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# Further Results on Odd Harmonious Labeling of Graphs

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**Abstract:** Liang and Bai proposed the notion of odd harmonious labeling of a graph in 2009. Since then, numerous papers have explored this topic. This study adds some new results to the existing literature. First, we provide a sufficient condition for an Eulerian graph to be an odd harmonious graph, which enhances the result from 2014. Furthermore, we identify several new classes of graphs that exhibit odd harmonious properties.

**Keywords:** disjoint union of graphs; odd harmonious labeling; path union of graphs; splitting graph; subdivided shell flower graph; vertex union of graphs.

**2010 Mathematics Subject Classification**. 26A25; 26A35.

## 1 Introduction

The concept of odd harmonious labeling of a graph was defined in 2009 by Liang and Bai [18] as follows: A graph $G$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0,1,2,....q\}$ then $f$ is called as strongly odd harmonious labeling and $G$ is called as strongly odd harmonious graph. Motivated by this concept, several researchers have studied this topic and relevant results have been published in many journals and proceedings, among others. Our published results on odd harmonious labeling of graphs [5]-[17], [19] might definitely be useful to the interested researchers.

In this paper, we add further results on this labeling. Throughout the paper, we refer to the vertices and edges of the graph $G$ by points and links respectively. We also introduce a new graph namely $D_2\&Spl(G)$. To construct the graph, first find shadow graph $D_2(G)$ and then for the original graph $G$, find splitting graph $Spl(G)$. The combination of $D_2(G)$ and $Spl(G)$ gives a new graph $D_2\&Spl(G)$. An example for the graph $D_2\&Spl(P_5)$ is shown in Figure 1.

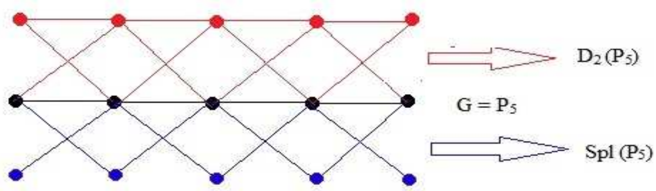


**Fig. 1:** $D_2\&Spl(P_5)$

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For the terminology, we refer the reader to [4]. A shell graph is a cycle $C_n$ with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n,n-3)$. A subdivided shell graph is a shell graph in which the edges in the path of the shell are subdivided. A subdivided shell flower graph is one vertex union of three subdivided shells and each subdivided shell in this graph called as a petal.
Let the points of $j^{th}$ copy of cycle $C_n$ be $v_1^j, v_2^j, \cdots, v_n^j$, $1 \le j \le t$. The graph $C_n^t$ is obtained by identifying the points $v_n^j$, $1 \le j \le t$ of each copy and denote it as the apex point $v_0$.
The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $v'$ in $G''$.
For a graph $G$ the split graph is obtained by adding to each vertex $v$ a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$. The resultant graph is denoted by $Spl(G)$. The $m$-splitting graph $Spl_m(G)$ of a graph $G$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v_1, v_2, \cdots, v_m$ such that $v_i$, $1 \le i \le m$ is adjacent to every vertex that is adjacent to $v$ in $G$.

## 2 Main Results

In the following theorem, we present a sufficient condition that significantly enhances the result previously established in [1].

**Theorem 1.** *If G is an Eulerian odd harmonious graph with q links, then* $q \equiv 0(mod\ 4)$.

*Proof.* Suppose that $G$ is an Eulerian odd harmonious graph with an odd harmonious labeling $f$. Then $G$ is a bipartite graph as $G$ is an odd harmonious graph. Since $2 \big| \sum_{v \in V(G)} deg(v)f(v)$ and $\sum_{e \in E(G)} f^*(e) = q^2 = \sum_{v \in V(G)} deg(v)f(v)$, $q$ is even. Moreover, since $G$ is an Eulerian graph, $G$ can be expressed as the union of edge-disjoint cycles, each is of even order as $G$ is a bipartite graph. Suppose that $G$ is the edge-disjoint union of cycles $C_1, C_2, \cdots, C_k, C_{k+1}, \cdots, C_{k+t}$, where the order of the first $k$ cycles is congruent to 0 modulo 4 and the order of the other $t$ cycles is congruent to 2 modulo 4. For $1 \le i \le k$ and $k+1 \le j \le k+t$, let $a_i = \sum_{e \in C_i} f^*(e)$ and $b_j = \sum_{e \in C_j} f^*(e)$ respectively. Now we have, $q^2 = \sum_{i=1}^{k} \sum_{e \in C_i} f^*(e) + \sum_{j=k+1}^{k+t} \sum_{e \in C_j} f^*(e) = \sum_{i=1}^{k} a_i + \sum_{j=k+1}^{k+t} b_j$ where $a_i \equiv 0(mod\ 4)$ for $1 \le i \le k$ and $b_j \equiv 2(mod\ 4)$ for $k+1 \le j \le k+t$. If $t$ is odd, then $\sum_{j=k+1}^{k+t} b_j \equiv 2(mod\ 4)$ and in this case, $q^2 \equiv 2(mod\ 4)$ which is a contradiction, hence $t$ is even and $q \equiv 0(mod\ 4)$.

**Theorem 2.** *The disjoint union of p copies of complete bipartite graphs,* $\bigcup_{i=1}^{p} K_{m_i,n_i}$, *is an odd harmonious graph for every positive integer p, and where* $m_i, n_i \ge 2$ *and* $1 \le i \le p$.

*Proof.* Let $G$ be a disjoint union of $p$ number of graphs $K_{m_1,n_1}, K_{m_2,n_2}, \cdots, K_{m_p,n_p}$.
$G$ has $m_1+n_1+m_2+n_2+\cdots+m_p+n_p$ points and $m_1n_1+m_2n_2+\cdots+m_pn_p$ links. Let $v_1^j, v_2^j, \cdots, v_{m_j}^j$ and $u_1^j, u_2^j, \cdots, u_{n_j}^j$ be the set of points of each $K_{m_j,n_j}$, where $1 \le j \le p$. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,2(m_1n_1+m_2n_2+\cdots+m_pn_p)-1\}$ as follows: $f(v_i^1) = -1+2i,\ 1 \le i \le m_1$; $f(u_i^1) = 2m_1(i-1),\ 1 \le i \le n_1$; $f(v_i^j) = 2m_1-1+4(j-1)+2[(m_2-1)+(m_3-1)+\cdots+(m_{j-1}-1)]+2(i-1)$, $1 \le i \le m_j$, $2 \le j \le p$; $f(u_i^j) = 2[m_1(n_1-1)+m_2(n_2-1)+\cdots+m_{j-1}(n_{j-1}-1)]-2(j-1)+2m_j(i-1)$, $2 \le j \le p$, $1 \le i \le n_j$. The induced link labels are as follows: $f^*(u_i^1v_k^1) = 2m_1(i-1)+2k-1$, $1 \le i \le n_1$, $1 \le k \le m_1$; $f^*(u_i^jv_k^j) = 2[m_1(n_1-1)+m_2(n_2-1)+\cdots+m_{j-1}(n_{j-1}-1)]-2(j-1)+2m_j(i-1)+2m_1-1+4(j-1)+2[(m_2-1)+(m_3-1)+\cdots+(m_{j-1}-1)]+2k-2$, $1 \le k \le m_j$, $1 \le i \le n_j$ and $2 \le j \le p$. Hence, the graph $\bigcup_{i=1}^{p} K_{m_i,n_i}$ is an odd harmonious graph.

**Corollary 1.** *The graph* $pK_{r,s}$ , $p, r \ge 2$ *and* $s \succ 2$ *is an odd harmonious graph.*

*Proof.* Let $G$ be a disjoint union of $p$ number of graphs $K_{r,s}$. This $G$ has $p(r+s)$ points and $prs$ links. Let $v_1^j, v_2^j, \cdots, v_r^j$ and $u_1^j, u_2^j, \cdots, u_s^j$ be the set of points of each $K_{r,s}$, where $1 \le j \le p$. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,2prs-1\}$ as follows: $f(v_i^j) = 2i-1+2(r+1)(j-1)$, $1 \le i \le r$, $1 \le j \le p$; $f(u_i^j) = 2r(i-1)+[2r(s-1)-2](j-1)$, $1 \le i \le s$, $1 \le j \le p$. The induced link labels are as follows:
$f^*(v_i^ju_k^j) = 2i-1+2(r+1)(j-1)+2r(k-1)+[2r(s-1)-2](j-1), 1 \le i \le r$, $1 \le k \le s$ and $1 \le j \le p$.
Hence, the graph $pK_{r,s}$ is an odd harmonious graph.

*Example 1.* An odd harmonious labeling of disjoint union of $K_{2,3}$, $K_{3,3}$, $K_{3,4}$ and $K_{2,4}$ is given in Figure 2.

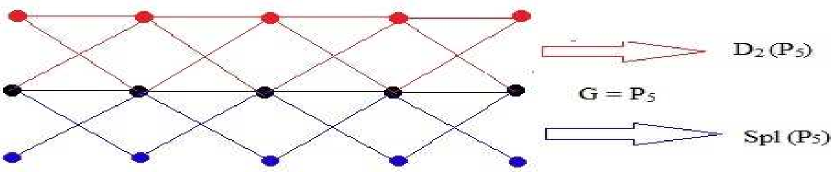


**Fig. 2:** An odd harmonious labeling of disjoint union of $K_{2,3}$, $K_{3,3}$, $K_{3,4}$ and $K_{2,4}$

*Example 2.* An odd harmonious labeling of $4K_{2,3}$ is given in Figure 3.

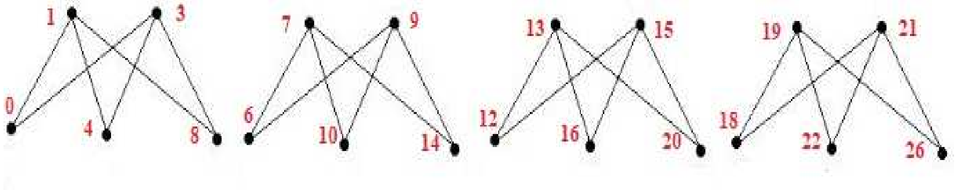


**Fig. 3:** An odd harmonious labeling of $4K_{2,3}$

**Theorem 3.** *The subdivided shell flower graph $SF(n,k)$ is an odd harmonious labeling for all $n \geq 2$ and $k \geq 1$.*

*Proof.* $SF(n,k)$ has $(2n-1)k+1$ points and $q = (3n-2)k$ links. Let the apex is denoted by $v_0$. $V\ (S(F(n,k)) = \{u_1^t, u_2^t, \cdots, u_{2n-1}^t : 1 \leq t \leq k\}$ and $E\ (S(F(n,k))) = \{v_0 v_{2r-1}^t : 1 \leq t \leq k, 1 \leq r \leq 2n-2\}$. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,2(3n-2)k-1\}$ as follows: $f(v_0) = 0$ and for $1 \leq t \leq k$, $f(v_{2r-1}^t) = 2n(t-1)+2r-1$; $1 \leq r \leq n$ $f(v_{2r}^t) = 2(q+1)-(6(n-1)+2)(t-1)-6r$; $1 \leq r \leq n-1$. The induced link labels are as follows: For $1 \leq t \leq k$, $\{f^*(v_0 v_{2r-1}^t) : 1 \leq r \leq n\} = \{2n(t-1)+2r-1 : 1 \leq r \leq n\} = \{1,3,\cdots,2n-1\} \cup 2n+\{1,3,\cdots,2n-1\} \cup \cdots \cup 2n(k-1)+\{1,3,\cdots,2n-1\} = \{1,3,\cdots,2kn-1\}$ and $\{f^*(v_r^t v_{r+1}^t) : 1 \leq r \leq 2n-2\} = \{f^*(v_{2r-1}^t v_{2r}^t) : 1 \leq r \leq n-1\} \cup \{f^*(v_{2r}^t v_{2r+1}^t) : 1 \leq r \leq n-1\}$ $= \{2(q+1)-(4n-8)(t-1)-4r-1 : 1 \leq r \leq n-1\} \cup \{2(q+1)-(4n-8)(t-1)-4r+1 : 1 \leq r \leq n-1\}$ $= \{2kn+1, 2kn+3, \cdots, 2q-1\}$. Evidently, the subdivided shell flower graph $SF(n,k)$ is an odd harmonious graph.

*Example 3.* An odd harmonious labeling of subdivided shell flower graph $SF(3,8)$ is given in Figure 4.

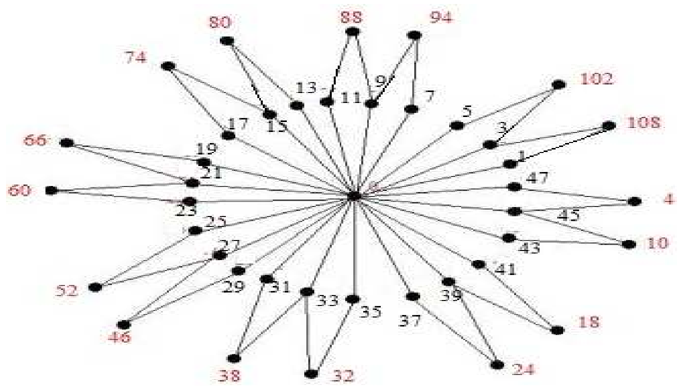


**Fig. 4:** An odd harmonious labeling of subdivided shell flower graph $SF(3,8)$

**Theorem 4.** *The disjoint union of $s$ copies of $P_r \times P_r$, $s \geq 2$ and $r \geq 3$ is an odd harmonious graph.*

*Proof.* Let $G$ be a disjoint union of $s$ copies of graph $P_r \times P_r$. Let $x^j_{i,1}, x^j_{i,2}, \cdots, x^j_{i,r}$ be the points of $i^{th}$ row of the $j^{th}$ copy of $G$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. This $G$ has $2sr(r-1)$ links and $sr^2$ points. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,4sr(r-1)-1\}$ as follows: $f(x^t_{i,j}) = h^2 + 2(j-h) - 1 + 2(r^2 - r - 1)(t-1)$; $(i,j) = (h,1),(h-1,2),\cdots,(2,h-1),(1,h)$, $h$ is odd and $1 \leq t \leq s$. $f(x^t_{i,j}) = h^2 + 2(j-h) - 1 + [4 + 2(r^2 - r - 1)](t-1)$; $(i,j) = (h,1),(h-1,2),\cdots,(2,h-1),(1,h)$, $h$ is even and $1 \leq t \leq s$.
We consider the following two cases:
Case I: If $r$ is even
$f(x^t_{i,j}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + 2(r^2-r-1)(t-1)$; $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $h$ is even and $1 \leq t \leq s$.
$f(x^t_{i,j}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + [4+2(r^2-r-1)](t-1)$; $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $h$ is odd and $1 \leq t \leq s$.
Case II: If $r$ is odd
$f(x^t_{i,j}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + [4+2(r^2-r-1)](t-1)$; $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $h$ is even and $1 \leq t \leq s$;
$f(x^t_{i,j}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + 2(r^2-r-1)(t-1)$; $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $h$ is odd and $1 \leq t \leq s$.
The induced link labels are as follows:
$f^(x^t_{i,j}x^t_{i,j+1}) = h^2 + k^2 + 4j - 2(h+k) + 4(t-1)(r^2-r)$, $(i,j) = (h,1),(h-1,2),\cdots,(2,h-1),(1,h)$, $(i,j+1) = (k,1),(k-1,2),\cdots,(2,k-1),(1,k)$;
$f^(x^t_{i,j}x^t_{i,j+1}) = 2(r-1)^2 + 2r(h+k) - 4r - h^2 - k^2 + 4j + 4(t-1)(r^2-r)$, $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $r$ is odd and $h$ is even; $(i,j+1) = (r,k),(r-1,k+1),\cdots,(k,r)$, $r$ is odd and $k$ is odd; $f^(x^t_{i,j}x^t_{i,j+1}) = h^2 + 2(j-h) - 1 + 2(r^2-r-1)(t-1) + (r-1)^2 + (2r-k+1)(k-1) + 2(j+1-h) + [4+2(r^2-r-1)](t-1)$, $(i,j) = (h,1),(h-1,2),\cdots,(2,h-1),(1,h)$, $h$ is odd, $(i,j+1) = (r,k),(r-1,k+1),\cdots,(k,r)$, $r$ is odd and $k$ is odd; $f^(x^t_{i,j}x^t_{i,j+1}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + 2(r^2-r-1)(t-1) + (r-1)^2 + (2r-k+1)(k-1) + 2(j+1-k) + [4+2(r^2-r-1)](t-1)$, $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $r$ is even and $h$ is even; $(i,j+1) = (r,k),(r-1,k+1),\cdots,(k,r)$, $r$ is even and $k$ is odd; $f^(x^t_{i,j}x^t_{i,j+1}) = (r-1)^2 + (2r-h+1)(h-1) + 2(j-h) + [4+2(r^2-r-1)](t-1) + (r-1)^2 + (2r-k+1)(k-1) + 2(j+1-k) + 2(r^2-r-1)(t-1)+$, $(i,j) = (r,h),(r-1,h+1),\cdots,(h,r)$, $r$ is even and $h$ is odd; $(i,j+1) = (r,k),(r-1,k+1),\cdots,(k,r)$, $r$ is even and $k$ is even. Hence, the disjoint union of $s$ copies of $P_r \times P_r$ is an odd harmonious graph.

*Example 4.* An odd harmonious Labeling of disjoint union of 3 copies of $P_3 \times P_3$ is given in Figure 5.

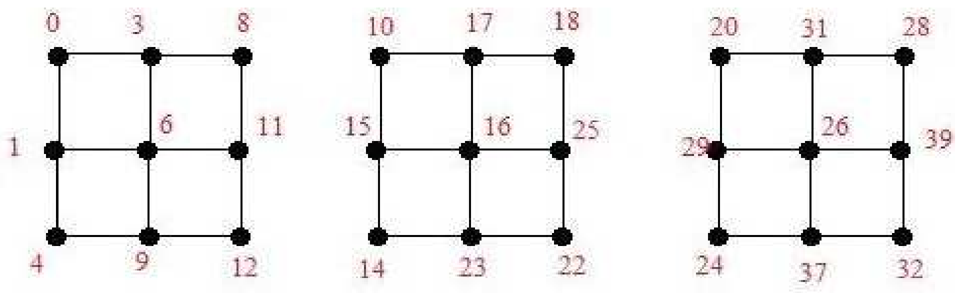


**Fig. 5:** An odd harmonious lableing of disjoint union of 3 copies of $P_3 \times P_3$

In [15], the authors demonstrated that $C_4^{(t)}$, $C_8^{(t)}$ for $t > 1$ and $C_6^{(t)}$ if $t$ is even, are odd harmonious graphs. Building on these results, in the following theorem we show that $C_{12}^{(t)}$ is also an odd harmonious graph.

**Theorem 5.** *The graph $C_{12}^{(t)}$ is an odd harmonious graph, $t \geq 1$.*

*Proof.* Let $G$ be a one point union of $t$ copies of cycle $C_{12}$. Let the points of $j^{th}$ copy of cycle $C_{12}$ be $v_1^j, v_2^j, \cdots, v_{12}^j$, $1 \leq j \leq t$. Identify the points $v_{12}^j$, $1 \leq j \leq t$ and denote the apex point as $v_0$. $G$ has $11t+1$ points and $12t$ links.
Define $f$ from $V(G)$ to $\{0,1,2,\cdots,24t-1\}$ as follows: $f(v_0) = 0$; $f(v_i^j) = 2t(i-1)+4j-3$, $i = 1,3,5$ and $1 \leq j \leq t$; $f(v_i^j) = 2(11-i)t+4j-1$, $i = 7,9,11$ and $1 \leq j \leq t$; $f(v_i^j) = 4it-8j+6$, $i = 2,4$ and $1 \leq j \leq t$; $f(v_i^j) = 4(10-i)t+8(t-j)+2$, $i = 8,10$ and $1 \leq j \leq t$; $f(v_6^j) = 4+8(t-j)$, $1 \leq j \leq t$. The induced link labels are
$f^*(v_0v_1^j) = 4j-3$, $1 \leq j \leq t$; $f^*(v_0v_{11}^j) = 4j-1$, $1 \leq j \leq t$; $f^*(v_iv_{i+1}^j) = 6(i-1)t+8t-4j+3$, $i = 1,3$ and $1 \leq j \leq t$;
$f^*(v_iv_{i+1}^j) = 6(i-2)t+12t-4j+3$, $i = 2,4$ and $1 \leq j \leq t$; $f^*(v_iv_{i+1}^j) = 16t-4j+1$, $i = 5$ and $1 \leq j \leq t$;
$f^*(v_iv_{i+1}^j) = 16t-4j+3$, $i = 6$ and $1 \leq j \leq t$; $f^*(v_iv_{i+1}^j) = 24t-6(i-7)t-4j+1$, $i = 7,9$ and $1 \leq j \leq t$;
$f^*(v_iv_{i+1}^j) = 20t-6(i-8)t-4j+1$, $i = 8,10$ and $1 \leq j \leq t$.
Hence $C_{12}^{(t)}$ is an odd harmonious graph.

*Example 5.* An odd harmonious labeling of $C_{12}^{(3)}$ is given in Figure 6.

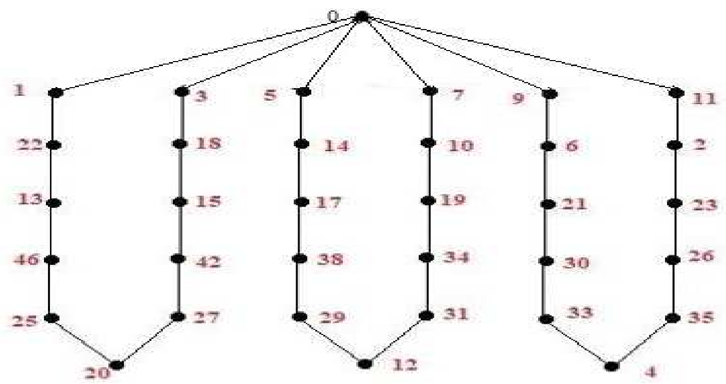


**Fig. 6:** An odd harmonious labeling of $C_{12}^{(3)}$

**Theorem 6.** *The graph constructed by joining n copies of graph $K_{p,q}$ by $n-1$ copies of path $P_m$, $p \leq q$ and m is odd, is an odd harmonious graph.*

*Proof.* Let $u_1^i, u_2^i, \cdots, u_p^i$, $v_1^j, v_2^j, \cdots, v_q^j$ be the points of $i^{th}$ copy of $K_{p,q}$ where $1 \leq i, j \leq n$. Let the points of $i^{th}$ copy of path $P_m$ be $w_1^k, w_2^k, \cdots, w_m^k$, $1 \leq i \leq n-1$, $1 \leq k \leq n-1$. Identify the points $u_p^i$ and $w_1^i$, $1 \leq i \leq n-1$ and also identify the points $w_m^i$ and $u_1^{i+1}$, $1 \leq i \leq n-1$. Then $|V| = n(p+q)+(n-1)(m-2)$ and $|E| = npq+(n-1)(m-1)$.
Define $f$ from $V(G)$ to $\{0,1,2,\cdots,2(npq+(n-1)(m-1))-1\}$ as follows: $f(v_i^1) = -2+2i$, $1 \leq i \leq q$;
$f(v_i^j) = 2(q-1)+(m-1)(j-1)+2(j-2)q+2i$ and $1 \leq i \leq q$, $1 \leq j \leq n$;
$f(u_i^j) = 1+(2j-2)q(p-1)+(m-1)(j-1)+2q(i-1)$ and $1 \leq i \leq p$, $1 \leq j \leq n$; $f(w_i^j) = 1+j2q(p-1)+(j-1)(m-1)+i-1$, $i = 1,3,5,\cdots$; $f(w_i^j) = (2q-2)+(j-1)(m-1)+(2j-2)q+i$, $i = 2,4,6,\cdots$.
The induced link labels are as follows: $f^*(u_i^1v_t^1) = 1+2q(i-1)+2t-2$, $1 \leq i \leq p$, $1 \leq j \leq q$;
$f^*(u_i^jv_t^k) = 1+(2j-2)q(p-1)+(m-1)(j-1)+2q(i-1)+(2q-2)+(k-1)(m-1)+2(k-2)q+2t$, $1 \leq i \leq p$, $1 \leq t \leq q$, $2 \leq i, j \leq n$;
$f^*(w_i^jw_{i+1}^j) = 2(2q-2)+4(m-1)(j-1)+4jq(p-1)+2(2j-2)q+4i+2$, $1 \leq j \leq n-1$, $1 \leq i \leq m-1$.
Hence, the joining $n$ copies of graph $K_{p,q}$ by $n-1$ copies of path $P_m$, $p \leq q$ and $m$ is odd, is an odd harmonious graph.

*Example 6.* An odd harmonious labeling of a graph constructed by joining 3 copies of $K_{4,6}$ by 2 copies of path $P_5$ is given in Figure 7.

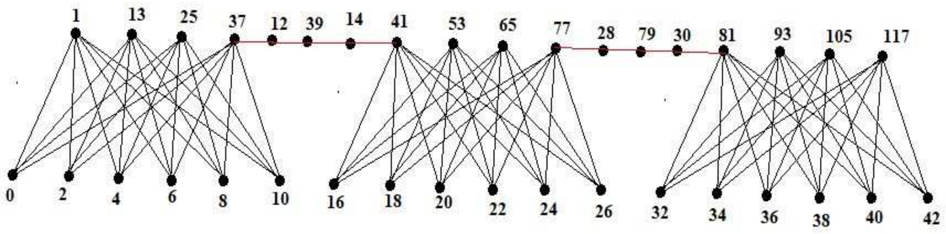


**Fig. 7:** An odd harmonious labeling of a graph joining 3 copies of $K_{4,6}$ by 2 copies of path $P_5$

In [17], the authors proved that the graph $Spl(Cb_n), n \geq 2$ is an odd harmonious graph and in continuation of this result, in the next theorem we show that $Spl_m(Cb_n), m \geq 1, n \geq 2$ is an odd harmonious graph.

**Theorem 7.** *The graph $Spl_m(Cb_n)$ for each $m \geq 1, n \geq 2$ is an odd harmonious graph.*

*Proof.* Let the points of comb be $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ in which $u_1, u_2, ..., u_n$ are the pendant points. Let $v_1^k, v_2^k, ..., v_n^k$ and $u_1^k, u_2^k, ..., u_n^k$, $1 \leq k \leq m$ be the $k^{th}$ points corresponding to $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ respectively, to obtain $Spl_m(Cb_n)$.The graph $Spl_m(Cb_n)$ is shown in Figure 8, then $|V(G)| = 2n(m+1)$ and

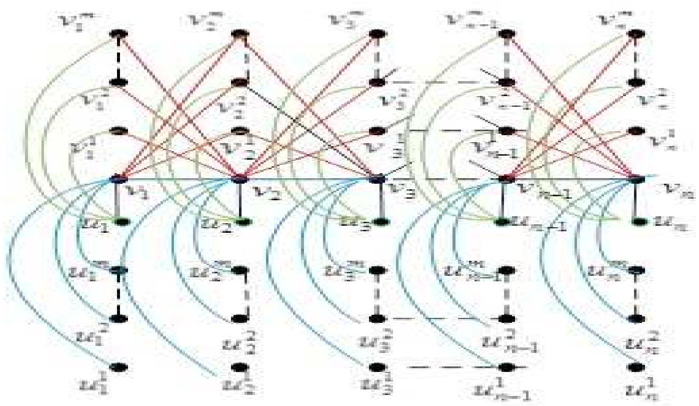

**Fig. 8:** $Spl_m(Cb_n)$

$|E(G)| = (4m+2)n - 2m - 1$. Define $f$ from $V(G)$ to $\{0, 1, 2, \cdots, 4[(2n-1)m+n] - 3\}$ as follows:

$$f(v_t) = \begin{cases} 2t-1 & \text{if } t \text{ is odd} \\ 2(t-1) & \text{if } t \text{ is even,} \end{cases} \quad f(u_t) = \begin{cases} 2(t-1) & \text{if } t \text{ is odd} \\ 2t-1 & \text{if } t \text{ is even,} \end{cases}$$

$$f(v_t^k) = \begin{cases} 4[n+t-k] + 6n(k-1) - 1 & \text{if } t \text{ is odd}, \quad 1 \leq k \leq m \\ 4[n+t-k] + 6n(k-1) - 2 & \text{if } t \text{ is even}, \quad 1 \leq k \leq m. \end{cases}$$

$$f(u_t^k) = \begin{cases} 2[(4m+2)n - (m+1)t - (m+k)] & \text{if } t \text{ is odd}, \quad 1 \leq k \leq m \\ 2[(4m+2)n - (m+1)t - (m+k)] + 1 & \text{if } t \text{ is even}, \quad 1 \leq k \leq m. \end{cases}$$

The induced link labels are as follows: $f^*(v_t v_{t+1}) = -1 + 4t, 1 \leq t \leq n-1$;
$f^*(v_t u_t) = -3 + 4t, 1 \leq t \leq n$; $f^*(v_t^k u_t) = 4(n+t-k) + 6n(k-1) + 2t - 3, 1 \leq t \leq n, 1 \leq k \leq m$;
$f^*(v_t^k v_{t+1}) = 4(n+t-k) + 6n(k-1) + 2t - 1, 1 \leq t \leq n-1, 1 \leq k \leq m$;
$f^*(v_t v_{t+1}^k) = 4(n+t-k+1) + 6n(k-1) + 2t - 3, 1 \leq t \leq n-1, 1 \leq k \leq m$;
$f^*(v_t u_t^k) = 2[(4m+2)n - m(t+1) - k] - 1, 1 \leq t \leq n, 1 \leq k \leq m$.
Hence, $Spl_m(Cb_n)$ for each $m \geq 1, n \geq 2$ is an odd harmonious graph.

*Example 7.* An odd harmonious labeling of $Spl_3(Cb_6)$ is given in Figure 9.

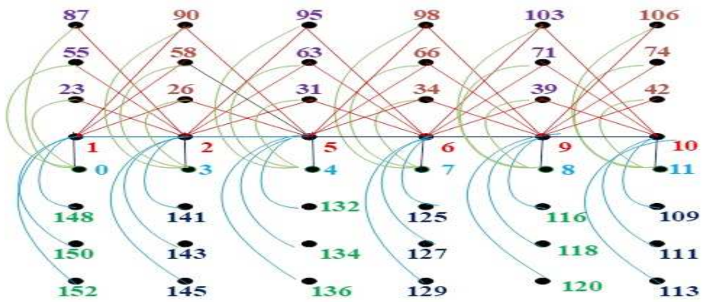


**Fig. 9:** An odd harmonious labeling of $Spl_3(Cb_6)$

In [3], the author proved that the graphs $(P_n \oplus \overline{K}_m)$, $Spl_m(P_n \oplus \overline{K}_2), n, m \geq 2$ are odd harmonious graphs and in continuation of these results, in the next two theorems we show that the point union and path union of $t$ copies of $(P_n \oplus \overline{K}_m)$ are odd harmonious graphs. Here point union and path union of $t$ copies of $(P_n \oplus \overline{K}_m)$ are denoted by $\mathscr{V}_t(P_n \oplus \overline{K}_m)$ and $\mathscr{P}_t(P_n \oplus \overline{K}_m)$ respectively.

**Theorem 8.** *The graph $\mathscr{V}_t(P_r \oplus \overline{K}_m)$ for each $t, m, r \geq 2$ is an odd harmonious graph.*

*Proof.* Let $G = \mathscr{V}_t(P_r \oplus \overline{K}_m)$ for each $t, m, r \geq 2$. Let $x^k_{i,1}, x^k_{i,2}, ..., x^k_{i,r}$ be the points of $k^{th}$ copy of the $i^{th}$ row points in $G$, where $1 \leq i \leq m$, $1 \leq k \leq t$. We join these successive copies of the graph $(P_r \oplus \overline{K}_m)$ by a point. Identify the points $x^k_{m,r}$ with $x^{k+1}_{1,1}$ for all $1 \leq k \leq t-1$ to form a point union of $t$ copies of the graph $(P_r \oplus \overline{K}_m)$. Then $|V(G)| = t(rm-1)+1$ and $|E(G)| = t(3m-2)(r-1)$. Define $f$ from $V(G)$ to $\{0, 1, 2, \cdots, 2t(3m-2)(r-1)-1\}$ as follows: We consider the following two cases:

Case I: If $r \equiv 1(mod\ 2), 1 \leq i \leq m, 1 \leq k \leq t$ :

$$f(x^k_{i,h}) = \begin{cases} (h-1)+(4m-3)(r-1)(k-1)+4(i-1)(r-1) & \text{if } h \text{ is odd,} \\ (h-1)+(2m-1)(r-1)(k-1)+2(i-1)(r-1) & \text{if } h \text{ is even.} \end{cases}$$

Case II: If $r \equiv 0(mod\ 2), \quad 1 \leq i \leq m$

We consider the following two subcases:

Subcase I: $k$ is even and $1 \leq k \leq t$

$$f(x^k_{i,h}) = \begin{cases} (h-1)+(2m-1)(r-1)+(3m-2)(r-1)(k-2)+4(i-1)(r-1), & \text{if } h \text{ is odd,} \\ (h-1)+(4m-3)(r-1)+(3m-2)(r-1)(k-2)+2(i-1)(r-1), & \text{if } h \text{ is even.} \end{cases}$$

Subcase II: $k$ is odd and $1 \leq k \leq t$

$$f(x^k_{i,h}) = \begin{cases} (h-1)+(3m-2)(r-1)(k-1)+4(i-1)(r-1), & \text{if } h \text{ is odd,} \\ (h-1)+(3m-2)(r-1)(k-1)+2(i-1)(r-1), & \text{if } h \text{ is even.} \end{cases}$$

The induced link labels are as follows:

Case I: If $r \equiv 1(mod\ 2)$ and for all $k$, $1 \leq k \leq t$

$$f^*(x^k_{i,h}x^k_{i,h+1}) = 2h-1+2(r-1)(k-1)(3m-2)+6(r-1)(i-1), 1 \leq i \leq m, 1 \leq h \leq r-1;$$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is odd}, 1 \leq h \leq r-1, 1 \leq i \leq m-1; \\ 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is even}, 1 \leq h \leq r-1, 1 \leq i \leq m-1. \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is odd}, 1 \leq h \leq r-1, 1 \leq i \leq m-1; \\ 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is even}, 1 \leq h \leq r-1, 1 \leq i \leq m-1. \end{cases}$$

Case II: If $r \equiv 0(mod\ 2)$

We consider the following subcases:

Subcase I: $k$ is even and $1 \leq k \leq t$

$$f^*(x^k_{i,h}x^k_{i,h+1}) = 2h-1+2(r-1)(3m-2)+2(r-1)(k-2)(3m-2)+6(r-1)(i-1), \quad 1 \leq i \leq m, 1 \leq h \leq r-1;$$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(r-1)(3m-2)+2(r-1)(k-2)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is odd,} \\ \quad 1 \le h \le r-1, \quad 1 \le i \le m-1; \\ 2h-1+2(r-1)(3m-2)+2(r-1)(k-2)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is even,} \\ \quad 1 \le h \le r-1, \quad 1 \le i \le m-1; \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(r-1)(3m-2)+2(r-1)(k-2)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is odd,} \\ \quad 1 \le h \le r-1, \quad 1 \le i \le m-1; \\ 2h-1+2(r-1)(3m-2)+2(r-1)(k-2)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is even,} \\ \quad 1 \le h \le r-1, \quad 1 \le i \le m-1; \end{cases}$$

Subcase II: $k$ is odd and $1 \le k \le t$

$f^*(x^k_{i,h}x^k_{i,h+1}) = 2j-1+2(r-1)(k-1)(3m-2)+6(r-1)(i-1), 1 \le i \le m, 1 \le h \le r-1;$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is odd}, 1 \le h \le r-1, 1 \le i \le m-1; \\ 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is even}, 1 \le h \le r-1, 1 \le i \le m-1. \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-1), & \text{if } h \text{ is odd}, 1 \le h \le r-1, 1 \le i \le m-1; \\ 2h-1+2(r-1)(k-1)(3m-2)+2(r-1)(3i-2), & \text{if } h \text{ is even}, 1 \le h \le r-1, 1 \le i \le m-1. \end{cases}$$

Hence, the graph $\mathscr{V}_t(P_r \oplus \overline{K}_m)$ for each $m, r \ge 2$ is an odd harmonious graph.

*Example 8.* An odd harmonious labeling of $\mathscr{V}_3(P_6 \oplus \overline{K}_4)$ and $\mathscr{V}_4(P_3 \oplus \overline{K}_3)$ are given in Figure 10 and Figure 11.

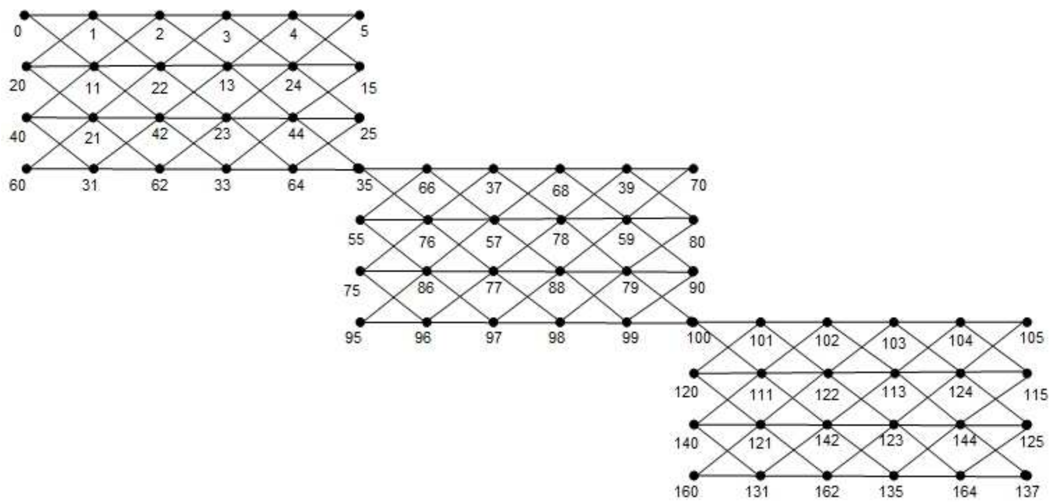


**Fig. 10:** An odd harmonious labeling of $\mathscr{V}_3(P_6 \oplus \overline{K}_4)$

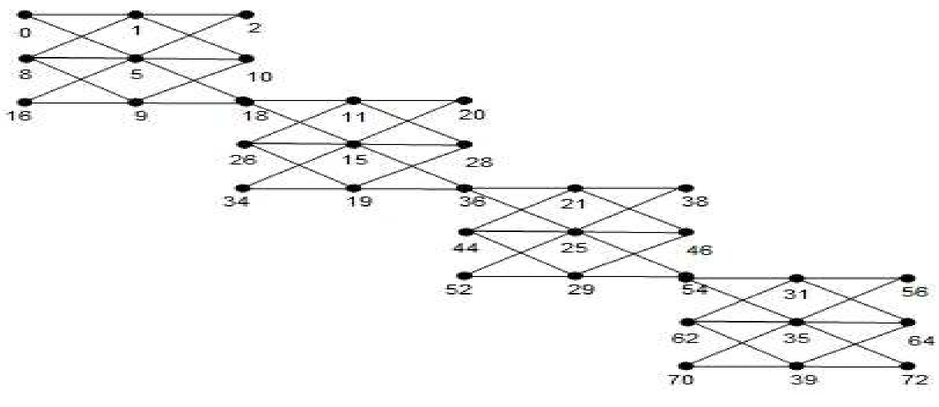


**Fig. 11:** An odd harmonious labeling of $\mathscr{V}_4(P_3 \oplus \overline{K}_3)$

**Theorem 9.** *The graph $\mathscr{P}_t(P_r \oplus \overline{K}_m)$ for each $t,m,r \geq 2$ is an odd harmonious graph.*

*Proof.* Let $G = \mathscr{P}_t(P_r \oplus \overline{K}_m)$ for each $t,m,r \geq 2$. Let the points of $k^{th}$ copy of $i^{th}$ row points in $G$ be $x^k_{i,1}, x^k_{i,2}, ..., x^k_{i,r}$ where $1 \leq i \leq m$, $1 \leq k \leq t$. We join these consecutive copies of the graph $(P_r \oplus \overline{K}_m)$ by a link. Join $x^k_{m,r}$ with $x^{k+1}_{1,1}$ for all $1 \leq k \leq t-1$ by a link to form a path union of $t$ copies of the graph $(P_r \oplus \overline{K}_m)$. This $G$ has $t[(3m-2)(r-1)+1]-1$ links and $trm$ points. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,2t[(3m-2)(r-1)+1]-1-1\}$ as follows: We consider the following two cases.
Case I: If $r \equiv 0(mod\ 2), 1 \leq i \leq m, 1 \leq k \leq t$ :

$$f(x^k_{i,h}) = \begin{cases} (h-1)+[(4m-3)(r-1)+1](k-1)+4(i-1)(r-1), \text{ if } h \text{ is odd}, 1 \leq h \leq r, \\ (h-1)+[(2m-1)(r-1)+1](k-1)+2(i-1)(r-1), \text{ if } h \text{ is even}, 1 \leq h \leq r. \end{cases}$$

Case II: If $r \equiv 1(mod\ 2)$, $\quad 1 \leq i \leq m$
We consider the following two subcases:
Subcase I: $k$ is even and $1 \leq k \leq t$

$$f(x^k_{i,h}) = \begin{cases} (h-1)+[(2m-1)(r-1)+1]+[(3m-2)(r-1)+1](k-2)+4(i-1)(r-1), & \text{if } h \text{ is odd}, 1 \leq h \leq r, \\ (h-1)+[(4m-3)(r-1)+1]+[(3m-2)(r-1)+1](k-2)+2(i-1)(r-1), & \text{if } h \text{ is even}, 1 \leq h \leq r. \end{cases}$$

Subcase II: $k$ is odd and $1 \leq k \leq t$

$$f(x^k_{i,h}) = \begin{cases} (h-1)+[(3m-2)(r-1)+1](k-1)+4(i-1)(r-1), \text{ if } h \text{ is odd}, 1 \leq h \leq r, \\ (h-1)+[(3m-2)(r-1)+1](k-1)+2(i-1)(r-1), \text{ if } h \text{ is even}, 1 \leq h \leq r. \end{cases}$$

The induced link labels are as follows:
Case I: If $r \equiv 0(mod\ 2)$ and for all $k$, $1 \leq k \leq t$

$$f^*(x^k_{i,h}x^k_{i,h+1}) = 6(r-1)(i-1)+2h-1+2(k-1)[(3m-2)(r-1)+1], 1 \leq i \leq m, 1 \leq h \leq r-1;$$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), & \text{if } h \text{ is odd}, \\ \quad 1 \leq i \leq m-1, 1 \leq h \leq r-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-1), & \text{if } h \text{ is even}, \\ \quad 1 \leq i \leq m-1, 1 \leq h \leq r-1. \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-1), & \text{if } h \text{ is odd}, \\ \quad 1 \leq i \leq m-1, 1 \leq h \leq r-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), & \text{if } h \text{ is even}, \\ \quad 1 \leq i \leq m-1, 1 \leq h \leq r-1. \end{cases}$$

Case II: If $r \equiv 1(mod\ 2)$
We consider the following two subcases:
Subcase I: $k$ is even and $1 \leq k \leq t$

$$f^*(x^k_{i,h}x^k_{i,h+1}) = 2h-1+6(r-1)(i-1)+2(k-1)[(3m-2)(r-1)+1], 1 \leq h \leq r-1, 1 \leq i \leq m;$$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), & \text{if } h \text{ is odd}, \\ \quad 1 \leq h \leq r-1, 1 \leq i \leq m-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-1), & \text{if } h \text{ is even}, \\ \quad 1 \leq h \leq r-1, 1 \leq i \leq m-1; \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]++2(r-1)(3i-1), & \text{if } h \text{ is odd}, \\ \quad 1 \leq h \leq r-1, \quad 1 \leq i \leq m-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), & \text{if } h \text{ is even}, \\ \quad 1 \leq h \leq r-1, \quad 1 \leq i \leq m-1; \end{cases}$$

Subcase II: $k$ is odd and $1 \leq k \leq t$

$$f^*(x^k_{i,h}x^k_{i,h+1}) = 2h-1+2(k-1)[(3m-2)(r-1)+1]+6(r-1)(i-1), 1 \leq i \leq m, 1 \leq h \leq r-1;$$

$$f^*(x^k_{i,h}x^k_{i+1,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), & \text{if } h \text{ is odd}, \\ \quad 1 \leq h \leq r-1, 1 \leq i \leq m-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-1), & \text{if } h \text{ is even}, \\ \quad 1 \leq h \leq r-1, 1 \leq i \leq m-1. \end{cases}$$

$$f^*(x^k_{i+1,h}x^k_{i,h+1}) = \begin{cases} 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-1), \\ \qquad \text{if } h \text{ is odd}, \quad 1 \le h \le r-1, \quad 1 \le i \le m-1; \\ 2h-1+2(k-1)[(3m-2)(r-1)+1]+2(r-1)(3i-2), \\ \qquad \text{if } h \text{ is even}, \quad 1 \le h \le r-1, \quad 1 \le i \le m-1. \end{cases}$$

$f^*(x^k_{m,n}x^{k+1}_{1,1}) = 2k[(3m-2)(r-1)+1]-1, \qquad 1 \le k \le t-1.$
Hence the graph $\mathscr{P}_t(P_r \oplus \overline{K}_m)$ for each $m,r,t \ge 2$ is an odd harmonious graph.

*Example 9.* An odd harmonious labeling of $\mathscr{P}_3(P_4 \oplus \overline{K}_3)$, $\mathscr{P}_4(P_5 \oplus \overline{K}_3)$ and $\mathscr{P}_4(P_5 \oplus \overline{K}_4)$ are given in Figure 12, Figure 13 and Figure 14 respectively.

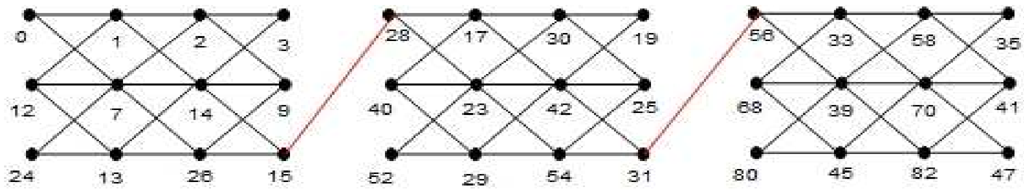


**Fig. 12:** An odd harmonious labeling of $\mathscr{P}_3(P_4 \oplus \overline{K}_3)$

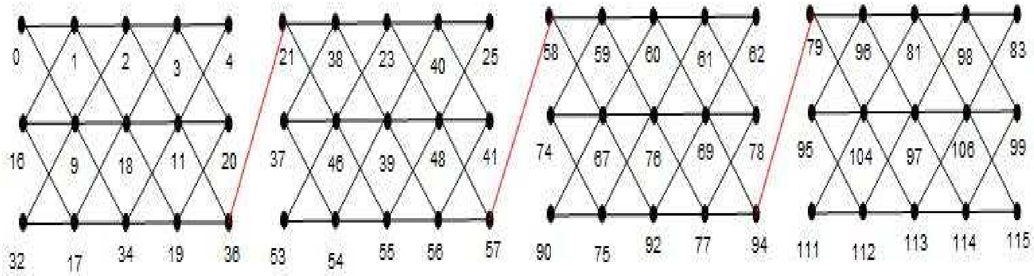


**Fig. 13:** An odd harmonious labeling of $\mathscr{P}_4(P_5 \oplus \overline{K}_3)$

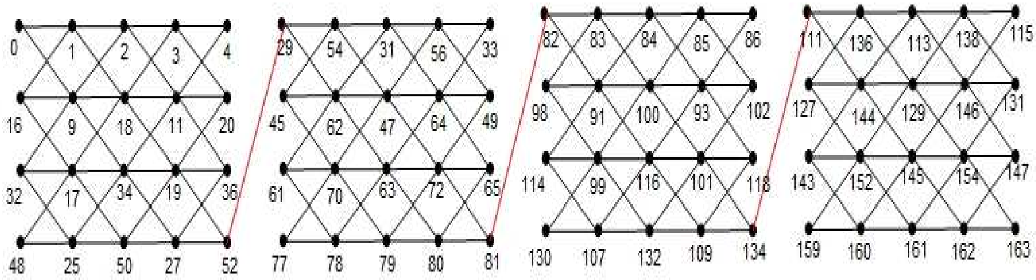


**Fig. 14:** An odd harmonious labeling of $\mathscr{P}_4(P_5 \oplus \overline{K}_4)$

**Theorem 10.** *The graph $D_2\&Spl(P_n)$, $n \ge 2$ is odd harmonious.*

*Proof.* Let $G = D_2\&Spl(P_n)$. Let $v_1, v_2, \cdots, v_n$ be the vertices of $P_n$. Let $v'_1, v'_2, \cdots, v'_n$ and $v''_1, v''_2, \cdots, v''_n$ are the vertices of $D_2(P_n)$ and $Spl(P_n)$ respectively. This graph $G$ has $3n$ vertices and $6(n-1)$ edges. Define $f$ from $V(G)$ to $\{0,1,2,\cdots,12(n-1)-1\}$ as follows: $f(v_i) = 6(i-1), i = 1,3,5,\cdots.$; $f(v_i) = 6i-9, i = 2,4,6,\cdots.$
$f(v'_i) = 6i, i = 1,3,5,\cdots.$; $f(v'_i) = 6i-7, i = 2,4,6,\cdots.$
$f(v''_i) = 6i-2, i = 1,3,5,\cdots.$; $f(v''_i) = 6i-11, i = 2,4,6,\cdots.$
The induced link labels are as follows:
$f^*(v_iv_{i+1}) = 12i-9$, $1 \le i \le n-1$; $f^*(v'_iv'_{i+1}) = 12i-1$, $1 \le i \le n-1$; $f^*(v_iv''_{i+1}) = 12i-11$, $1 \le i \le n-1$ and $i$ is odd; $f^*(v_iv''_{i+1}) = 12i-5$, $1 \le i \le n-1$ and $i$ is even; $f^*(v''_iv_{i+1}) = 12i-5$, if $i$ is odd; $f^*(v''_iv_{i+1}) = 12i-11$, if $i$ is

even; $f^*(v_i^{'}v_{i+1}) = 12i-3$, if $i$ is odd; $f^*(v_i^{'}v_{i+1}) = 12i-7$, if $i$ is even; $f^*(v_iv_{i+1}^{'}) = 12i-7$, if $i$ is odd; $f^*(v_iv_{i+1}^{'}) = 12i-3$, if $i$ is odd. Hence the graph $D_2\&Spl(P_n)$, $n \geq 2$ is odd harmonious.

*Example 10.* An odd harmonious labeling of $D_2\&Spl(P_5)$ is shown in Figure 15.

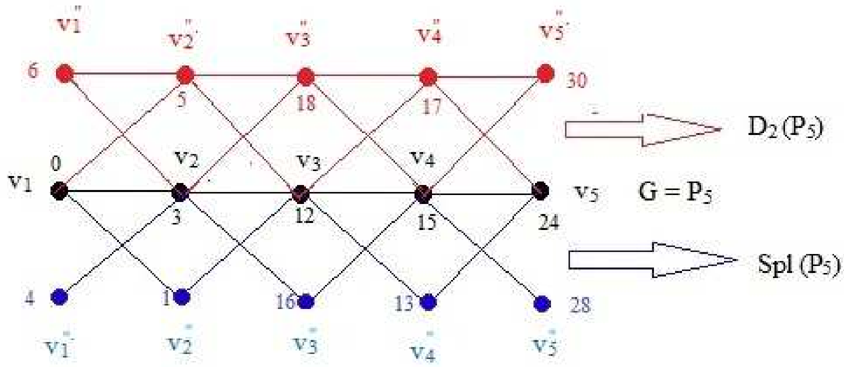


**Fig. 15:** An odd harmonious labeling of $D_2\&Spl(P_5)$

**Theorem 11.** *The graph $D_2\&Spl(C_n)$, $n \equiv 0(mod\ 4)$ is odd harmonious.*

*Proof.* Let $G = D_2\&Spl(C_n)$. Let $v_1, v_2, \cdots, v_n$ be the vertices of $C_n$. Let $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ and $v_1^{''}, v_2^{''}, \cdots, v_n^{''}$ are the vertices of $D_2(C_n)$ and $Spl(C_n)$ respectively, to obtain $D_2\&Spl(C_n)$. This graph $G$ has $3n$ vertices and $6n$ edges.
Define $f$ from $V(G)$ to $\{0, 1, 2, \cdots, 12n-1\}$ as follows: $f(v_i) = i-1$, if $1 \leq i \leq n/2$;

$$f(v_i) = \begin{cases} i+1, & \text{if } i \text{ is odd}, \quad \frac{n}{2}+1 \leq i \leq n, \\ i-1, & \text{if } i \text{ is even}, \quad \frac{n}{2}+1 \leq i \leq n. \end{cases}; \quad f(v_i^{'}) = \begin{cases} i-1+2n, & \text{if } i \text{ is odd}, \\ i-1+4n, & \text{if } i \text{ is even}, \end{cases} \quad 1 \leq i \leq \frac{n}{2}.$$

$$f(v_i^{'}) = \begin{cases} i+1+2n, & \text{if } i \text{ is odd}, \\ i-1+4n, & \text{if } i \text{ is even}, \end{cases} \quad \frac{n}{2}+1 \leq i \leq n.; \quad f(v_i^{''}) = \begin{cases} i-1+8n, & \text{if } i \text{ is odd}, \\ i-1+10n, & \text{if } i \text{ is even}, \end{cases} \quad 1 \leq i \leq \frac{n}{2}.$$

$$f(v_i^{''}) = \begin{cases} i+1+8n, & \text{if } i \text{ is odd}, \\ i-1+10n, & \text{if } i \text{ is even}, \end{cases} \quad \frac{n}{2}+1 \leq i \leq n.$$

The induced link labels are as follows: $f^*(v_iv_{i+1}) = 2i-1,\ 1 \leq i \leq \frac{n}{2}-1$;
$f^*(v_iv_{i+1}) = 2i+1,\ \frac{n}{2} \leq i \leq n-1$. $f^*(v_nv_1) = n-1$; $f^*(v_n^{'}v_1^{'}) = 7n-1$; $f^*(v_iv_{i+1}^{'}) = 2i-1+4n$, $1 \leq i \leq \frac{n}{2}-1$; $f^*(v_i^{'}v_{i+1}) = 2i+1+4n$, $\frac{n}{2} \leq i \leq n$ and $i$ is even; $f^*(v_i^{'}v_{i+1}) = 2i+1+2n$, $\frac{n}{2} \leq i \leq n$ and $i$ is odd;

$$f^*(v_iv_{i+1}^{'}) = \begin{cases} 2i-1+4n, & \text{if } i \text{ is odd}, \quad 1 \leq i \leq \frac{n}{2}-1 \\ 2i-1+2n, & \text{if } i \text{ is even}, \quad 1 \leq i \leq \frac{n}{2}-1 \end{cases}$$

$$f^*(v_iv_{i+1}^{'}) = \begin{cases} 2i+1+4n, & \text{if } i \text{ is odd}, \quad \frac{n}{2} \leq i \leq n-1 \\ 2i+1+2n, & \text{if } i \text{ is even}, \quad \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$f^*(v_n^{'}v_1) = 5n-1$; $f^*(v_nv_1^{'}) = 3n-1$; $f^*(v_n^{''}v_1) = 11n-1$; $f^*(v_nv_1^{''}) = 9n-1$;

$$f^*(v_iv_{i+1}^{''}) = \begin{cases} 2i-1+10n, & \text{if } i \text{ is odd}, \quad 1 \leq i \leq \frac{n}{2}-1 \\ 2i-1+8n, & \text{if } i \text{ is even}, \quad 1 \leq i \leq \frac{n}{2}-2 \end{cases}$$

$$f^*(v_iv_{i+1}^{''}) = \begin{cases} 2i+1+10n, & \text{if } i \text{ is odd}, \quad \frac{n}{2} \leq i \leq n-1 \\ 2i+1+8n, & \text{if } i \text{ is even}, \quad \frac{n}{2} \leq i \leq n-1 \end{cases}$$

$$f^*(v_i^{''}v_{i+1}) = \begin{cases} 2i-1+8n, & \text{if } i \text{ is odd}, \\ 2i-1+10n, & \text{if } i \text{ is even}, \end{cases} \quad 1 \leq i \leq \frac{n}{2}-1$$

$f^*(v_i^{''}v_{i+1}) = \begin{cases} 2i+1+8n, & \text{if } i \text{ is odd}, \\ 2i+1+10n, & \text{if } i \text{ is even}, \end{cases}$ $\frac{n}{2} \leq i \leq n-1$. Hence the graph $D_2\&Spl(C_n)$, $n \equiv 0(mod\ 4)$ is odd harmonious.

*Example 11.* An odd harmonious labeling of $D_2\&Spl(C_8)$ is shown in Figure 16.

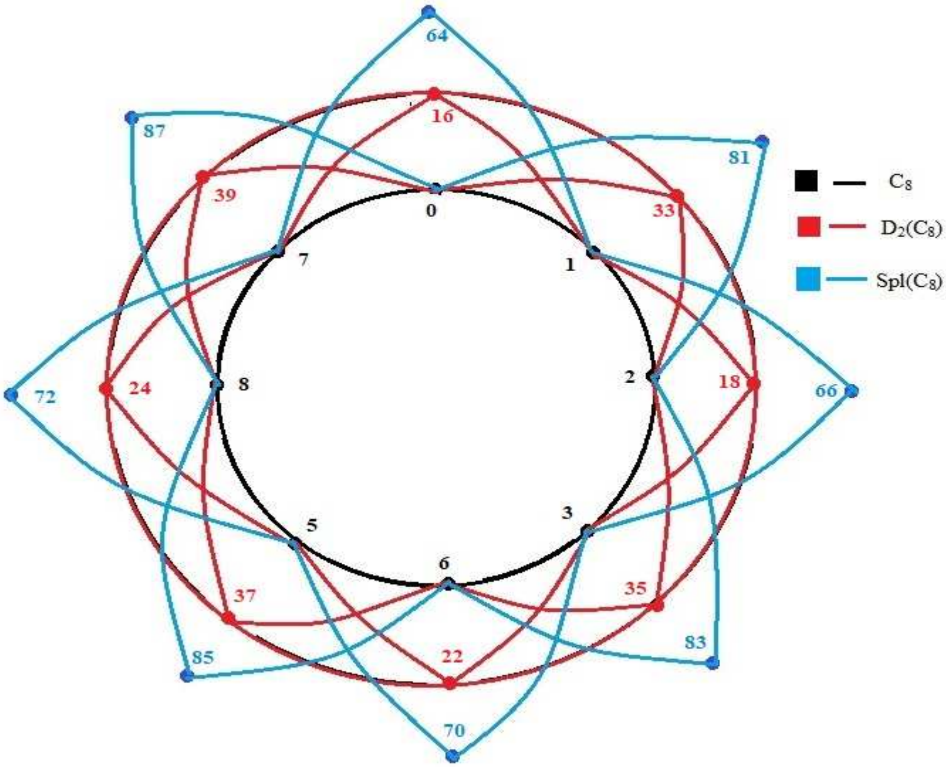


**Fig. 16:** An odd harmonious labeling of $D_2\&Spl(C_8)$

**Theorem 12.** *The graph $D_2\&Spl(K_{1,n})$, $n \geq 2$ is odd harmonious.*

*Proof.* Let $G = D_2\&Spl(K_{1,n})$. Let $v, v_1, v_2, \cdots, v_n$ be the vertices of $K_{1,n}$. Let $v', v'_1, v'_2, \cdots, v'_n$ and $v'', v''_1, v''_2, \cdots, v''_n$ are the vertices of $D_2(K_{1,n})$ and $Spl(K_{1,n})$ respectively, to obtain $D_2\&Spl(K_{1,n})$. The graph $D_2\&Spl(K_{1,n})$ is shown in Figure 17, then this graph $G$ has $3(n+1)$ vertices and $6n$ edges. Define $f$ from $V(G)$ to $\{0, 1, 2, \cdots, 12n-1\}$ as follows:

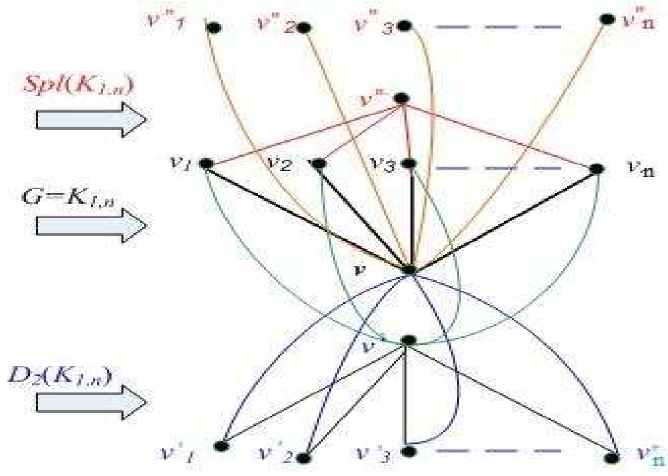


**Fig. 17:** $D_2\&Spl(K_{1,n})$

$f(v) = 0$; $f(v^{'}) = 2$; $f(v^{''}) = 8$;
$f(v_i) = 10i-9, i = 1,2,3,\cdots,n$; $f(v_i^{'}) = 10i-5, i = 1,2,3,\cdots,n$;
$f(v_i^{''}) = 10n+2i-1, i = 1,2,3,\cdots,n$;
The induced link labels are as follows:
$f^*(vv_i) = 10i-9,\ 1 \le i \le n$; $f^*(v^{'}v_i) = 10i-7,\ 1 \le i \le n$; $f^*(vv_i^{'}) = 10i-5,\ 1 \le i \le n$ ; $f^*(v^{'}v_i^{'}) = 10i-3,\ 1 \le i \le n$ ; $f^*(v^{''}v_i) = 10i-1, 1 \le i \le n$ ; $f^*(vv_i^{''}) = 10n+2i-1,\ 1 \le i \le n$. Hence the graph $D_2\&Spl(K_{1,n}), n \ge 2$ is odd harmonious.

*Example 12.* An odd harmonious labeling of $D_2\&Spl(K_{1,4})$ is shown in Figure 18.

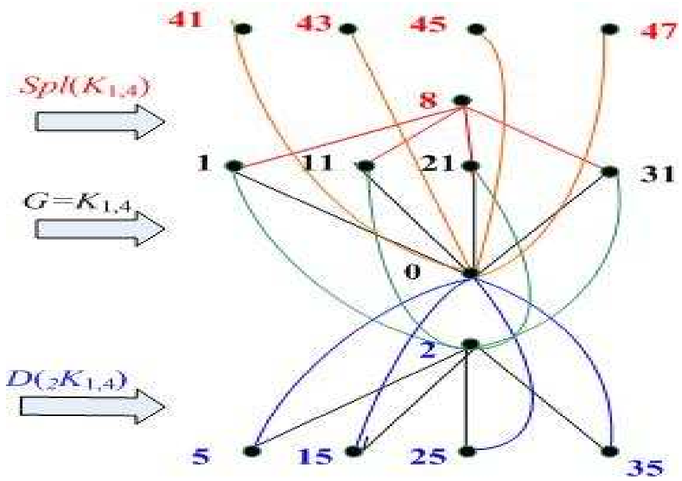


**Fig. 18:** An odd harmonious labeling of $D_2\&Spl(K_{1,4})$

## Declarations

**Competing interests**: The authors declare that they have no conflict of interest.
**Authors' contributions**: The new results are obtained by S. Philo and M. E. Abdel-Aal. The first draft of the paper is prepared by S. Philo and P. Jeyanthi and it is thoroughly checked by Maged Z Youssef. All the authors have checked, all the previous versions of the paper and approved the final version of the paper before submission.
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