Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/18.2.1

An Efficient New Ratio-Type and Ratio-Type Exponential Estimator For Population Mean in Sample Surveys

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Received: Dec. 14, 2023 Accepted: Oct. 24,2024

Abstract: This paper addresses the problem of estimating the finite population mean of the study variable using information on auxiliary variable in sample surveys. A class of ratio-type and ratio-type exponential formulae for estimating finite population mean is defined. The bias and mean squared error of the proposed class of estimators are obtained up to terms of order n^{-1} under simple random sampling without replacement (SRSWOR) sampling scheme. The optimum conditions are obtained at which the mean squared error is minimum. It has been shown theoretically that at the optimum conditions, the proposed class of estimators is more efficient than the customary unbiased estimator, ratio and regression estimators. We have also obtained the condition in which the proposed class of estimators is superior to Rao's (1991) estimator. Two numerical exemplifications are given in support of the present study.

Keywords: Population Mean; Bias; Mean squared error; Ratio-type exponential estimator

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

It is tradition to use auxiliary information at the estimation stage to obtain better estimators of the population mean, catching the advantage of the correlation between the study and auxiliary variables. The ratio, product and regression estimators and their modified versions have been proposed by several authors, including Srivastava (1967, 1971, 1980), Reddy (1973, 1974), Walsh (1970), Sahai (1979), Adhvaryu and Gupta (1983), Upadhyaya et al. (1985), Singh (1986), Kothawala and Gupta (1988), Upadhyaya and Singh (1999), Singh and Agnihotri (2008), Singh and Nigam (2020), Pal and Singh (2022), among others, for estimating the population mean. These above authors have proposed modified classes of ratio and product type estimators and studied their properties up to terms of order O $\left(n^{-1}\right)$. The estimators developed by these authors have common minimum mean squared error (MSE) (up to terms of order n^{-1}) (at optimum conditions) equivalent to the approximate MSE/variance of the
ordinary linear regression estimator. Thus, a question arises that can we develop an estimator whose MSE is smaller than the MSE of the conventional regression estimator? Answer of this quest is given in this paper.

This paper considers a class of estimators for population mean along with their properties up to the first order of approximation (FOA). It has been demonstrated that the newly developed class of estimators has MSE smaller than that of the regression (or difference) estimator. The theoretical outcomes have been supported through an empirical study.

Let $\Omega=(\Omega_1,\Omega_2,\ldots,\Omega_N)$ be a finite population of N units. Let (y,x) be the (study, auxiliary) variables. Let $(\bar{Y},\bar{X})$ be the population means of the study and auxiliary variables respectively. It is presumed that the population mean $\bar{X}$ of x is known in advance. In such a situation, the problem of the estimation of $\bar{Y}$ of y has been considered. For this, it is required to draw a simple random sample (SRS) of size n without replacement (WOR) from the population Ω.

An unbiased estimator for the population mean $\bar{Y}$ of the study variable y is accustomed by

$$t_0=\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i \tag{1.1}$$

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with MSE/Variance under SRSWOR sampling design:

$$\mathrm{MSE}(t_0) = \mathrm{Var}(\bar{y}) = \lambda \bar{Y}^2 C_y^2, = \lambda S_y^2, \tag{1.2}$$

where$\lambda = \left(n^{-1} - N^{-1}\right), C_y = \frac{S_y}{\bar{Y}}$ and $S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})^2$

The ordinary ratio estimator for $\bar{Y}$ with known $\bar{X}$ of x is stated as

$$t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right). \tag{1.3}$$

whose MSE of t_R up to FOA is given by

$$\mathrm{MSE}(t_R) = \lambda \bar{Y}^2 \left[C_y^2 + C_x^2(1-2k)\right], \tag{1.4}$$

where $k = \rho \frac{C_y}{C_x}, C_x = \frac{S_x}{\bar{X}}, \rho = \frac{S_{yx}}{S_y S_x}, S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_i - \bar{X})^2,$

$$S_{yx} = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) \text{ and } \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

.

A class of ratio estimators, for $\bar{Y}$ due to Walsh (1970) and Reddy (1973), is delineated as

$$t_{WR} = \bar{y}[1 + \eta(u-1)]^{-1}, \tag{1.5}$$

with $u = \frac{\bar{x}}{\bar{X}}$ and ' η ' being a constant. Up to FOA, the MSE of t_{WR} is provided as

$$\mathrm{MSE}(t_{WR}) = \lambda \bar{Y}^2 \left[C_y^2 + \eta C_x^2(\eta - 2k)\right] \tag{1.6}$$

The 'best' value of ' η ' that minimizes the MSE of t_{WR} is:

$$\eta = k. \tag{1.7}$$

This yields the minimum MSE of t_{WR} as

$$MSE_{\min}(t_{WR}) = \lambda S_y^2 \left[1 - \rho^2\right]. \tag{1.8}$$

On the line of Bahl and Tuteja (1991), the exponential version of the ratio-type estimator t_{WR} at (1.5) is defined by

$$t_{W\mathrm{Re}} = \bar{y}\exp\left\{\frac{\eta(1-u)}{2 + \eta(u-1)}\right\} \tag{1.9}$$

The approximate MSE of $t_{W\,\mathrm{Re}}$ is obtained as:

$$\mathrm{MSE}(t_{W\mathrm{Re}}) = \lambda \bar{Y}^2 \left[C_y^2 + \left(\frac{1}{4}\right) \eta C_x^2(\eta - 4k)\right] \tag{1.10}$$

which is minimized for

$$\eta = 2k. \tag{1.11}$$

Insertion of (1.11) in (1.10) yields the least MSE of t_{WRe} as

$$MSE_{\min}(t_{W\,\mathrm{Re}}) = \lambda S_y^2 \left(1 - \rho^2\right). \tag{1.12}$$

It is to be mentioned that for $\eta = 1$, the estimator $t_{W\,\mathrm{Re}}$ boils down to the estimator of population mean $\bar{Y}$ as

$$t_{\mathrm{Re}} = \bar{y}\exp\left(\frac{1-u}{1+u}\right). \tag{1.13}$$

which is due to Bahl and Tuteja (1991).
The MSE of t_{Re} up to FOA is presented as

$$\mathrm{MSE}(t_{\mathrm{Re}}) = \lambda \bar{Y}^2 \left[C_y^2 + \frac{1}{4} C_x^2 (1 - 4K)\right]. \tag{1.14}$$

The ordinary regression estimator for $\bar{Y}$ is defined as

$$\bar{y}_{lr} = \bar{y} + \hat{\beta}(\bar{X} - \bar{x}), \tag{1.15}$$

where $\hat{\beta} = \frac{s_{yx}}{s_x^2}, s_{yx} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})$, and $s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2$.
The approximate MSE of $\bar{y}_{lr}$ is stated as

$$\text{MSE}(\bar{y}_{lr}) = \lambda S_y^2\left(1 - \rho^2\right). \tag{1.16}$$

Thus from (1.8), (1.12) and (1.14) we have

$$MSE_{\min}(t_{WR}) = MSE_{\min}(t_{WRe}) = \text{MSE}(\bar{y}_{lr}) = \lambda S_y^2\left(1 - \rho^2\right). \tag{1.17}$$

Rao (1991) envisaged a difference-type estimator for $\bar{Y}$ as

$$t_{\text{Rao}} = w_1\bar{y} + w_2(\bar{X} - \bar{x}), \tag{1.18}$$

(w_1, w_2) being suitably constants.
The MSE of t_{Rao} is obtained as

$$MSE_{\min}(t_{\text{Rao}}) = \bar{Y}^2\left[1 + w_1^2 a_1 + w_2^2 a_2 + 2w_1 w_2 a_3 - 2w_1\right], \tag{1.19}$$

where $a_1 = \left(1 + \lambda C_y^2\right)$, $a_2 = \left(\frac{\lambda}{R^2}\right)C_x^2, a_3 = -\left(\frac{\lambda}{R}\right)\rho C_y C_x$.
The 'best' values of (w_1, w_2) that minimizes the MSE(t_{Rao}) are:

$$\left.\begin{array}{l} w_1 = \frac{a_2}{\left(a_1 a_2 - a_3^2\right)} = w_{1(opt)} \\ w_2 = \frac{a_3}{\left(a_1 a_2 - a_3^2\right)} = w_{2(opt)} \end{array}\right\} \tag{1.20}$$

Substituting (1.20) in (1.19) we get the least MSE of t_{Rao} :

$$MSE_{\min}(t_{\text{Rao}}) = \bar{Y}^2\left[1 - \frac{a_2}{\left(a_1 a_2 - a_3^2\right)}\right]. \tag{1.21}$$

In this paper, we have discussed the problem of estimating the population mean $\bar{Y}$ of the study variable y using information on the known population mean $\bar{X}$ of the auxiliary variable x. The objective of this paper is to develop an efficient class of estimators for the population mean $\bar{Y}$ of the study variable y with the help of known population mean $\bar{X}$ of the auxiliary variable *x*. In the introduction section of the paper, some existing mean estimators are briefly reviewed. A new class of estimators is suggested in Section 2, along with its bias and mean squared error FOA, optimum conditions and minimum mean square error. In subsection 2.1, we have given a member of the suggested class of estimators along with its properties. Theoretical comparisons of the proposed class of estimators with other existing estimators have been dealt in Section 3. In Section 4, a computational study has been carried out over two natural population data sets. At the end of the paper, Section 5 provides conclusion of this study.

2 Developed Class of Estimators

It is observed from (1.15), that the minimum MSEs of the estimators t_{WR} and t_{WRe} are same and equal to the approximate MSE of the ordinary regression estimator $\bar{y}_{lr}$. This led authors to search an estimator of population mean $\bar{Y}$ of the study variable y using information on auxiliary variable x which is better than the ordinary regression estimator $\bar{y}_{lr}$. Rao (1991) made an effort to develop such an estimator. Keeping the above discussions in view we have made an effort to develop an estimator that is better than the regression estimator $\bar{y}_{lr}$ and the Rao's (1991) estimator t_{Rao} under certain conditions.

Thus, taking clues from Upadhyaya et al. (1985) and Rao (1991), combining the two estimators t_{WR} and t_{WRe} given at (1.5) and (1.9) respectively of the population mean $\bar{Y}$ of the study variable y, we put forward a class of estimators for the finite population mean $\bar{Y}$ of y as
$T = \bar{y}\left[\alpha_1 \frac{1}{\{1+\eta(u-1)\}} + \alpha_2 \exp\left\{\frac{\eta(1-u)}{2+\eta(u-1)}\right\}\right]$,
where (α_1, α_2) being suitable chosen scalars. The proposed class of estimators T reduces to the following set of known estimators of population mean $\bar{Y}$:
(i) $T \rightarrow t_0 = \bar{y}$ for $(\alpha_1, \alpha_2, \eta) = (1, 0, 0)$,

(ii) $T \to t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$ for $(\alpha_1, \alpha_2, \eta) = (1,0,1)$,
(iii) $T \to t_{\text{WR}} = \bar{y}[1+\eta(u-1)]^{-1}$ for $(\alpha_1, \alpha_2, \eta) = (1,0,\eta)$,
(iv) $T \to t_{\text{Re}} = \bar{y}\exp\left\{\frac{1-u}{1+u}\right\}$ (Bahl and Tuteja (1991) ratio-type exponential estimator) for $(\alpha_1, \alpha_2, \eta) = (0,1,1)$,
(v) $T \to t_{\text{Re}} = \bar{y}\exp\left\{\frac{\eta(1-u)}{2+\eta(u-1)}\right\}$ for $(\alpha_1, \alpha_2, \eta) = (0,1,\eta)$.

Thus we see that the proposed class of estimators T is very wide and includes various estimators as its members. Also, the study of the properties of the proposed class of estimators T unifies several results in one place which justifies the proposal of the class of estimators T.

For deriving bias and MSE of T, we write

$$\bar{y} = \bar{Y}(1+e_0), \bar{x} = \bar{X}(1+e_1)$$

such that

$$E(e_0) = E(e_1) = 0$$

and

$$E\left(e_0^2\right) = \lambda C_y^2, E\left(e_1^2\right) = \lambda C_x^2 \text{ and } E(e_0 e_1) = \lambda\rho C_y C_x.$$

Putting $\bar{y} = \bar{Y}(1+e_0)$ and $\bar{x} = \bar{X}(1+e_1)$ in T at (2.1), we have

$T = \bar{Y}\left[\alpha_1(1+e_0)(1+\eta e_1)^{-1} + \alpha_2(1+e_0)\exp\left\{\frac{-\eta e_1}{2}\left(1+\frac{\eta e_1}{2}\right)^{-1}\right\}\right]$

which is approximated as

$T \cong \bar{Y}\left[\alpha_1\left\{1+e_0-\eta e_1-\eta e_0 e_1+\eta^2 e_1^2\right\}+\alpha_2\left\{1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0 e_1}{2}+\frac{3\eta^2 e_1^2}{8}\right\}\right]$
or

$$(T-\bar{Y}) \cong \bar{Y}\left[\alpha_1\left\{1+e_0-\eta e_1-\eta e_0 e_1+\eta^2 e_1^2\right\}+\alpha_2\left\{1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0 e_1}{2}+\frac{3\eta^2 e_1^2}{8}\right\}-1\right] \tag{2.3}$$

$$\Rightarrow B(T) \cong E(T-\bar{Y})$$
$$= \bar{Y}(\alpha_1 A_4 + \alpha_2 A_5 - 1) \tag{2.4}$$

which is bias of T up to FOA, where

$$A_4 = \left[1+\lambda C_x^2\eta(\eta-k)\right], \quad A_5 = \left[1+\lambda\frac{\eta C_x^2}{2}\left(\frac{3}{4}\eta-k\right)\right].$$

Approximating the square of $(T-\bar{Y})$ at (2.3), we have

$$(T-\bar{Y})^2 \cong \bar{Y}^2\begin{bmatrix} 1+\alpha_1^2\left(1+2e_0-2\eta e_1+e_0^2-4\eta e_0 e_1+3\eta^2 e_1^2\right) \\ +\alpha_2^2\left(1+2e_0-\eta e_1+e_0^2-2\eta e_0 e_1+\eta^2 e_1^2\right) \\ +2\alpha_1\alpha_2\left(1+2e_0-\frac{3}{2}\eta e_1-3\eta e_0 e_1+e_0^2+\frac{15}{8}\eta^2 e_1^2\right) \\ -2\alpha_1\left(1+e_0-\eta e_1-\eta e_0 e_1+\eta^2 e_1^2\right) \\ -2\alpha_2\left(1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0 e_1}{2}+\frac{3}{8}\eta^2 e_1^2\right) \end{bmatrix} \tag{2.5}$$

$$\Rightarrow \text{MSE}(T) = E(T-\bar{Y})^2 = \bar{Y}^2\left[1+\alpha_1^2 A_1+\alpha_2^2 A_2+2\alpha_1\alpha_2 A_3-2\alpha_1 A_4-2\alpha_2 A_5\right] \tag{2.6}$$

which is MSE of T up to FOA, where
$A_1 = \left[1+\lambda\left\{C_y^2+\eta C_x^2(3\eta-4k)\right\}\right], A_2 = \left[1+\lambda\left\{C_y^2+\eta C_x^2(\eta-2k)\right\}\right]$,
$A_3 = \left[1+\lambda\left\{C_y^2+3\eta C_x^2\left(\frac{5}{8}\eta-k\right)\right\}\right]$, A_4 and A_5 are same as defined earlier.

Now, setting $\frac{\partial MSE(T)}{\partial \alpha_i} = 0$, i $= 1,2$; we have

$$\begin{bmatrix} A_1 & A_3 \\ A_3 & A_2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} A_4 \\ A_5 \end{bmatrix}. \tag{2.7}$$

Simplifying (2.7), the optimum values of (α_1, α_2) are:

$$\left.\begin{array}{l}\alpha_{1(opt)} = \frac{(A_2A_4 - A_3A_5)}{(A_1A_2 - A_3^2)} \\ \alpha_{2(opt)} = \frac{(A_1A_5 - A_3A_4)}{(A_1A_2 - A_3^2)}\end{array}\right\}.$$

Insertion of (2.8) in (2.6) yields the least MSE of T as

$$MSE_{\min}(T) = \bar{Y}^2\left(1 - \frac{\Delta}{\Delta^*}\right) \tag{2.9}$$

which holds when

$$\left.\begin{array}{c}0 < \frac{\Delta}{\Delta^*} < 1 \\ \text{and} \quad \Delta^* > 0\end{array}\right\} \tag{2.10}$$

,

where
$\Delta = \left(A_2A_4^2 - 2A_3A_4A_5 + A_1A_5^2\right)$, and $\Delta^* = \left(A_1A_2 - A_3^2\right)$.
Thus we reached to the theorem stated below:
Theorem 2.1- The MSE$(T) \geq MSE_{\min}(T)$

$$= \bar{Y}^2\left(1 - \frac{\Delta}{\Delta^*}\right)$$

with equality holding if

$$\alpha_1 = \alpha_{1(opt)},$$
$$\alpha_2 = \alpha_{2(opt)},$$

where $\alpha_{i(opt)}$, i $= 1,2$ is given by (2.8).

2.1 Particular Case

If we set $\alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_2 = (1 - \alpha_1)$ in (2.1) then
$T \to T_{(1)} = \bar{y}\left[\eta_1 \frac{1}{\{1+\eta(u-1)\}} + (1-\eta_1)\exp\left\{\frac{\eta(1-u)}{2+\eta(u-1)}\right\}\right]$
Inserting $\alpha_2 = (1 - \alpha_1)$ in (2.4) and (2.6), the bias and MSE of $T_{(1)}$ are:

$$B\left(T_{(1)}\right) = \bar{Y}\left[\alpha_1(A_4 - A_5) + A_5 - 1\right] \tag{2.12}$$

$$\text{MSE}\left(T_{(1)}\right) = \bar{Y}^2\left[1 + A_2 - 2A_5 + \alpha_1^2(A_1 + A_2 - 2A_3) - 2\alpha_1(A_2 - A_3 + A_4 - A_5)\right] \tag{2.13}$$

$$= \lambda\bar{Y}^2\left[C_y^2 + \frac{\eta C_x^2}{4}(\eta - 4k) + \alpha_1^2\left(\frac{\eta^2C_x^2}{4}\right) + \alpha_1\left(\frac{\eta C_x^2}{2}\right)(\eta - 2k)\right] \tag{2.14}$$

which is minimized for

$$\alpha^*_{1(\text{ opt })} = \frac{(A_2 - A_3 + A_4 - A_5)}{(A_1 + A_2 - 2A_3)}$$
$$= \frac{(2k - \eta)}{\eta} = \left(\frac{2k}{\eta} - 1\right). \tag{2.15}$$

Posing (2.15) in (2.13) (or (2.14)) we find the least MSE of $T_{(1)}$ as

$$MSE_{\min}\left(T_{(1)}\right) \cong \bar{Y}^2\left[1 + A_2 - 2A_5 - \frac{(A_2 - A_3 + A_4 - A_5)^2}{(A_1 + A_2 - 2A_3)}\right],$$
$$= \lambda\bar{Y}^2\left[C_y^2 + \frac{\eta C_x^2}{4}(\eta - 4k) - \frac{(2k-\eta)^2}{4}C_x^2\right],$$
$$= \lambda S_y^2\left(1 - \rho^2\right) = \text{MSE}(\bar{y}_{lr}). \tag{2.16}$$

3 Theoretical Comparison

The theoretical comparison plays an important role in obtaining the conditions of preference under which the proposed class of estimators T is more efficient than the estimators $\bar{y}, t_R, t_{\mathrm{Re}}, t_{WR}, t_{W\mathrm{Re}}, \bar{y}_{lr}, T_{(1)}$ and the estimator t_{Rao} due to Rao (1991). This fact has been dealt under this section.

From (1.2), (1.4), (1.14) and (2.16), we have

$$\mathrm{MSE}(\bar{y}) - \mathrm{MSE}_{\min}\left(T_{(1)}\right) = \lambda S_y^2 \rho^2 \geq 0 \tag{3.1}$$

$$\mathrm{MSE}\left(t_R\right) - \mathrm{MSE}_{\min}\left(T_{(1)}\right) = \lambda \bar{Y}^2 C_x^2 (1-k)^2 \geq 0 \tag{3.2}$$

$$\mathrm{MSE}\left(t_{\mathrm{Re}}\right) - MSE_{\min}\left(T_{(1)}\right) = \lambda \bar{Y}^2 C_x^2 \left(\frac{1}{2} - k\right)^2 \geq 0 \tag{3.3}$$

Further from (1.8), (1.12), (1.16) and (2.16), we have

$$MSE_{\min}\ (t_{WR}) = MSE_{\min}\ (t_{W\mathrm{Re}}) = MSE_{\min}\ \left(T_{(1)}\right) = \mathrm{MSE}\left(\bar{y}_{lr}\right) = \lambda S_y^2 \left(1-\rho^2\right) \tag{3.4}$$

Expressions (3.1), (3.2) and (3.3) exhibit that the envisaged formula $T_{(1)}$ is superior to the formulae $\bar{y}$ and t_R. That is, the proposed estimator $T_{(1)}$ is more efficient than the conventional unbiased estimator $\bar{y}$, ordinary ratio estimator t_R and the ratio-type exponential estimator t_{Re} due to Bahl and Tuteja (1991). Equation (3.4) demonstrates that the proposed estimator $T_{(1)}$ is at par with $t_{WR}, t_{W\,\mathrm{Re}}$ (at optimum condition) and the regression estimator $\bar{y}_{lr}$.

Subtracting (2.9) from (2.16), we get

$$MSE_{\min}\left(T_{(1)}\right) - MSE_{\min}(T) = \bar{Y}^2 \frac{\left[A_2\left(A_4 - A_1\right) + A_3\left(A_3 - A_4\right) + A_5\left(A_1 - A_3\right)\right]^2}{\left(A_1 + A_2 - 2A_3\right)\left(A_1 A_2 - A_3^2\right)} \geq 0$$

which yields the inequality

$$MSE_{\min}(T) < MSE_{\min}\left(T_{(1)}\right) \tag{3.5}$$

This leads to a conclusion from (3.5) that the developed class of estimators T is more accurate than the estimators $\bar{y}, t_R, t_{\mathrm{Re}}, t_{\mathrm{WR}}, t_{WRe}, T_{(1)}$ and the regression estimator $\bar{y}_{lr}$.

We express $MSE_{\min}\ \left(T_{(1)}\right)$ at (2.16) in terms of (a_1, a_2, a_3) as

$\mathrm{MSE}\,E_{\min}\left(T_{(1)}\right) = \bar{Y}^2\left(a_1 - 1 - \frac{a_3^2}{a_2}\right)$

So from (1.16) and (3.6), we note that

$$MSE_{\min}\left(T_{(1)}\right) - MSE_{\min}\left(t_{\mathrm{Rao}}\right) = \bar{Y}^2 \frac{\left(a_1 a_2 - a_3^2 - a_2\right)^2}{a_2\left(a_1 a_2 - a_3^2\right)} \geq 0 \tag{3.7}$$

$$\Rightarrow MSE_{\min}\left(t_{\mathrm{Rao}}\right) \leq MSE_{\min}\left(T_{(1)}\right) \tag{3.8}$$

which shows that Rao's (1991) estimator t_{Rao} is more precise than the estimators $\bar{y}, t_R, t_{\mathrm{Re}}, t_{WR}, t_{WRe}, \bar{y}_{lr}$ and $T_{(1)}$.

From (1.21) and (2.9) the superiority of T over t_{Rao} can be seen if the following inequality:

$$\frac{\left(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2\right)}{\left(A_1 A_2 - A_3^2\right)} > \frac{a_2}{\left(a_1 a_2 - a_3^2\right)} \tag{3.9}$$

holds good.

4 Empirical Study

To see the achievement of the developed formula T over other formulae $\bar{y}, t_R, t_{WR}, t_{W\,\mathrm{Re}}$ and $T_{(1)}$, we contemplate the two population data sets given below.

Population I (Source: Cochran (1977, p.172)).

y (study variable): The estimated production in bushels of peach.

x (auxiliary variable): The number of peach trees in an orchard.

The values of required parameters are:

$\bar{Y} = 56.47, \bar{X} = 44.45, \quad S_y^2 = 6409, S_x^2 = 3898, S_{xy} = 4434, \rho = 0.887,$ N $= 257,$ n $= 25.$

Population II (Source: MFA (2004)).

y: District wise tomato production in tonnes of Pakistan 2003.

x: District wise tomato production in tonnes of Pakistan 2002.

$\bar{Y} = 3135.6186, \bar{X} = 3050.2784, \quad S_y^2 = 4785598, \quad S_x^2 = 50993978, \quad S_{xy} = 39875735.79, \rho = 0.8072,$ N $= 97,$ n $=$ 30.

We have computed the percent relative efficiencies (PRE's) of different estimators t_R, $\left(t_{WR}, t_{W\,\mathrm{Re}}, T_{(1)}, \bar{y}_{lr}\right)$, t_{Rao} and T relative to $\bar{y}$ through the following formulae:

$$\mathrm{PRE}\left(t_R, \bar{y}\right) = \frac{C_y^2}{C_y^2 + C_x^2(1-2k)]} \times 100 \tag{4.1}$$

$$\mathrm{PRE}\left(t_{WR} \text{ or } t_{WRe} \text{ or } T_{(1)} \text{ or } \bar{y}_{lr}, \bar{y}\right) = \left(1-\rho^2\right)^{-1} \times 100 \tag{4.2}$$

$$\mathrm{PRE}\left(t_{\mathrm{Rao}}, \bar{y}\right) = \frac{\lambda C_y^2}{\left[1 - \frac{a_2}{\left(a_1 a_2 - a_3^2\right)}\right]} \times 100 \tag{4.3}$$

$$\mathrm{PRE}\left(t_{\mathrm{Rao}}, \bar{y}\right) = \frac{\lambda C_y^2}{\left[1 - \frac{\left(A_2 A_4^2 - 2A_3A_4A_5 + A_1A_5^2\right)}{\left(A_1A_2 - A_3^2\right)}\right]} \times 100 \tag{4.4}$$

The PRE values of t_R, $\left(t_{WR}, t_{WRe}, T_{(1)}, \bar{y}_{lr}\right)$ and t_{Rao} are depicted in Table 4.1.

We have computed the PRE $(T, \bar{y})$ for selected values of η and outcomes are depicted in Table 4.2.

Table 4.1 shows that t_{Rao} is better than $\bar{y}, t_R$ and $\left(t_{WR}, t_{WRe}, T_{(1)}, \bar{y}_{lr}\right)$ in both populations I and II. Results of Tables 4.1 and 4.2 demonstrate that the formulated estimator T is more precise than Rao's (1991) formula t_{Rao} with sizeable gain in efficiency for a wide range of η. Hence, the developed formula T is superior to $\left(\bar{y}, t_R, t_{WR}, t_{WRe}, T_{(1)}, \bar{y}_{ir}\right)$ with considerable gain in efficiency. The largest PRE $(T, \bar{y}) = 2997439.00\%$ (in population I) and PRE $(T, \bar{y}) = 1565.13$ % (in population II) are recorded for $\eta = -0.6034$ which are very large as compared to the PRE $(t_{\mathrm{Rao}}, \bar{y}) = 476.23\%$ (in population I) and PRE $(t_{\mathrm{Rao}}, \bar{y}) = 298.21\%$ (in population II) of Rao's estimator t_{Rao}. Thus there are several η-values for which the developed formula T is superior to the estimators $\left(\bar{y}, t_R, t_{WR}, t_{WRe}, T_{(1)}, \bar{y}_{ir}, t_{\mathrm{Rao}}\right)$. We, therefore, conclude that the developed class of estimators T is beneficial over the estimators (discussed here) in practice.

5 Conclusion

In this study an improved class of estimators of finite population mean has been suggested. Asymptotic expressions for bias and MSE have been derived. The optimum conditions are obtained under which formulated class of estimators T has least MSE. Theoretical comparisons of the developed formula T has been made with existing competitors. We have performed an empirical study through two natural population data sets and entries are given in Tables 4.1 and 4.2. It is observed from Tables 4.1 and 4.2 that the developed formula T has PRE larger than $\left(\bar{y}, t_R, t_{WR}, t_{WRe}, T_{(1)}, \bar{y}_{ir}\right)$ and Rao's (1991) estimator t_{Rao}. Table 4.2 also exhibits that the PRE of the proposed class of estimators T over other existing estimators are considerable for several other values of η. Thereby meaning is that there is enough scope of selecting the values of η for obtaining estimators from the suggested class of estimators T better than the existing estimators. For obtaining larger gain in efficiency one has to choose the value of η in the vicinity of $\eta = -0.6034$ for both the populations, see Table 4.2. Such an estimation procedure can be used in future where the study variable is sensitive and auxiliary variable is sensitive/nonsensitive. Thus, it is worth advisable to recommend the use of the present study in practice.

Table 4.1- PRE's of t_R, $\left(t_{\mathrm{WR}}, t_{WRe}, T_{(1)}, \bar{y}_{lr}\right)$ and t_{Rao} with respect to $\bar{y}$

Estimator	PRE $(.,\bar{y})$	
	Population I	Population II
$\bar{y}$	100.00	100.00
t_R	446.43	242.18
$\left(t_{\mathrm{WR}}, t_{W\,\mathrm{Re}}, T_{(1)}, \bar{y}_{lr}\right)$	468.97	287.00
t_{Rao}	476.23	298.21

Table 4.2- PRE $(T,\bar{y})$ for different values of η for population I and II.

Scalar 'η'	PRE $(T,\bar{y})$	
	Population I	Population II
-0.10	584.35	309.04
-0.20	654.43	328.35
-0.30	776.70	362.39
-0.40	1027.65	428.81
-0.50	1779.53	592.90
-0.60	46522.68	1465.97
-0.6025	167926.90	1537.44
-0.6030	352887.20	1552.69
-0.6034	**2997439.00**	**1565.13**
1.790531=2 K (Population I)	540.70	355.49
1.521374=2 K (Population II)	519.08	339.35
1.70	533.53	350.05
1.60	525.46	344.05
1.50	517.36	338.08
1.40	509.38	332.14
1.30	501.66	326.28
1.20	494.35	320.54
1.10	487.60	314.97
1.00	481.58	309.63
0.90	476.47	304.60

Declarations

Competing interests: No
Authors' contributions: Prof. Housila P. Singh has Formulated the problem, Prof. Rajesh Tailor has prepared the manuscript and Priyanka Malviya has done Empirical study, Discussion and Conclusion.
Funding: No
Availability of data and materials: Data of Population I (Source: Cochran (1977,p.172)) and Population II (Source: MFA (2004))
Acknowledgments: Authors are thankful to the learned referee for their valuable suggestions regarding improvement of the paper.

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