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The Set of Values of Functionals in the Classes of Functions Having Integral Representation

Hassan Baddour

ABSTRACT:

This paper presents a certain method to determine the set of values of the functional

$$J(f) = \sum_{k=0}^{n} a_k(z)\ f^{(k)}(z)$$

defined on the family of analytic functions in the unit disc possessing an integral representation with kernel $q(z,t)$ in the interval $[\alpha,\beta]$.

It has been shown that the set of values of the linear functional $J[f(z)]$at the point z_0 is the convex hull of the curve Γ, given by equation

$$\Gamma :\ w(t) = \sum_{k=0}^{n} a_k(z_0) q^{(k)}(z_0,t)\ , \quad \alpha \le t \le \beta\ .$$

The Class of Functions with Limited Rotation was cosidered as a special case and some new results were obtained .

1. Introduction

One of the fundamental extremal problems considered in the domain of complex functions is concerned with determining the range of variability of the functional

$$J(f) = F\left[f(z_0), f'(z_0), \dots, f^{(n)}(z_0) \right]$$

defined on some family E of analytic functions where z_0 is a fixed point of the domain in which the functions f are defined. A survey of methods and results can be found among others in [1] and [5].

In this paper we shall present a certain method to determine the range of variability (or the set of values) of the linear functional

$$J(f) = \mathrm{F}\left(f, f', \dots, f^{(\mathrm{n})}\right) = \sum_{k=0}^{n} a_k(z_0)\ f^{(k)}(z_0)$$

defined in the family of functions possessing an integral representation in the unit disk (the precise definition of will be given in the next paragraph).This method can be also applied to solve the coefficients problem in classes considered , i.e. if the function is given by Taylor Series then the coefficients can be strictly estimated in some special cases .

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2. The problem in the Class E_q

Let E_q denote the class of functions f given by the formula

(1) $$f(z)=\int_{\alpha}^{\beta} q(z,t)\,d\mu(t)$$

where $q(z,t)$ is an analytic function in the unit disc $D\,(|z|<1)$ for every fixed $t\in[\alpha,\beta]$ and $\mu(t)$ is a nondecreasing function in the interval $[\alpha,\beta]$ such that $\mu(\beta)-\mu(\alpha)=1$.
This class is known as the family of functions possessing an integral representation .It is to be proved that [6]:
1- The class E_q is compact and connected in the topology of almost uniform convergence.
2- The set of values of functional $F(f)=f(z_0)$ where $f\in E_q$ is closed , convex and connected and it is at the same time the convex hull of the curve $w(t)=q(z,t), \alpha\le t\le\beta$ (The convex hull of the set W is the smallest convex set containing the set W).

An essential example of this class is the Caratheodory Class (denoted by C) i.e. the class of analytic functions f in the unit disk D with a positive real part and $f(0)=1$. It is well known ([2] and [4]) that for the functions of this class the following integral representation holds:

(2) $$f(z)=\int_{-\pi}^{+\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)$$

where $\mu(t)$ in this case is the set of nondecreasing functions on the interval $[-\pi,\pi]$ such that $\mu(+\pi)-\mu(-\pi)=1$). We see that the class C coincides with the family E_q when we put

$$q(z,t)=\left(e^{it}+z\right)_0\Big/\left(e^{it}-z_0\right) \text{ and } \alpha=-\pi\,,\ \ \beta=\pi\,.$$

We shall make use of the following lemma:

Lemma1. If $W(t)$ is a complex and continuous function with a real variable t on the interval $[a,b]$ and $U[a,b]$ the set of nondecreasing functions $\mu(t)$ on this interval such that $\mu(a)=0$, $\mu(b)=1$, then the set of values of the integral

$$J(\mu)=\int_a^b W(t)\,d\mu(t)\ ,\quad \mu\in U[a,b]$$

is the convex hull of the curve Γ given by equation

$$\Gamma: W=W(t),\ \ a\le t\le b\,.$$

The proof of this lemma can be found in [4] .

Making use of this lemma we shall prove the following:

Theorem 1. If $a_k(z), k=0,1,...n$ are some continuous functions on the unit disc D and z_0 is some fixed point in this disc, then the set of values of the linear functional

(3) $$J(f)=\sum_{k=0}^{n} a_k(z_0)\ f^{(k)}(z_0)\ ,\quad f\in E_q$$

is the convex hull of the curve Γ, given by equation

$$W(z_0,t)=\sum_{k=0}^{n} a_k(z_0)q_z^{(k)}(z_0,t)$$

where $q(z_0,t)$ are given by (1) .

Proof. Differentation of both sides in the formula (1) gives the relations:

(4) $$f^{(k)}(z)=\int_{\alpha}^{\beta} q_z^{(k)}(z,t)d\mu(t) \quad k=0,1,2,3,\dots,$$

provided that $f^{(0)}(z)=f(z)$ .And so

$$J(f)=\sum_{k=0}^{n} a_k(z)\int_{\alpha}^{\beta} q_z^{(k)}(z,t)d\mu(t)=\int_{\alpha}^{\beta}\sum_{k=0}^{n} a_k(z)q_z^{(k)}(z,t)d\mu(t).$$

Taking

(5) $$W(z_0,t)=\sum_{k=0}^{n} a_k(z_0)q^{(k)}(z_0,t)$$

we obtain the functional (3) depends only on $\mu(t)$ and so we can write

$$J=\Phi(\mu)=\int_{\alpha}^{\beta} W(z_0,t)d\mu(t).$$

Substituting $a=\alpha, b=\beta$ and $W(t)=W(z_0,t)$ in Lemma 1 we get the thesis of theorem.

3. The problem in the class of functions with limited rotation.

Let V denote the family of analytic functions f in the unit disk D satisfying the conditions:

$$f(0)=0\,,\ f'(0)=1\,,\quad |\arg f'(z)|\le \pi/2$$

This family is known as the class of functions with limited rotation.

Lemma 2. If $f\in V$ then :

a) $f'\in C$.

b) $f\in E_q$

Proof. a) If $f\in V$ then (by the definition) $f'(0)=1$. On the other side the condition $-\pi/2<\arg f'(z)<\pi/2$ implies that

$$re\ f'(z)=re|f'|e^{i\arg f'(z)}=|f'|\cos\arg f'(z)>0$$

and hence $f'\in C$.

b) From the fact that $f'\in C$ and formula (2) we conclude that

$$f(z)=\int_{0}^{z}\Big[\int_{-\pi}^{+\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\Big]dz$$

And hence the functions of V has the following integral representation:

$$f(z)=\int_{-\pi}^{+\pi}\left[\int_{0}^{z}\frac{e^{it}+z}{e^{it}-z}\,dz\right]d\mu(t)$$

In this way we see that $f\in E_q$ with $\alpha=-\pi,\beta=\pi$, and

$$q(z,t)=\int_{0}^{z}\frac{e^{it}+z}{e^{it}-z}\,dz\,.$$

Theorem 2 . The set of values of the functional

$$J(f)=\sum_{k=0}^{n}a_k(z_0)\ f^{(k)}(z_0)\ ,\quad f\in V \tag{6}$$

is the convex hull of the curve Γ given by the equation

$$W(z_0,t)=a_0(z_0)\int_{0}^{z_0}\frac{e^{it}+z}{e^{it}-z}+a_1(z_0)\frac{e^{it}+z}{e^{it}-z}+\sum_{k=2}^{n}a_k(z_0)\frac{2(k-1)!\,e^{it}}{\left(e^{it}-z\right)^k} \tag{7}$$

where $-\pi\le t\le\pi$.

Proof. Notice that the equations (4) have , in this case, the form

$$f^{(0)}(z)=f(z),\qquad f'(z)=\int_{-\pi}^{+\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)$$

$$f^{(k)}(z)=\int_{-\pi}^{+\pi}\frac{2(k-1)!\,e^{it}}{\left(e^{it}-z\right)^k}\ d\mu(t),\ k=2,3,\ldots$$

and thus putting

$$q(z_0,t)=\int_{0}^{z}\frac{e^{it}+z}{e^{it}-z}\,dz,$$

$$q'(z_0,t)=\frac{e^{it}+z}{e^{it}-z},$$

$$q^{(k)}(z_0,t)=\frac{2(k-1)!\,e^{it}}{\left(e^{it}-z\right)^k},\ k=2,3,\ldots,n$$

and substituting in (5) we get the curve Γ in the form (7) .And so the convex hull of this curve is the set of values of the functional (6). This ends the proof.

4. Applications.

In what follows we will give some propositions as an application of the above theorem .

Proposition 1. The sets of values of functional $F(f)=f(z_0)$ with $f \in V$ and $z_0 \in D$ is the closed set G which is connected , symmetric with respect to the real axis and contained in the disc $|w-z_0| \le 2d$ where $d = \max_{\varsigma \in G} |\varsigma|$ and $\varsigma = Log\left(1-e^{-it} z_0\right)$.

Proof. The curve Γ in this case has the form

$$\Gamma : W(z_0,t) = \int_0^{z_0} \frac{e^{it}+z}{e^{it}-z} dz , \ -\pi \le t \le +\pi$$

Putting $W(t) = W(z_0,t)$ and integrating with respect to z we get

(8) $$W(t) = z_0 - 2e^{it} Log\left(1-e^{-it} z_0\right).$$

Its known that [4] the principle branch of logarithm $\varsigma = Log(1-z)$ maps conformally the closed disc $|z| \le r,\ r<1$ into the connected domain G which is closed and symmetric with respect to the real axis . If δ is the maximum of the magnitude of ς when ς varies on G (e.i. $d = \max_{\varsigma \in G} |\varsigma|$) then the set of values of $\varsigma = Log(1-z)$ exists in the disc $|\varsigma| \le d$. This implies that the set of values of $-2e^{it} Log\left(1-e^{-it} z_0\right)$ exists in the disc $|\varsigma| \le 2d$ and hence the convex hull of the curve (8) would be contained the closed disc $|W - z_0| \le 2d$ where $d = \max_{\varsigma \in G} |\varsigma|$ and $\varsigma = Log\left(1-e^{-it} z_0\right)$.

Proposition 2 The sets of values of functional

$$J(f) = f(z) + f'(z) + f^{(n)}(z) \quad f \in V$$

at the point $z_0 = 0$ is the closed disc with center at 1 and rarius $2(n-1)!$.

Proof. Since

$$a_0(z) = a_1(z) = a_n(z) = 1, \quad a_k(z_0) = 0,\ k = 2,3,...,n-1,$$

And $z_0 = 0$ in theorem 2 , then the equation of Γ in this case has the form

$$W(t) = q(0,t) + q'(0,t) + q^{(n)}(0,t) = \int_0^{z_0} dz + 1 + \frac{2(n-1)!\, e^{it}}{\left(e^{it}\right)^n}$$

which implies that Γ is the circle

$$W(t) = 1 + 2(n-1)!\, e^{i(-n+1)t} , \ -\pi < t \le \pi$$

And so the set of values of the given functional is the convex hull of Γ which represents in this case the closed disc $|W-1| \le 2(n-1)!$.

Now, let $f \in V$ and let

(9) $$f(z) = z + a_2 z^2 + ... + a_n z^n + ... ,\ |z| < 1$$

be expantion of f in Taylor Series . It is well known [6] that the coefficients a_n in this case satisfy the estimates $|a_n| \le 2/n$ for every n . We will show this fact using the above theorems.

Proposition 3. If (9) is the Taylor Expantion of $f \in V$ in the neighborhood of 0 then the following estimates hold:

$$|a_n| \le 2/n\ ,\quad n = 1,2,3,...$$

Proof. Let us first find the set of values of the functional

$$F(f) = f^{(n)}(z)\ ,\ f \in V$$

at the point $z = 0$. Noticing that the curve Γ in this case is given by the equation

$$W(t) = W(0,t) = q^{(n)}(0,t) = 2(n-1!)e^{-i(n-1)t}\ ,\quad -\pi \le t \le +\pi$$

(which represents a circle $|W| = 2(n-1)!$) we can conclude that the closed disc $|W| \le 2(n-1)!$ is the convex hull of Γ and it is at the same time the set of values of the given functional . Now if (9) is the expantion of the function $f(z)$ in Taylor Series then

$$f^{(n)}(0) = n!\,a_n\ .$$

On the other hand we have shown that the value $f^{(n)}(0)$ lies in the disc $|W| \le 2(n-1)!$ for every $f \in V$. This implies the relation

$$\left|f^{(n)}(0)\right| = |n!\,a_n| \le 2(n-1)!$$

which shows that the estimates $|a_n| \le \frac{2(n-1)!}{n!} = \frac{2}{n}$ hold for every n .

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Department of Mathematics
University of Tishreen
Email: hbadour@scs-net.org