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Optimal Maintenance Strategy of a System Which Deteriorates in Accordance With a Gamma Process Over a Bounded Time Horizon

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Abstract: This article discusses the mean and variance of discounted total maintenance cost of a system intended to operate during the finite time interval $[0,t]$. We implement the age replacement strategy under the assumption that the system deteriorates in accordance with a stochastic gamma process. Laplace transforms for the mean and second moment of the total maintenance cost are obtained. We apply the result to obtain an optimal period for preventive maintenance which minimizes the mean of the discounted total maintenance cost in the time interval $[0,t]$. Standard deviation for the optimal period for preventive maintenance is also calculated.

Keywords: Age replacement strategy; Gamma process; Discount factor; Renewal process; Laplace transform.

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1 Introduction

Consider a system, for example a production machine, that is expected to operate continuously. Due to various factors such as environmental conditions or aging, the system's performance decreases over time and can even break down. To ensure that the system works properly, system maintenance is required. An optimal system maintenance strategy is an important topic because it can minimize the total maintenance cost of the system in some time period.

The simplest system maintenance strategy is to replace the system when it breaks down. This strategy is called a replacement strategy. Assume that the system starts operating at time 0 and is immediately replaced with a new one when it fails. We can model the number of system replacements in the time interval $[0,t]$ with a renewal process. Furthermore, if each system replacement is charged, then the total cost of system replacement can be modeled a renewal reward process.

Typically system performance degrades stochastically over time. Many researchers model the performance deterioration of systems using a stochastic gamma process. In this model, it is usually assumed that the system will fail when its deterioration exceeds a certain level. So the time to failure of the system equals epoch since the last replacement until the system deterioration exceeds a specified level. For more information about the application of stochastic gamma processes in maintenance see [11].

Replacing a failed system by the new one is often incurred high cost so that if we use the replacement strategy the total replacement cost can be very high in some time interval. To reduce the total maintenance cost we may apply the following alternative maintenance strategy. Soon after its failure the system is replaced by the new one. We call this type of maintenance a corrective maintenance. If within some given fixed time period since the last maintenance action the system did not fail, a preventive maintenance action is performed which brings the system performs as good as new. This maintenance scenario is referred to as age replacement strategy, see [4].

In the age replacement strategy an interesting issue is to find an optimal period for preventive maintenance so that the mean of the discounted total maintenance costs over some time interval is minimum. Some authors have paid attention to this issue. Van Noortwijk [9] discussed optimal replacement decisions for structure under a gamma process deterioration over unbounded time horizon where the time to failures are modeled as a discrete time renewal process, the corrective

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and preventive maintenance costs are constants, and a discount factor is included. In another article Van Noortwijk [10] discussed the asymptotic variance of discounted total maintenance costs. In [9] and [10] it was assumed that the maintenance actions performed such that the system like the new one. Suyono and Van der Weide [7] discussed the expected value of the discounted total maintenance cost in a finite time interval $[0,t]$ where the deterioration of the system follows a gamma process, the maintenance times is modeled as a continuous-time renewal process, and the costs for corrective and preventive maintenance are constants.

Since the publication of the article [7] as far as we know there is no article which discuss the variance of the discounted total maintenance cost under assumptions as in [7]. In [8] the mean and variance of total discounted cost were discussed but they did not consider the system deterioration. Cheng and Li [2] discussed an optimal replacement policy for a degenerative system with two-types of failure states. They assume that the system cannot be as good as new after repaired, and the deterioration process is stochastic. Corset, Fouladirad, Paroissin [3] considered a condition-based maintenance policy with perfect corrective and an imperfect preventive actions where the degradation is modelled by a gamma process. They derived the maintenance cost by using a Markov-renewal process. This paper discusses the mean and variance of the discounted total maintenance cost in the finite time interval $[0,t]$ where the system deterioration is modeled by a gamma process, maintenance times is modeled as a continuous-time renewal process, and the corrective and preventive costs are assumed to be random variables. This setting is different from those of Van Noortwijk [9]. This paper can also be considered as a generalization of the model in [7] because we allow the corrective and preventive maintenance costs are random variables.

This article is organized as follows. In Section 2 we review the notion of the stochastic gamma process. In Section 3 we define mathematical models for the discounted total maintenance cost in the finite time interval $[0,t]$ where the system deterioration follows a gamma process defined in Section 2. Section 4 and 5 deal with the mean and variance of the discounted total maintenance cost. In Section 6 we give an example where we can determined an optimal time period for preventive maintenance action. The conclusion is given in the last section.

2 Gamma Process

Recall that a random variable X has a gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$, if its probability density function (p.d.f.) is given by

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} 1_{[0,\infty)}(x) \tag{1}$$

where $1_A(x) = 1$ if $x \in A$ and zero otherwise, and

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

is the gamma function for $a > 0$. Related to the gamma distribution we may define a gamma process as follows. Let $\{X(t), t \geq 0\}$ be a continuous-time stochastic process having stationary increment property. The stochastic process $\{X(t), t \geq 0\}$ is called a gamma process with the shape function $a(t) > 0$, $t > 0$, and scale parameter $b > 0$, if it satisfies the following properties:

(i) $X(0) = 0$ with probability 1,
(ii) $X(t) - X(s)$ is gamma distributed with the shape parameter $a(t) > 0$ and the scale parameter $b > 0$, for all $t \geq s \geq 0$,
(iii) $X(t)$ has an independent increment property,

see [5] for example. The gamma process is widely used to model deterioration of systems.

If the gamma process $\{X(t), t \geq 0\}$ has the mean function $E[X(t)] = \mu t$ and variance $Var[X(t)] = \sigma^2 t$, where $\mu > 0$, $\sigma^2 > 0$, then we may write the p.d.f. of $X(t)$ as

$$f_{X(t)}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} 1_{[0,\infty)}(x) \tag{2}$$

where $a = \frac{\mu}{\sigma^2} t$ and $b = \frac{\mu^2}{\sigma^2}$.

3 Modeling total maintenance cost

Suppose that a new system starts operating at time $S_0 = 0$. Over time, the system performance decreases stochastically, and eventually break down. We will model the system deterioration as a gamma process having the marginal distribution defined in (2). We suppose that the system will fail if the deterioration exceeds a specified level $y > 0$. Thus, if the lifetime of the system is denoted by T, then

$$T > s \quad \text{if and only if} \quad X(s) < y. \tag{3}$$

In this paper we apply the age replacement strategy as follows. If no failure after a time period L since the time 0 or the last maintenance, a preventive maintenance is carried out perfectly so that the system is back to a state as good as new. If the system fails within the period L time units after the time 0 or the last maintenance action, a corrective maintenance is performed by replacing the system with the new one. So after both types of maintenance actions the system is in a good condition as a new one.

Let $0 < S_1 < S_2 < ...$ be the times at which maintenance actions, both preventive and corrective, take place. Since we assume that after maintenance the system is as good as new, we model the sequence $\{S_j : j \geq 1\}$ as a renewal process. The times T_j between successive maintenance actions, i.e. $T_j = S_j – S_{j-1}, j = 1,2,3,...$ are are assumed to be i.i.d., strictly positive random variables. Since we model the deterioration of the system by the gamma process defined in (2), it follows from (3) that

$$P(T_j > s) = P(X(s) < y), s < L \tag{4}$$

and $P(T_j = 0)$ otherwise. We will denote by $N(t)$ the number of maintenance actions during the time interval $[0,t]$, that is

$$N(t) = \sup\{j : S_j \leq t\}.$$

Obviously,

$$N(t) \geq n \quad \text{if and only if} \quad S_n \leq t.$$

We will denote by C_P and C_F the preventive and corrective maintenance costs respectively. We assume that (C_P,C_F) is a random vector with non-negative components and independent of the sequence $\{S_j : j \geq 1\}$ of maintenance times. When $T_j < L$ for some j the corrective maintenance cost C_F has to be paid at time S_j, and when $T_j = L$ the preventive maintenance cost C_P is incurred at time S_j. So the total maintenance costs during the time interval $[0,t]$ is given by

$$K(t) = \sum_{i=1}^{N(t)} [C_P 1_{\{T_i=L\}} + C_F 1_{\{T_i<L\}}]. \tag{5}$$

If we apply the exponential discount rate $r > 0$ to each maintenance cost, then the amount of money that has to be deposited at time 0 to cover the total maintenance cost during the bounded time interval $[0,t]$ is formulated by

$$K(t,r) = \sum_{i=1}^{N(t)} [C_P e^{-rS_i} 1_{\{T_i=L\}} + C_F e^{-rS_i} 1_{\{T_i<L\}}]. \tag{6}$$

In the next sections we will derive the Laplace transforms of the mean and second moment of $K(t)$ and $K(t,r)$.

4 Mean of the total maintenance cost

The first result in this article is the Laplace transform of the mean of the total maintenance cost $K(t,r)$ defined in (6) as stated in the following theorem.

Theorem 1. *For $\alpha > 0$,*

$$\int_0^\infty E[K(t,r)]e^{-\alpha t}dt = \frac{E[C_P]e^{-(\alpha+r)L}P(T_1 = L) + E[C_F]E[e^{-(\alpha+r)T_1} 1_{\{T_1<L\}}]}{\alpha(1 - E[e^{-(\alpha+r)T_1}])}. \tag{7}$$

Proof. Let $Y_i = C_P e^{-rS_i} 1_{\{T_i=L\}} + C_F e^{-rS_i} 1_{\{T_i<L\}}$. Then we may write $K(t,r)$ defined in (6) as

$$K(t,r) = \sum_{i=1}^{N(t)} Y_i. \tag{8}$$

The expected value of $K(t,r)$ is

$$E[K(t,r)] = E\left[\sum_{i=1}^{N(t)} Y_i\right] = E\left[\sum_{i=1}^{\infty} Y_i 1_{\{N(t)\geq i\}}\right] = E\left[\sum_{i=1}^{\infty} Y_i 1_{\{S_i \leq t\}}\right] \tag{9}$$

where we have used the relation $N(t) \geq i$ if and only if $S_i \leq t$ in the last equality. Generally it is difficult to calculate directly the expected value $E[K(t,r)]$. One way to overcome this difficulty is by taking the Laplace transform of $E[K(t,r)]$ considered as a function of $t > 0$. Since we assume Y_i is nonnegative, by using Tonelli's theorem, see for example [6], the Laplace transform of $E[K(t,r)]$ can be written as follows: for $\alpha > 0$,

$$\int_0^\infty E[K(t,r)]e^{-\alpha t}dt = \int_0^\infty E\left[\sum_{i=1}^{\infty} Y_i 1_{\{S_i\leq t\}}\right] e^{-\alpha t}dt = \sum_{i=1}^{\infty} E\left[\int_0^\infty Y_i 1_{\{S_i\leq t\}} e^{-\alpha t}dt\right] = \frac{1}{\alpha}\sum_{i=1}^{\infty} E[Y_i e^{-\alpha S_i}]. \tag{10}$$

Now,

$$\begin{aligned} E[Y_i e^{-\alpha S_i}] &= E[C_P e^{-rS_i} 1_{\{T_i=L\}} e^{-\alpha S_i}] + E[C_F e^{-rS_i} 1_{\{T_i<L\}} e^{-\alpha S_i}] \\ &= E[C_P]E[1_{\{T_i=L\}} e^{-(\alpha+r)S_i}] + E[C_F]E[1_{\{T_i<L\}} e^{-(\alpha+r)S_i}]. \end{aligned}$$

Writing $S_i = T_i + S_{i-1}$, and using the fact that T_i and S_{i-1} are independent, we have

$$\begin{aligned} E[1_{\{T_i=L\}} e^{-(\alpha+r)S_i}] &= E[1_{\{T_i=L\}} e^{-(\alpha+r)(T_i+S_{i-1})}] \\ &= E[1_{\{T_i=L\}} e^{-(\alpha+r)T_i}]E[e^{-(\alpha+r)S_{i-1})}] \\ &= e^{-(\alpha+r)L}P(T_1 = L)E[e^{-(\alpha+r)S_{i-1})}]. \end{aligned}$$

Note that

$$E[e^{-(\alpha+r)S_{i-1})}] = E[e^{-(\alpha+r)(T_1+T_2+\ldots+T_{i-1})}] = (E[e^{-(\alpha+r)T_1})^{i-1}$$

since (T_i) are i.i.d. random variables. Thus

$$E[1_{\{T_i=L\}} e^{-(\alpha+r)S_i}] = e^{-(\alpha+r)L}P(T_1 = L)(E[e^{-(\alpha+r)T_1}])^{i-1}$$

Similar argument leads

$$E[1_{\{T_i<L\}} e^{-(\alpha+r)S_i}] = E[e^{-(\alpha+r)T_1} 1_{\{T_1<L\}}](E[e^{-(\alpha+r)T_1}])^{i-1}.$$

It follows that

$$E[Y_i e^{-\alpha S_i}] = \left(E[C_P]e^{-(\alpha+r)L}P(T_1 = L) + E[C_F]E[e^{-(\alpha+r)T_1} 1_{\{T_1<L\}}]\right)(E[e^{-(\alpha+r)T_1}])^{i-1}. \tag{11}$$

Substituting (11) into (10) we get

$$\int_0^\infty E[K(t,r)]e^{-\alpha t}dt = \frac{1}{\alpha}\sum_{i=1}^{\infty}\left(E[C_P]e^{-(\alpha+r)L}P(T_1 = L) + E[C_F]E[e^{-(\alpha+r)T_1} 1_{\{T_1<L\}}]\right)(E[e^{-(\alpha+r)T_1}])^{i-1}.$$

Since $0 < e^{-(\alpha+r)T_1} < 1$ with probability 1, and hence $0 < E[e^{-(\alpha+r)T_1} < E[1] = 1$, it follows that

$$\int_0^\infty K(t,r)e^{-\alpha t}dt = \frac{E[C_P]e^{-(\alpha+r)L}P(T_1 = L) + E[C_F]E[e^{-(\alpha+r)T_1} 1_{\{T_1<L\}}]}{\alpha(1 - E[e^{-(\alpha+r)T_1}])}.$$

As a corollary, in case of no discounting, the Laplace transform of $K(t)$ defined in (5) is

$$\int_0^\infty E[K(t)]e^{-\alpha t}dt = \frac{E[C_P]e^{-\alpha L}P(T_1 = L) + E[C_F]E[e^{-\alpha T_1} 1_{\{T_1<L\}}]}{\alpha(1 - E[e^{-\alpha T_1}])}.$$

5 Variance of the total maintenance costs

The second result in this paper is the Laplace transform of the second moment of the total maintenance costs defined in (6) as stated in the following theorem.

Theorem 2. *For $\alpha > 0$,*

$$\int_0^\infty E[K(t,r)^2]e^{-\alpha t}dt = \frac{E[C_P^2]A_{01} + E[C_F^2]A_{02}}{\alpha(1-B)} + \frac{2(E[C_P^2]A_1 + E[C_PC_F]A_2 + E[C_PC_F]A_3 + E[C_F^2]A_4)}{\alpha(1-B)(1-C)} \quad (12)$$

where

$A_{01} = e^{-(\alpha+2r)L}P(T_1 = L)$, $\quad A_{02} = E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$,
$A_1 = e^{-(2\alpha+3r)L}P(T_1 = L)$, $\quad A_2 = e^{-(\alpha+2r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+r)T_1}]$,
$A_3 = e^{-(\alpha+r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$, $\quad A_4 = e^{-(\alpha+r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$,
$B = E[e^{-(\alpha+2r)T_1}]$, *and* $C = E[e^{-(\alpha+r)T_1}]$.

Proof. Let $Y_i = C_Pe^{-rS_i}1_{\{T_i=L\}} + C_Fe^{-rS_i}1_{\{T_i<L\}}$. Then from (6) we have

$$K(t,r)^2 = \left(\sum_{i=1}^{N(t)} Y_i\right)^2 = \sum_{i=1}^{N(t)} Y_i^2 + 2\sum_{i=1}^{N(t)-1}\sum_{j=i+1}^{N(t)} Y_iY_j \quad (13)$$

Note that

$$\begin{aligned} Y_i^2 &= [C_P1_{\{T_i=L\}} + C_F1_{\{T_i<L\}}]^2(e^{-rS_i})^2 \\ &= [C_P^21_{\{T_i=L\}} + 2C_PC_F1_{\{T_i=L\}}1_{\{T_i<L\}} + C_F^21_{\{T_i<L\}}]e^{-2rS_i} \end{aligned} \quad (14)$$

and

$$\begin{aligned} Y_iY_j &= [C_P1_{\{T_i=L\}} + C_F1_{\{T_i<L\}}]e^{-rS_i}[C_P1_{\{T_j=L\}} + C_F1_{\{T_j<L\}}]e^{-rS_j} \\ &= [C_P^21_{\{T_i=L\}}1_{\{T_j=L\}} + C_PC_F1_{\{T_i=L\}}1_{\{T_j<L\}} + C_PC_F1_{\{T_i<L\}}1_{\{T_j=L\}} + C_F^21_{\{T_i<L\}}1_{\{T_j<L\}}]e^{-rS_i}e^{-rS_j} \end{aligned} \quad (15)$$

From (13) we get

$$\begin{aligned} E[K(t,r)^2] &= E\left[\sum_{i=1}^{N(t)} Y_i^2 + 2\sum_{i=1}^{N(t)-1}\sum_{j=i+1}^{N(t)} Y_iY_j\right] \\ &= E\left[\sum_{i=1}^{\infty} Y_i^21_{\{N(t)\geq i\}} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} Y_iY_j1_{\{N(t)-1\geq i\}}1_{\{N(t)\geq j\}}\right] \\ &= E\left[\sum_{i=1}^{\infty} Y_i^21_{\{S_i\leq t\}} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} Y_iY_j1_{\{S_{i+1}\leq t\}}1_{\{S_j\leq t\}}\right]. \end{aligned}$$

The Laplace transform of $E[K(t,r)^2]$ can be calculated as follows.

$$\begin{aligned} \int_0^\infty E[K(t,r)^2]e^{-\alpha t}dt &= \int_0^\infty E\left[\sum_{i=1}^{\infty} Y_i^21_{\{S_i\leq t\}} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} Y_iY_j1_{\{S_{i+1}\leq t\}}1_{\{S_j\leq t\}}\right]e^{-\alpha t}dt \\ &= \sum_{i=1}^{\infty} E\left[\int_0^\infty Y_i^21_{\{S_i\leq t\}}e^{-\alpha t}dt\right] + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} E\left[\int_0^\infty Y_iY_j1_{\{S_{i+1}\leq t\}}1_{\{S_j\leq t\}}e^{-\alpha t}dt\right] \\ &= \frac{1}{\alpha}\sum_{i=1}^{\infty} E\left[Y_i^2e^{-\alpha S_i}\right] + \frac{2}{\alpha}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} E\left[Y_iY_je^{-\alpha S_j}\right] \end{aligned} \quad (16)$$

Now, from (14) we get

$$\begin{aligned} E[Y_i^2e^{-\alpha S_i}] &= E[(C_P^21_{\{T_i=L\}} + 2C_PC_F1_{\{T_i=L\}}1_{\{T_i<L\}} + C_F^21_{\{T_i<L\}})e^{-(\alpha+2r)S_i}] \\ &= E[C_P^21_{\{T_i=L\}}e^{-(\alpha+2r)S_i}] + 2E[C_PC_F1_{\{T_i=L\}}1_{\{T_i<L\}}e^{-(\alpha+2r)S_i}] + E[C_F^21_{\{T_i<L\}})e^{-(\alpha+2r)S_i}] \end{aligned}$$

Note that

$$E[C_P^2 1_{\{T_i=L\}} e^{-(\alpha+2r)S_i}] = E[C_P^2] e^{-(\alpha+2r)L} P(T_i = L) E[e^{-(\alpha+2r)S_{i-1}}] = E[C_P^2] e^{-(\alpha+2r)L} P(T_1 = L)(E[e^{-(\alpha+2r)T_1}])^{i-1}.$$

Clearly,

$$E[C_P C_F 1_{\{T_i=L\}} 1_{\{T_i<L\}} e^{-(\alpha+2r)S_i}] = 0$$

since $1_{\{T_i=L\}=1}$ and $1_{\{T_i<L\}} =$ cannot occur at the same poin. The last term,

$$\begin{aligned} E[C_F^2 1_{\{T_i<L\}}) e^{-(\alpha+2r)S_i}] &= E[C_F^2] E[1_{\{T_i<L\}} e^{-(\alpha+2r)T_i}] E[e^{-(\alpha+2r)S_{i-1}}] \\ &= E[C_F^2] E[1_{\{T_1<L\}} e^{-(\alpha+2r)T_1}] (E[e^{-(\alpha+2r)T_1}])^{i-1}. \end{aligned}$$

So we have

$$\begin{aligned} E[Y_i^2 e^{-\alpha S_i}] &= E[C_P^2] e^{-(\alpha+2r)L} P(T_1 = L)(E[e^{-(\alpha+2r)T_1}])^{i-1} + E[C_F^2] E[1_{\{T_1<L\}} e^{-(\alpha+2r)T_1}] (E[e^{-(\alpha+2r)T_1}])^{i-1} \\ &= \left(E[C_P^2] e^{-(\alpha+2r)L} P(T_1 = L) + E[C_F^2] E[1_{\{T_1<L\}} e^{-(\alpha+2r)T_1}] \right) B^{i-1} \end{aligned} \quad (17)$$

where $B = E[e^{-(\alpha+2r)T_1}]$.

Next, using (15) we get

$$\begin{aligned} E[Y_i Y_j e^{-\alpha S_i}] &= E[(C_P^2 1_{\{T_i=L\}} 1_{\{T_j=L\}} + C_P C_F 1_{\{T_i=L\}} 1_{\{T_j<L\}} + C_P C_F 1_{\{T_i<L\}} 1_{\{T_j=L\}} + C_F^2 1_{\{T_i<L\}} 1_{\{T_j<L\}}) e^{-rS_i} e^{-rS_j} e^{-\alpha S_j}] \\ &= E[C_P^2] E[1_{\{T_i=L\}} 1_{\{T_j=L\}} e^{-rS_i} e^{-(\alpha+r)S_j}] + E[C_P C_F] E[1_{\{T_i=L\}} 1_{\{T_j<L\}} e^{-rS_i} e^{-(\alpha+r)S_j}] \\ &\quad + E[C_P C_F] E[1_{\{T_i<L\}} 1_{\{T_j=L\}} e^{-rS_i} e^{-(\alpha+r)S_j}] + E[C_F^2] E[1_{\{T_i<L\}} 1_{\{T_j<L\}} e^{-rS_i} e^{-(\alpha+r)S_j}]. \end{aligned} \quad (18)$$

Next, we calculate the expectations in (18) as follows.

First,

$$\begin{aligned} E[1_{\{T_i=L\}} 1_{\{T_j=L\}} e^{-rS_i} e^{-(\alpha+r)S_j}] &= E\left[1_{\{T_i=L\}} e^{-rS_i} 1_{\{T_j=L\}} e^{-(\alpha+r)[S_i+T_{i+1}+\ldots+T_j]}\right] \\ &= E\left[1_{\{T_i=L\}} e^{-(\alpha+2r)S_i} 1_{\{T_j=L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\ &= E\left[1_{\{T_i=L\}} e^{-(\alpha+2r)T_i} e^{-(\alpha+2r)S_{i-1}} 1_{\{T_j=L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\ &= E[1_{\{T_i=L\}} e^{-(\alpha+2r)T_i}] E[e^{-(\alpha+2r)S_{i-1}}] E[1_{\{T_j=L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}] \\ &= e^{-(\alpha+2r)L} P(T_i = L)(E[e^{-(\alpha+2r)T_1}])^{i-1} e^{-(\alpha+r)L} P(T_j = L)(E[e^{-(\alpha+r)T_1}])^{j-i-1} \\ &= e^{-(2\alpha+3r)L} P(T_1 = L)^2 (E[e^{-(\alpha+2r)T_1}])^{i-1} (E[e^{-(\alpha+r)T_1}])^{j-i-1} \\ &= A_1 B^{i-1} C^{j-i-1} \end{aligned}$$

where $A_1 = e^{-(2\alpha+3r)L} P(T_1 = L)^2$, $B = E[e^{-(\alpha+2r)T_1}]$, and $C = E[e^{-(\alpha+r)T_1}]$.

Second,

$$\begin{aligned} E\left[1_{\{T_i=L\}} 1_{\{T_j<L\}} e^{-rS_i} e^{-(\alpha+r)S_j}\right] &= E\left[1_{\{T_i=L\}} e^{-rS_i} 1_{\{T_j<L\}} e^{-(\alpha+r)[S_i+T_{i+1}+\ldots+T_j]}\right] \\ &= E\left[1_{\{T_i=L\}} e^{-(\alpha+2r)S_i} 1_{\{T_j<L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\ &= E\left[1_{\{T_i=L\}} e^{-(\alpha+2r)T_i} e^{-(\alpha+2r)S_{i-1}} 1_{\{T_j<L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\ &= E[1_{\{T_i=L\}} e^{-(\alpha+2r)T_i}] E[e^{-(\alpha+2r)S_{i-1}}] E[1_{\{T_j<L\}} e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}] \\ &= e^{-(\alpha+2r)L} P(T_i = L)(E[e^{-(\alpha+2r)T_1}])^{i-1} (E[e^{-(\alpha+r)T_1}])^{j-i-1} E[1_{\{T_j<L\}} e^{-(\alpha+r)T_j}] \\ &= e^{-(\alpha+2r)L} P(T_1 = L) E[1_{\{T_1<L\}} e^{-(\alpha+r)T_1}] (E[e^{-(\alpha+2r)T_1}])^{i-1} (E[e^{-(\alpha+r)T_1}])^{j-i-1} \\ &= A_2 B^{i-1} C^{j-i-1} \end{aligned}$$

where $A_2 = e^{-(\alpha+2r)L} P(T_1 = L) E[1_{\{T_1<L\}} e^{-(\alpha+r)T_1}]$.

Third,

$$\begin{aligned}
E[1_{\{T_i<L\}}1_{\{T_j=L\}}e^{-rS_i}e^{-(\alpha+r)S_j}] &= E\left[1_{\{T_i<L\}}e^{-rS_i}1_{\{T_j=L\}}e^{-(\alpha+r)[S_i+T_{i+1}+\ldots+T_j]}\right] \\
&= E\left[1_{\{T_i<L\}}e^{-(\alpha+2r)S_i}1_{\{T_j=L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\
&= E\left[1_{\{T_i<L\}}e^{-(\alpha+2r)T_i}e^{-(\alpha+2r)S_{i-1}}1_{\{T_j=L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\
&= E[1_{\{T_i<L\}}e^{-(\alpha+2r)T_i}]E[e^{-(\alpha+2r)S_{i-1}}]E[1_{\{T_j=L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}] \\
&= E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}](E[e^{-(\alpha+2r)T_1}])^{i-1}e^{-(\alpha+r)L}P(T_1=L)(E[e^{-(\alpha+r)T_1}])^{j-i-1} \\
&= e^{-(\alpha+r)L}P(T_1=L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}](E[e^{-(\alpha+2r)T_1}])^{i-1}(E[e^{-(\alpha+r)T_1}])^{j-i-1} \\
&= A_3B^{i-1}C^{j-i-1}
\end{aligned}$$

where $A_3 = e^{-(\alpha+r)L}P(T_1=L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$.

Fourth,

$$\begin{aligned}
E[1_{\{T_i<L\}}1_{\{T_j<L\}}e^{-rS_i}e^{-(\alpha+r)S_j}] &= E\left[1_{\{T_i<L\}}e^{-rS_i}1_{\{T_j<L\}}e^{-(\alpha+r)[S_i+T_{i+1}+\ldots+T_j]}\right] \\
&= E\left[1_{\{T_i<L\}}e^{-(\alpha+2r)S_i}1_{\{T_j<L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\
&= E\left[1_{\{T_i<L\}}e^{-(\alpha+2r)T_i}e^{-(\alpha+2r)S_{i-1}}1_{\{T_j<L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}\right] \\
&= E[1_{\{T_i<L\}}e^{-(\alpha+2r)T_i}]E[e^{-(\alpha+2r)S_{i-1}}]E[1_{\{T_j<L\}}e^{-(\alpha+r)[T_{i+1}+\ldots+T_j]}] \\
&= E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}](E[e^{-(\alpha+2r)T_1}])^{i-1}E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}](E[e^{-(\alpha+r)T_1}])^{j-i-1} \\
&= E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]E[1_{\{T_1<L\}}e^{-(\alpha+r)T_1}](E[e^{-(\alpha+2r)T_1}])^{i-1}(E[e^{-(\alpha+r)T_1}])^{j-i-1} \\
&= A_4B^{i-1}C^{j-i-1}
\end{aligned}$$

where $A_4 = E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]E[1_{\{T_1<L\}}e^{-(\alpha+r)T_1}]$. So we have

$$E[Y_iY_je^{-\alpha S_j}] = (E[C_P^2]A_1 + E[C_PC_F]A_2 + E[C_PC_F]A_3 + E[C_F^2]A_4)B^{i-1}C^{j-i-1} \tag{19}$$

Substituting (17) and (19) into (16) we get

$$\begin{aligned}
\int_0^\infty E[K(t,r)^2]e^{-\alpha t}dt &= \frac{1}{\alpha}\sum_{i=1}^{\infty}E\left[Y_i^2e^{-\alpha S_i}\right] + \frac{2}{\alpha}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}E\left[Y_iY_je^{-\alpha S_j}\right] \\
&= \frac{1}{\alpha}\sum_{i=1}^{\infty}\left(E[C_P^2]e^{-(\alpha+2r)L}P(T_1=L) + E[C_F^2]E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]\right)B^{i-1} \\
&\quad + \frac{1}{\alpha}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(E[C_P^2]A_1 + E[C_PC_F]A_2 + E[C_PC_F]A_3 + E[C_F^2]A_4)B^{i-1}C^{j-i-1} \\
&= \frac{E[C_P^2]e^{-(\alpha+2r)L}P(T_1=L) + E[C_F^2]E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]}{\alpha(1-B)} \\
&\quad + \frac{2(E[C_P^2]A_1 + E[C_PC_F]A_2 + E[C_PC_F]A_3 + E[C_F^2]A_4)}{\alpha(1-B)(1-C)} \\
&= \frac{E[C_P^2]A_{01} + E[C_F^2]A_{02}}{\alpha(1-B)} + \frac{2(E[C_P^2]A_1 + E[C_PC_F]A_2 + E[C_PC_F]A_3 + E[C_F^2]A_4)}{\alpha(1-B)(1-C)}
\end{aligned} \tag{20}$$

where $A_{01} = e^{-(\alpha+2r)L}P(T_1=L)$ and $A_{02} = E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$.

As a corollary, in case of without discounting, the Laplace transform of $E[K(t)^2]$ can be obtained by simply replacing r with 0 in (20). The mean $E[K(t,r)]$ and the second moment $E[K(t,r)^2]$ can be obtained by inverting their Laplace transforms as stated in Theorem 1 and 2, and then we can calculate $Var[K(t,r)]$ using the well known formula.

6 A numerical example: optimal maintenance strategy

Suppose that performance of a system deteriorates in accordance with a stochastic gamma process $(X(t), t \geq 0)$ with the mean function $E[X(t)] = \mu t = 6t$ and the variance $Var[X(t)] = \sigma^2 t = 2t$. Hence, $\mu = 6$ and $\sigma^2 = 2$. Suppose also that the failure of the system occurs when the deterioration level exceeds a specified level $y = 15$. We assume that the cost for preventive maintenance C_P and the cost for corrective maintenance C_F are independent random variables with $E[C_P] = 1$ currency unit and $E[C_F] = 3$ currency unit. Setting the discount factor $r = 0.1$ we will calculate the mean and variance of $K(t,r)$ in the finite time interval [0,50].

Recall from (7) that for $\alpha > 0$,

$$\int_0^\infty E[K(t,r)]e^{-\alpha t}dt = \frac{E[C_P]e^{-(\alpha+r)L}P(T_1 = L) + E[C_F]E[e^{-(\alpha+r)T_1}1_{\{T_1<L\}}]}{\alpha(1 - E[e^{-(\alpha+r)T_1}])}. \tag{21}$$

The Laplace transform of $E[K(t,r)$ can be calculated as follows. Firstly, note that $T_1 = L$ if and only if $X(L) \leq y$. It follows that

$$P(T_1 = L) = P(X(L) \leq y) = \int_0^y \frac{b^{a_1}x^{a_1-1}e^{-bx}}{\Gamma(a_1)}dx = \frac{1}{\Gamma(a_1)}\int_0^{by} z^{a_1-1}e^{-z}dz = G(b_1, a_1) \tag{22}$$

where $a_1 = \mu^2 L/\sigma^2 = 18L$, $b_1 = by = \mu y/\sigma^2 = 3y$, and

$$G(x,a) = \frac{1}{\Gamma(a)}\int_0^x z^{a-1}e^{-z}dz$$

is the regularized lower incomplete gamma function.

Next,

$$E[e^{-(\alpha+r)T_1}1_{\{T_1<L\}}] = \int_0^{L^-} e^{-(\alpha+r)z}dF(z)$$

where $F(z) = P(T_1 \leq z)$. We need a numerical integration method to calculate this integral. Using the trapezoidal rule we get

$$\int_0^{L^-} e^{-(\alpha+r)z}dF(z) \approx \sum_{j=1}^n e^{-(\alpha+r)(j-0.5)L/n}[F(jL/n) - F((j-1)L/n)].$$

Since $T_1 > s$ if and only if $X(s) < y$, it follows that

$$\begin{aligned} F(jL/n) &= P(T_1 \leq jL/n) = 1 - P(T_1 > jL/n) = 1 - P(X(jL/n) \leq y) \\ &= 1 - \int_0^y \frac{b^{a_2}x^{a_2-1}e^{-bx}}{\Gamma(a_2)}dx = 1 - \frac{1}{\Gamma(a_2)}\int_0^{by} z^{a_2-1}e^{-z}dz = 1 - G(b_1, a_2) \end{aligned}$$

where $a_2 = \mu^2 jL/(n\sigma^2) = 18jL/n$. Similarly

$$F((j-1)L/n) = 1 - G(b_1, a_3)$$

where $a_3 = \mu^2(j-1)L/(n\sigma^2) = 18(j-1)L/n$. So we get

$$E[e^{-(\alpha+r)T_1}1_{\{T_1<L\}}] \approx \sum_{j=1}^n e^{-(\alpha+r)(j-0.5)L/n}[G(b_1, a_3) - G(b_1, a_2)]. \tag{23}$$

The expected value $E[e^{-(\alpha+r)T_1}]$ in the denominator of (21) can be calculated as follows.

$$E[e^{-(\alpha+r)T_1}] = E[e^{-(\alpha+r)T_1}1_{\{T_1<L\}}] + E[e^{-(\alpha+r)T_1}1_{\{T_1=L\}}] + E[e^{-(\alpha+r)T_1}1_{\{T_1>L\}}]. \tag{24}$$

The first term can be approximated by (23). The second term is

$$E[e^{-(\alpha+r)T_1}1_{\{T_1=L\}}] = e^{-(\alpha+r)L}P(T_1 = L). \tag{25}$$

Clearly the last term is equal to zero. So we get

$$E[e^{-(\alpha+r)T_1}] \approx \sum_{j=1}^n e^{-(\alpha+r)(j-0.5)L/n}[G(b_1, a_3) - G(b_1, a_2)] + e^{-(\alpha+r)L}G(b_1, a_1). \tag{26}$$

Substituting (22), (23), and (26) into (21) we get

$$\int_0^\infty E[K(t,r)]e^{-\alpha t}dt \approx \frac{E[C_P]e^{-(\alpha+r)L}G(b_1,a_1)+E[C_F]\sum_{j=1}^{n} e^{-(\alpha+r)(j-0.5)L/n}[G(b_1,a_3)-G(b_1,a_2)]}{\alpha\left(1-\sum_{j=1}^{n} e^{-(\alpha+r)(j-0.5)L/n}[G(b_1,a_3)-G(b_1,a_2)]+e^{-(\alpha+r)L}G(b_1,a_1)\right)}. \tag{27}$$

Next we calculate the Laplace transform of $E[K(t,r)^2]$. We have from (12) that

$$\int_0^\infty E[K(t,r)^2]e^{-\alpha t}dt = \frac{E[C_P^2]A_{01}+E[C_F^2]A_{02}}{\alpha(1-B)} + \frac{2(E[C_P^2]A_1+E[C_PC_F]A_2+E[C_PC_F]A_3+E[C_F^2]A_4)}{\alpha(1-B)(1-C)}$$

where

$A_{01} = e^{-(\alpha+2r)L}P(T_1 = L)$, $\quad A_{02} = E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$,
$A_1 = e^{-(2\alpha+3r)L}P(T_1 = L)$, $\quad A_2 = e^{-(\alpha+2r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+r)T_1}]$,
$A_3 = e^{-(\alpha+r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$, $\quad A_4 = e^{-(\alpha+r)L}P(T_1 = L)E[1_{\{T_1<L\}}e^{-(\alpha+2r)T_1}]$,
$B = E[e^{-(\alpha+2r)T_1}]$, and $C = E[e^{-(\alpha+r)T_1}]$.

We need to calculate $P(T_1 = L)$ and expectations of the form $E[e^{-\beta T_1}]$ and $E[1_{\{T_1<L\}}e^{-\beta T_1}]$ where $\beta = \alpha + r$ or $\beta = \alpha + 2r$. But we can use the results in formulae (22), (23), and (26).

To calculate $E[K(t,r)]$ and $E[K(t,r)^2]$ we use the numerical inversion algorithm in [1] to invert their Laplace transforms. As an ilustration we calculate $E[K(50,0.1)]$ for several values of L, and the result is graphed in Figure 1.

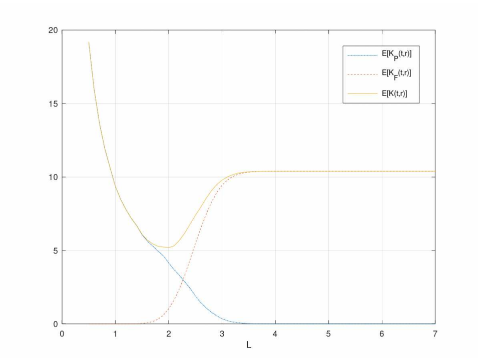

Fig. 1: The graph of $E[K(50,0.1)]$

From Figure 1 we see that the optimal period L for preventive maintenance which minimize the expected value $E[K(50,0.1)]$ over the bounded time interval [0,50] is approximately 2. Setting $L = 2$ and the other parameters remain the same we get $E[K(50,0.1)] = 5.1901$, $E[K(50,0.1)^2] = 27.618$. So we get the variance $Var[K(50,0.1)] = E[K(50,0.1)^2] - (E[K(50,0.1)])^2 = 27.618–5.1901^2 = 0.681$, and hence the standard deviation of $K(50,0.1)$ is equal to 0.825.

7 Conclusion

In this paper we derived the formulae for the Laplace transforms of the mean and second moment of the discounted total maintenance cost of a system in the time interval $[0,t]$ defined in (6) where the system degradation is modelled by a gamma stochastic process. Using these results we may obtain an optimal period for the preventive maintenance period which minimizes the mean of the discounted total maintenance cost in the time interval $[0,t]$, and also calculate the variance (or standard deviation) of the discounted total maintenance cost. An example is given to illustrate how to find the optimal period of the preventive maintenance. For further research one may consider the probability distribution of the discounted total maintenance cost and its limiting properties.

Declarations

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