa_{A_i}$ are the characteristic functions of the measurable sets A_i in I and $y_i^j \in X$. We can assume that $\sum_{i=1}^{n} \varkappa_{A_i} = 1$ and $\mu(A_i) > 0$.

Given $\epsilon > 0$ for each $i = 1, 2, ..., n$, select $g_i \in G$ such that:

$$\sum_{j=1}^{m} \varphi \left\|y_i^j - g_i\right\| < dist_{\varphi}(y_i^1, y_i^2, ..., y_i^m, G) + \frac{\epsilon}{n\mu(A_i)}.$$

Let $g(t)=\sum\limits_{i=1}^{n}\varkappa_{A_i}(t)g_i$. Clearly $g\in L^{\varphi}(I,G)$ and

$$\begin{aligned}
dist_{\varphi}\left(f_1,...,f_m,L^{\varphi}(I,G)\right) &\leq \sum_{j=1}^{m}\left\|f_j-f_j^{'}\right\|_{\varphi} \\
&\quad + \text{dist}_{\varphi}\left(f_1^{'},f_2^{'},...,f_m^{'},L^{\varphi}(I,G)\right) \\
&\leq \epsilon+\sum_{j=1}^{m}\left\|f_j^{'}-g\right\|_{\varphi} \\
&= \epsilon+\sum_{j=1}^{m}\int_I \varphi\left\|f_j^{'}(s)-g(s)\right\|d\mu(s) \\
&= \epsilon+\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{A_i}\varphi\left\|f_j^{'}(s)-g(s)\right\|d\mu(s) \\
&= \epsilon+\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\varphi\left\|y_i^{j}-g_i\right\|\right)\mu(A_i) \\
&= \epsilon+\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\varphi\left\|y_i^{j}-g_i\right\|\right)\mu(A_i) \\
&\leq \epsilon+\sum_{i=1}^{n}\mu(A_i)\text{dist}_{\varphi}(y_i^1,y_i^2...,y_i^m,G)+\frac{\epsilon}{n} \\
&\leq 2\epsilon+\sum_{i=1}^{n}\int_{A_i}\text{dist}_{\varphi}(y_i^1,y_i^2,...,y_i^m,G)d\mu(s) \\
&= 2\epsilon+\int_I \text{dist}_{\varphi}\left(f_1^{'}(s),f_2^{'}(s),...,f_m^{'}(s),G\right)d\mu(s).
\end{aligned}$$

Since

$$\begin{aligned}
dist_{\varphi}\left(f_1^{'}(s),f_2^{'}(s),...,f_m^{'}(s),G\right) &\leq \text{dist}_{\varphi}\left(f_1(s),f_2(s),...,f_m(s),G\right) \\
&\quad +\sum_{j=1}^{m}\varphi\left\|f_j^{'}(s)-f(s)\right\|
\end{aligned}$$

then,

$$\begin{aligned} dist_{\varphi}\left(f_1, f_2, ..., f_m, L^{\varphi}(I,G)\right) &\leq 2\epsilon + \sum_{j=1}^{m} \int_I \varphi \left\| f_j'(s) - f_j(s) \right\| d\mu(s) \\ &\quad + \int_I \mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s) \\ &= 2\epsilon + \sum_{j=1}^{m} \left\| f_j - f_j' \right\|_{\varphi} \\ &\quad + \int_I \mathrm{dist}_{\varphi}(f_1(s), f_2(s), ..., f_m(s), G) d\mu(s) \\ &\leq 3\epsilon + \int_I \mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s), \end{aligned}$$

which (since ϵ is arbitrary) implies that

$$dist_{\varphi}\left(f_1, f_2, ..., f_m, L^{\varphi}(I,G)\right) \leq \int_I \mathrm{dist}_{\varphi}\left(f_1(s), f_2(s), ..., f_m(s), G\right) d\mu(s). \tag{2}$$

Hence by 1 and 2 the proof is complete. □

An application of Theorem 2.1 is

Corollary 2.2. *An element $g \in L^{\varphi}(I,G)$ is a best simultaneous approximation of $f_1, f_2, ..., f_m \in L^{\varphi}(I,X)$ if and only if $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), ..., f_m(t) \in X$ for almost all $t \in I$.*

3. Best Simultaneous Approximation in $L^{\varphi}(I,X)$

The main result in this section is, for a modulus function φ and a closed separable subspace G of a Banach space X, $L^{\varphi}(I,G)$ is simultaneously proximinal in $L^{\varphi}(I,X)$ if and only if G is simultaneously proximinal in X. We begin with the following:

Theorem 3.1. *If G is simultaneously proximinal in X, then for every m-tuple of simple function $f_1, f_2, ..., f_m \in L^{\varphi}(I,X)$, $P(f_1, f_2, ..., f_m, L^{\varphi}(I,X))$ is not empty, where $P(f_1, f_2, ..., f_m, L^{\varphi}(I,X))$ is the set of all elements $g \in L^{\varphi}(I,G)$ such that g is a best simultaneous approximation of m-tuple of the elements $f_1, f_2, ..., f_m$.*

Proof. Let $f_1, f_2, ..., f_m$ be an m-tuple of simple functions in $L^{\varphi}(I,X)$. With no loss of generality we can assume that $f_j(t) = \sum_{i=1}^{n} \varkappa_{A_i}(t) y_i^j$, where A_i are disjoint measurable sets such that $\bigcup_{i=1}^{n} A_i = I$. Pick $g_i \in G$ such that g_i is a best simultaneous approximation of the m-tuple of elements $y_i^1, y_i^2, ..., y_i^m \in X, i = 1, 2, ..., n$. Set $g(t) = \sum_{i=1}^{n} \varkappa_{A_i}(t) g_i$. Then for any $h \in L^{\varphi}(I, X)$ we have:

$$\begin{aligned}
\sum_{j=1}^{m} \|f_j - h\|_{\varphi} &= \sum_{j=1}^{m} \int_I \varphi \|f_j(s) - h(s)\| \, d\mu(s) \\
&= \int_I \sum_{j=1}^{m} \varphi \|f_j(s) - h(s)\| \, d\mu(s) \\
&= \sum_{i=1}^{n} \int_{A_i} \sum_{j=1}^{m} \varphi \left\|y_i^j - h(s)\right\| d\mu(s) \\
&\geq \sum_{i=1}^{n} \int_{A_i} \sum_{j=1}^{m} \varphi \left\|y_i^j - g_i\right\| d\mu(s) \\
&= \int_I \sum_{j=1}^{m} \varphi \|f_j(s) - g(s)\| \, d\mu(s).
\end{aligned}$$

Hence $\sum_{j=1}^{m} \|f_j - g\|_{\varphi} = \inf_{h \in L^{\varphi}(I,G)} \sum_{j=1}^{m} \|f_j - h\|_{\varphi}$ □

Theorem 3.2. *If φ is a modulus function, then G is simultaneously proximinal in X if $L^{\varphi}(I, G)$ is simultaneously proximinal in $L^{\varphi}(I, X)$.*

Proof. Let $x_1, x_2, ..., x_m \in X$. Set $f_j = 1 \otimes x_j,\ j = 1, 2, ..., m$, where 1 is the constant function 1. Clearly for each $j = 1, 2, ..., m$, $f_j \in L^{\varphi}(I, X)$. By assumption there exists $g \in L^{\varphi}(I, G)$ such that for any $h \in L^{\varphi}(I, G)$

$$\sum_{j=1}^{m} \|f_j - g\|_{\varphi} \leq \sum_{j=1}^{m} \|f_j - h\|_{\varphi}.$$

By Theorem 2.1

$$\sum_{j=1}^{m} \varphi \|f_j(t) - g(t)\| \leq \sum_{j=1}^{m} \varphi \|f_j(t) - h(t)\|$$

a.e. in I. Or

$$\sum_{j=1}^{m} \varphi \|x_j - g(t)\| \leq \sum_{j=1}^{m} \varphi \|x_j - h(t)\|.$$

Let h run over all functions $1 \otimes z$, for $z \in G$, we get

$$\sum_{j=1}^{m} \varphi \|x_j - g(t)\| \leq \sum_{j=1}^{m} \varphi \|x_j - z\|.$$

□

Now we pose the following problem: If G is separable is it true that $L^{\varphi}(I,G)$ is simultaneously proximinal in $L^{\varphi}(I,X)$? to solve this problem we begin by the following:

Lemma 3.3. [*Lemma 2.9 of* [9]] *Assume $\mu(I) < +\infty$. Suppose (M,d) is a metric space and A is a subset of I such that $\mu^*(A) = \mu(I)$, where μ^* denotes the outer measure associated to μ. If g is a mapping from I to M with separable range, then for any $\epsilon > 0$ there exists a countable partition $\{E_n\}$ of I in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and diam$(g(A_n)) < \epsilon$ for all n.*

Theorem 3.4. *Let G be a closed separable subspace of X. Let us suppose that G is simultaneously proximinal in X and $f_1, f_2, ..., f_m : I \to X$ be measurable functions. Then there is a measurable function $g : I \to X$ such that $g(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$ in G for almost all t.*

Proof. Let $f_1, f_2, ..., f_m : I \to X$ be measurable functions. So we may assume that $f_1(I), f_2(I), ..., f_m(I)$ are separable sets in X. Using the fact that μ is σ-finite we can find countable partitions $\{I_{1n}\}_{n=1}^{\infty}, \{I_{2n}\}_{n=1}^{\infty}, ..., \{I_{mn}\}_{n=1}^{\infty}$of I in measurable sets such that $\text{diam}_{\varphi}(f_i(I_{in}) < \frac{1}{2}$ and $\mu(I_{in}) < \infty, i = 1, 2, ..., m$, for all n, where

$$\text{diam}_{\varphi} A = \sup \{\varphi \|x - y\| : x, y \in A\}.$$

Consider the partition $\{I_{n_1,n_2,...,n_m}\}_{n_i=1,i=1}^{\infty,\, m}$, where $I_{n_1,n_2,...,n_m} = \bigcap_{i=1}^{m} I_{in}$, for $1 \le n_i < \infty$. Then $\text{diam}_{\varphi}(f_i(I_{n_1,n_2,...,n_m})) < \frac{1}{2}, i = 1, 2, ..., m$. For simplicity we write $\{I_{n_1,n_2,...,n_m}\}_{n_i,i=1}^{\infty,m}$ as $\{I_n\}_{n=1}^{\infty}$. For each $t \in I$, let $g_0(t)$ be a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$ in G. Define g_0 from I into G such that $g_0(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$. Applying Lemma 3.3 to the mapping g_0 in each I_n taking $\epsilon = \frac{1}{2}$ and $I = A = I_n$. We get a countable partition in each I_n and therefore a countable partition in the whole of I. Thus we get a countable partition $\{E_n\}_{n=1}^{\infty}$ of I in measurable sets and a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of I such that

$$\begin{aligned} A_n &\subseteq E_n,\ \mu^*(A_n) = \mu(E_n) < +\infty, \\ \text{diam}_{\varphi}(g_0(A_n)) &< \frac{1}{2},\ \text{diam}_{\varphi}(f_i(E_n)) < \frac{1}{2},\ i = 1, 2, ..., m. \end{aligned}$$

Let us apply again the same argument in each E_n with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each n we get a countable partition $\{E_{n_k} : 1 \le k < \infty\}$of E_n in measurable sets and a sequence $\{A_{n_k} : 1 \le k < \infty\}$ of subsets of I such that

$$\begin{aligned} A_{n_k} &\subseteq E_{n_k} \cap A_n,\ \mu^*(A_{n_k}) = \mu(E_{n_k}), \\ \text{diam}_{\varphi}(g_0(A_{n_k})) &< \frac{1}{2^2} \text{ and } \text{diam}_{\varphi}(f_i(E_{n_k})) < \frac{1}{2^2}, i = 1, 2, ..., m, \end{aligned}$$

for all n and k. Let us proceed by induction. Now for each natural number k, let $\triangle_k$ be the set of k-tuples of natural numbers and let $\triangle = \bigcup_{k=1}^{\infty} \triangle_k$. On this $\triangle$ consider the partial order defined by $(m_1, m_2, ..., m_i) \le (n_1, n_2, ..., n_j)$ if and only if $i \le j$ and $m_k = n_k$

for $k = 1, 2, ..., i$. Then by induction for each natural number k, we can take a partition $\{E_\alpha : \alpha \in \triangle_k\}$ of subsets of I and a collection $\{A_\alpha\}_{\alpha\in\triangle_k}$such that:

(1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$ for each α.

(2) $A_\alpha \subseteq A_\beta$ and $E_\alpha \subseteq E_\beta$ if $\beta \leq \alpha$.

(3) $\mathrm{diam}_\varphi(f_i(E_\alpha)) < \frac{1}{2^k}$ for $i = 1, 2, ..., m$ and $\mathrm{diam}_\varphi(g_0(A_\alpha)) < \frac{1}{2^k}$ if $\alpha \in \triangle_k$.

We may assume that $A_\alpha \neq \emptyset$ for all α (forget the $\alpha' s$ for which $A_\alpha = \emptyset$). For each $\alpha \in \triangle$ take $t_\alpha \in A_\alpha$ and define g_k from I into G by $g_k(.) = \sum\limits_{\alpha\in\triangle_k} \varkappa_{E_\alpha}(.)g_0(t_\alpha)$. Then for each $t \in I$ and $n \leq k$ we have:

$$\varphi \|g_n(t) - g_k(t)\| = \varphi \left\| \sum_{\alpha\in\triangle_n} \varkappa_{E_\alpha}(t)g_0(t_\alpha) - \sum_{\beta\in\triangle_k} \varkappa_{E_\beta}(t)g_0(t_\beta) \right\|.$$

But since $n \leq k$ by 1 and 2 we have:

$$\begin{aligned}
\varphi \|g_n(t) - g_k(t)\| &\leq \varphi \left\| \sum_{\beta\in\triangle_k} \varkappa_{E_\beta}(t) \left(g_0(t_\alpha) - g_0(t_\beta)\right) \right\| \\
&\leq \sum_{\beta\in\triangle_k} \phi \left\| \left(g_0(t_\alpha) - g_0(t_\beta)\right) \right\| \mu(E_\beta) \\
&\leq \frac{1}{2^n}.
\end{aligned}$$

Therefore $(g_k(t))$ is a Cauchy sequence in X for every $t \in I$. Consequently $(g_k(t))$ is a convergent sequence for every $t \in I$. Let $g : I \to G$ be the point wise limit of (g_k). Since g_k is measurable for each k, g is measurable. Let $t \in I$ and let n be a natural number. Suppose $t \in E_\alpha$. We have:

$$\begin{aligned}
\sum_{i=1}^{m} \varphi \|f_i(t) - g_n(t)\| &= \sum_{i=1}^{m} \varphi \|f_i(t) - g_0(t_\alpha)\| \\
&\leq \sum_{i=1}^{m} \varphi \|f_i(t) - f_i(t_\alpha)\| + \varphi \|f_i(t_\alpha) - g_0(t_\alpha)\| \\
&\leq \sum_{i=1}^{m} \frac{1}{2^n} + \varphi \|f_i(t_\alpha) - g_0(t_\alpha)\| \\
&\leq \frac{m}{2^n} + \mathrm{dist}_\varphi((f_1(t_\alpha), f_2(t_\alpha), ..., f_m(t_\alpha)), G) \\
&\leq \frac{m}{2^n} + \sum_{i=1}^{m} \varphi \|f_i(t) - f_i(t_\alpha)\| \\
&\quad + \mathrm{dist}_\varphi((f_1(t), f_2(t), ..., f_m(t)), G) \\
&\leq \frac{m}{2^{n-1}} + \mathrm{dist}_\varphi((f_1(t), f_2(t), ..., f_m(t)), G).
\end{aligned}$$

Letting $n \to \infty$ we get:

$$\begin{aligned}\sum_{i=1}^{m} \varphi \left\| f_i(t) - g(t) \right\| &= \lim_{n\to\infty} \sum_{i=1}^{m} \varphi \left\| f_i(t) - g_n(t) \right\| \\ &= \operatorname{dist}_{\varphi}((f_1(t), f_2(t), ..., f_m(t)), G).\end{aligned}$$

and so $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), ..., f_m(t)$ in G. □

Theorem 3.5. *Let φ be a modulus function and G be a closed separable subspace of X. Then $L^{\varphi}(I, G)$ is simultaneously proximinal in $L^{\varphi}(I, X)$ if and only if G is simultaneously proximinal in X.*

Proof. Necessity is in Theorem 3.2 Let us show sufficiency. Suppose that G is simultaneously proximinal in X, and let $f_1, f_2, ..., f_m$ be functions in $L^{\varphi}(I, X)$. Theorem 3.4 guarantees that there exists a measurable function g defined on I with values in X such that $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), ...f_m(t)$ in G for almost all t. It follows from Corollary 2.2 that g is a best simultaneous approximation of $f_1, f_2, ..., f_m$ in $L^{\varphi}(I, G)$ □

Theorem 3.6. *Let φ be a modulus function. Then if $g \in L^{\varphi}(I, G)$ is a best simultaneous approximation from $L^{\varphi}(I, G)$ of an m-tuple of elements $f_1, f_2, ..., f_m \in L^{\varphi}(I, X)$ then for every measurable subset A of I and every $h \in L^{\varphi}(I, G)$,*

$$\int_A \varphi \left(\left\| f_{j_0}(s) - g(s) \right\| \right) d\mu(s) \le \int_A \varphi \left(\left\| f_{j_0}(s) - h(s) \right\| \right) d\mu(s),$$

for some $j_0 \in \{1, 2, ..., m\}$.

Proof. If $\mu(A) = 0$ then there is nothing to prove. Suppose that for some A satisfying $\mu(A) > 0$ and for some $h_0 \in L^{\varphi}(I, G)$, the inequality does not hold for $J = 1,\ 2, ..., m$. Now, define $g_0 \in L^{\varphi}(I, G)$ by

$$g_0(s) := \begin{cases} g(s) & \text{if} \quad s \in I - A \\ h_0(s) & \text{if} \quad s \in A \end{cases}$$

Then we have for $j = 1, 2, ..., m$

$$\begin{aligned}\int_I \varphi\left(\|f_j(s) - g_0(s)\|\right) d\mu &= \int_A \varphi\left(\|f_j(s) - h_0(s)\|\right) d\mu(s) \\ &\quad + \int_{I-A} \varphi\left(\|f_j(s) - g(s)\|\right) d\mu(s) \\ &< \int_A \varphi\left(\|f_j(s) - g(s)\|\right) d\mu(s) \\ &\quad + \int_{I-A} \varphi\left(\|f_j(s) - g(s)\|\right) d\mu(s) \\ &= \int_I \varphi\left(\|f_j(s) - g(s)\|\right) d\mu(s).\end{aligned}$$

This implies that

$$\sum_{j=1}^{m} \|f_j - g_0\|_\varphi < \sum_{j=1}^{m} \|f_j - g\|_\varphi$$

which contradict the fact that g is a best simultaneous approximation from $L^\varphi(I, G)$ of the m-tuple of elements $f_1, f_2, ..., f_m$. □

As a corollary we get:

Corollary 3.7. *If g is a best simultaneous approximation from $L^\varphi(I, G)$ of an m-tuple of elements $f_1, f_2, ..., f_m \in L^\varphi(I, X)$ then, for every measurable subset A if I,*

$$\int_A \varphi\left(\|g(s)\|\right) d\mu(s) \leq 2 \max_{1\leq j\leq m} \left(\int_A \varphi\left(\|f_j(s)\|\right) d\mu(s) \right).$$

Proof. Since, for $j = 1, 2, ..., m$

$$\int_A \varphi\left(\|g(s)\|\right) d\mu(s) \leq \int_A \varphi\left(\|f_j(s) - g(s)\|\right) d\mu(s) + \int_A \varphi\left(\|f_j(s)\|\right) d\mu(s),$$

we obtain, by using Theorem 3.6 with $h =: 0$, that for $j_0 \in \{1, 2, ..., m\}$

$$\begin{aligned}\int_A \varphi\left(\|g(s)\|\right) d\mu(s) &\leq 2 \int_A \varphi\left(\|f_{j_0}(s)\|\right) d\mu(s) \\ &\leq 2 \max_{1\leq j\leq m} \left(\int_A \varphi\left(\|f_j(s)\|\right) d\mu(s) \right),\end{aligned}$$

which completes the proof. □

We end this paper with the following result on best simultaneous approximation of $l^{\varphi}(X)$ in $l^{\varphi}(G)$.

Theorem 3.8. *Let φ be a modulus function. Then $l^{\varphi}(G)$ is simultaneously proximinal in $l^{\varphi}(X)$ if G is simultaneously proximinal in X.*

Proof. Let $f_1,\ f_2,...,f_m \in l^{\varphi}(X)$. Since G is simultaneously proximinal in X, for each n, there exists $g(n) \in G$ such that for every $y \in G$

$$\sum_{j=1}^{m} \varphi \left\| f_j(n) - g(n) \right\| \leq \sum_{j=1}^{m} \varphi \left\| f_j(n) - y \right\|.$$

Since $y = 0 \in G$, we get

$$\sum_{j=1}^{m} \varphi \left\| f_j(n) - g(n) \right\| \leq \sum_{j=1}^{m} \varphi \left\| f_j(n) \right\|.$$

But φ is increasing and subadditive so

$$\begin{aligned} m\varphi \left\| g(n) \right\| &= \sum_{j=1}^{m} \varphi \left\| g(n) - f_j(n) + f_j(n) \right\| \\ &\leq \sum_{j=1}^{m} \varphi \left\| g(n) - f_j(n) \right\| + \varphi \left\| f_j(n) \right\| \leq 2 \sum_{j=1}^{m} \varphi \left\| f_j(n) \right\|. \end{aligned}$$

Consequently $g = (g(n)) \in l^{\varphi}(G)$. We claim that g is a best simultaneous approximation for $f_1,\ f_2,...,f_m \in l^{\varphi}(X)$ in $l^{\varphi}(G)$. To see that let $h \in l^{\varphi}(G)$. Then

$$\begin{aligned} \sum_{j=1}^{m} \left\| f_j - h \right\|_{\varphi} &= \sum_{j=1}^{m} \sum_{n=1}^{\infty} \varphi \left\| f_j(n) - h(n) \right\| \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{m} \varphi \left\| f_j(n) - h(n) \right\| \\ &\geq \sum_{n=1}^{\infty} \sum_{j=1}^{m} \varphi \left\| f_j(n) - g(n) \right\| \\ &= \sum_{j=1}^{m} \sum_{n=1}^{\infty} \varphi \left\| f_j(n) - g(n) \right\| \\ &= \sum_{j=1}^{m} \left\| f_j - g \right\|_{\varphi}. \end{aligned}$$

□

References

[1] I. Abu-Sarhan and R. Khalil, Best simultaneous approximation in vector valued function spaces. Int. Journal of Mathematical Analysis 2(2008) 207-212.

[2] L. Chong and G.A. Watson, On the best simultaneously approximation, J. Approx. Theory 91(1997) 332-348.
[3] W. Deeb and R. Khalil, On the tensor product of non-locally convex topological vector spaces, Illinois J. Math. Proc. 30 (1986), 594- 601.
[4] W. Deeb and R. Khalil, best approximation in $L^p(I,X), 0 < p < 1$ J. Approx. Theory 58 (1989), 68-77.
[5] W. Deeb and R. Younis, Extreme points in a class of non-locally convex topological vector spaces, Math. Reb. Toyama Univ. 6(1983),95-103.
[6] R. Khalil and W. Deeb, best approximation in $L^p(I,X), ii$ J. Approx. Theory 59(1989), 296-299.
[7] R. Khalil and F. Saidi, best approximation in $L^1(I,X)$, Proc. Amer. Math. Soc. 123(1995), 183-190.
[8] W. Light, Proximinality in $L^p(S,Y)$, Rocky Mountain J. Math. (1989).
[9] J. Mach, Best simultaneous approximation of vector valued functions with values in certain Banach spaces, Math. Ann. 240(1979), 157-164.
[10] J. Mendoza, Proximinality in $L^p(\mu,X)$, *J. Approx. Theory,* **93** (1998), 331-343.
[11] P.D.Milman, On best simultaneous approximation in normed linear spaces, J. Approx. Theory 20(1977), 223-238.
[12] A. Pinkus, Uniqueness in vector valued approxomation, J. Approx. Theory 73(1993), 17-92.
[13] B.N.Sahney and S.P.Singh, On Best simultaneous approximation in Banach spaces, J. Approx. Theory 35(1982), 222-224.
[14] F. Saidi, D. Hussein and R. Khalil, Best simultaneous approximation in $L^p(I,X)$, J. Approx. Theory 116(2002), 369-379.
[15] S. Tanimoto, on of best simultaneous approximation, Math. Japonica 48(1998), 275-279.
[16] G.A. Watson,, A characterization of best simultaneous approximation, J. Approx. Theory 75(1993),175-182.

Department of Mathematics, Yarmouk University,Irbid Jordan
E-mail address: sharifa@yu.edu.jo