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New Robust Weighted Grouping Method for Multiple Models

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Abstract: In this paper, three new estimation methods are proposed to fit a multiple structural measurement error model with two independent variables when all variables are subject to errors. The first two procedures are modifications of the Theil and Siegel estimators, where they involved the proposed Weighted Latent Variables method, while the third procedure is Iterative Weighted Grouping, an extension of Wald estimation that involved the Weighted Grouping method. A Monte Carlo experiment is performed to investigate the performance of the new estimators compared with the classical estimation methods; the Maximum Likelihood Estimator and Method of Moment, in terms of root mean square error and its bias. The outcomes of the simulation demonstrated that the suggested estimators are more effective than conventional estimators. In addition, real data analysis is discussed to examine the relationship between national gross domestic product, unemployment rate, and human development index, after applying the proposed estimation methods.

Keywords: Model of Measurement Error; Robust Estimators; Iterative Estimator; Human Development Index; Unemployment Rate; National Gross Domestic Product; Monte Carlo Simulation.

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

When modeling the relationship between two variables, one can use the structural Measurement Error Model (MEM) [26, 39] as an extension of the simple linear regression model by assuming both variables (response and predictor) are measured with independent errors. This paper discusses linear MEMs with vector-valued explanatory variables, that is, with more than one x variable. This is an extension of the simple MEM model.

Consider the equation error model as:

$$\eta_i \;=\; \alpha \;+\; \beta\;\xi_i i1 \;++\beta\;\xi_i i2\;+.....++\beta\;\xi_i ik\; i=1,2,...,n,\, j=1,2,....,n \tag{1}$$

where

$$y_i=\eta_i+\varepsilon_i \quad \text{and} \quad x_{ij}=\xi_{ij}+\delta_{ij} \quad \text{for } i=1,2,\ldots,n \text{ and } j=1,2,\ldots,k. \tag{2}$$

The measurement errors $(\delta_{ij},\varepsilon_i)$ are independent and identically distributed random vectors, and the latent variable ξ_{ij} is assumed to be independent and normally distributed in general. However, when there is skewness, outliers, or multimodality, the true distribution of the latent variable ξ_{ij} deviates from normality. As a result, selecting more flexible models can be a useful alternative to the standard one [14].

The main problem in Equation (1) is estimating the unknown parameters α and β, and comprehensive reviews of relevant techniques can be found in [12]. The classical estimation method such as the Maximum Likelihood Estimation (MLE) assumes that the measurement error is normally distributed and independent of the true value, and thus is not well-suited for estimating measurement error models. Therefore, researchers seek to find alternative estimation procedures

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to solve the problem that occurs when using the MLE [13,26,39]. In this paper, an iterative estimation and weighted grouping method are proposed to fit the multiple structural MEM.

Although there are numerous methods for correcting the effects of measurement error, they become unreliable when their underlying assumptions are violated. When assumptions about the distribution of error terms are difficult or impossible to test with the available data, the problem of inaccurate modeling is further compounded. In the literature on measurement error, it is common to assume an additive model with normally distributed errors, which is a simple and attractive assumption. However, in many practical applications, this assumption is frequently found to be incorrect [15].

Many authors have discussed several estimation methods to fit the structural MEM. The common ones are the least squares and the MLE methods. After considering some prior assumptions, the MLE method is used in [36]. Also, [9] wrote a long summary, detailing how to fit a straight-line problem by using MLE when both variables are measured with errors. Additionally, [26] presented a general review of normal theory for the structural MEM. Moreover, [5] showed that the General Maximum Entropy (GME) approach outperforms the Partial Least Squares (PLS) in terms of mean squares of errors (MSE) when investigating the distributions without relying on the classical assumptions. According to [42], adaptation to abnormal errors is a significant area of research, and the nonparametric or semiparametric methods are substantial in providing flexible ways to correct the effects of measurement error because they avoid making assumptions on the distribution of the error terms. [20,36,44] are among those who participated in this work. On the other hand, [12] proposed the use of an empirical Bayesian approach with discretionary Expectation-Maximization (EM) algorithms to compute MLE for MEMs with or without equation error. Similar results were obtained by [11] in utilizing the EM algorithm for heteroscedastic MEM to derive iterative MLE formulas. However, [35] proposed a generalized method of average grouping as another type of estimation approach. The approach suggests plotting the points of the first-third and last-third means of the whole observations to get a more accurate estimate for the slope compared to the Walds method. Meanwhile, [5,7] have recently used information theory concepts such as entropy and mutual information to assess the quality of the information provided by the observed data and the extent to which it can be used to estimate the true values of the underlying variables. The non-parametric approaches have also been proposed for modeling and correcting for measurement error in several contexts, including measurement error models (MEMs) as in [3,4,6,34,44]. More information on various estimation methods in the context of the MEM can be further found in [16,22,23,26,31,41, 42].

This paper introduces three new non-parametric estimating approaches: an iterative weighted procedure (IWP) and two modifications of Theil and Siegel. The IWP proposed is based on the multiplication of the weighted latent variables by the observation to estimate parameters, which differs from [43], in which the estimation was based on a multivariate median.

The rest of the paper is organized as follows: Section 2 reviews the two classical estimation methods; the MLE and Method of Moments (MOM). Meanwhile, the three new procedures: the iterative weighted and the modifications of Theil and Siegel are presented in Section 3. The performances of the new procedures are illustrated in Section 4 by conducting a Monte Carlo simulation, and a real data application is presented in Section 5. Section 6 concludes the article.

2 Classical Estimation Methods for Multiple MEM

This section briefly discusses the common estimation techniques used for fitting a model with structural measurement error. The techniques are the Maximum Likelihood Estimation and Method of Moment.

2.1 Maximum Likelihood Estimator

The MLE is a classical and widely used estimation method for MEMs. However, MLE may not always be the best method for estimating parameters in MEMs, particularly if the assumptions of the model are not well-established or if the measurement error distribution is not known [38,29]. Consider equations (1) and (2) that can be rewritten in matrix form as:

$$\eta = \xi' \beta; \quad Z_t = z_t + \varepsilon_t. \tag{3}$$

where

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}; \quad \varepsilon_t = \begin{pmatrix} \delta_t \\ \varepsilon_t \end{pmatrix}.$$

Then, under the multivariate normal distribution assumption (i.e., $\varepsilon_t \sim \mathrm{NI}(0, \Sigma_{\varepsilon\varepsilon})$), the variance-covariance matrix is known and given as:

$$\Sigma_{\varepsilon\varepsilon} = \Upsilon_{\varepsilon\varepsilon} \sigma^2.$$

The unknown parameters can be estimated by finding the first derivative of the logarithm of the likelihood function, which is based on a random sample of size n. This is written as

$$\log L = c - \frac{n}{2}\log|2\pi\Upsilon_{\varepsilon\varepsilon}\sigma^2| - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(Z_t - z_t)'\Upsilon_{\varepsilon\varepsilon}^{-1}(Z_t - z_t). \tag{3}$$

Solving the first-order conditions of the log-likelihood function, the unknown parameters can be estimated as:

$$\hat{\beta} = \left[M_{XX} - \left(\hat{\lambda} - \frac{1}{n}\alpha\right)S_{\delta\delta}\right]^{-1}\left[M_{XY} - \left(\hat{\lambda} - \frac{1}{n}\alpha\right)S_{\delta\varepsilon}\right].$$

where

$$M_{zz} = \frac{1}{n-1}(Z_t - \bar{Z})'(Z_t - \bar{Z}),$$

$\hat{\lambda}$ is the smallest root of $|M_{XX} - \lambda S_{\varepsilon\varepsilon}| = 0$, and $S_{\varepsilon\varepsilon}$ is an unbiased estimator of $\Sigma_{\varepsilon\varepsilon}$.

2.2 Method of Moments

The method of moments (MOM) is an approach for estimating model parameters by equating sample moments, such as mean and variance, to population moments and solving for unknowns. In measurement error models (MEM), MOM effectively accounts for measurement error and potential biases, providing an estimate of the true variable value. Its advantages include computational simplicity and broad applicability. Additionally, MOM shows greater resilience to non-classical measurement errors compared to maximum likelihood estimation (MLE), as it does not assume independence between the measurement error and the true value. The MOM was used in [33] by utilizing the sample and population moments, while [17]utilized estimation of the parameters of a straight line and of the variances of the variables if they are both subject to error. In more recent years, [24,25,27,30] have explored the use of moments to develop optimal estimators, particularly those based on higher moments. [19] has developed several estimators of slope using the MOM but has not provided information about estimators based on higher moments. Following [30] the MOM estimator of the model given in eq. (1) can be derived by computing the deviation of all variables given in the model. This is written as:

$$\eta' = \eta - \bar{\eta}; \quad \xi_i' = \xi_i - \bar{\xi}_i; \quad y' = y - \bar{y}; \quad x_i' = x_i - \bar{x}_i. \tag{4}$$

Also, if the error terms are assumed to be symmetrically distributed, then

$$E(y'x_i'^2) = \sum_{j=1}^{k}\beta_j E(x_j'x_i'^2); \quad i = 1,2,\ldots,m.$$

Therefore, from

$$A\beta = B,$$

we have

$$\beta = A^{-1}B.$$

where

$$A = \left(a_{ij}\right) \text{ with } a_{ij} = E(x_i'^2 x_j'^2),$$
$$B' = \left(E(y'x_1'^2), E(y'x_2'^2), \ldots, E(y'x_m'^2)\right),$$

and

$$\beta' = (\beta_1, \beta_2, \ldots, \beta_m).$$

Finally, $\hat{\beta} = \hat{A}^{-1}\hat{B}$ is a consistent estimator provided that $|A| \neq 0$, where $\hat{A}$ and $\hat{B}$ are the sample estimates of A and B. Therefore, unless additional information about the relationship beyond the observations is available, only MOM estimators can be used, and the variances of such estimators remain unknown.

3 The New Proposed Procedures.

This section discusses the proposes three new procedures for fitting a multiple structural measurement error model. The new procedures are modifications of the Theil and Siegel methods, which involved the Weighted Latent Variables procedures; thirdly, the proposed Iterative Weighted procedure involves the weighted grouping method as opposed to the Wald-type grouping method.

3.1 The Weighted Latent Variables Method

The Weighted Latent Variable Method, on the other hand, assumes that the measurement error is non-classical and may be correlated with the true value of the independent variable. It involves estimating a latent variable model that includes the true value of the independent variable and the observed variable as well as their error terms. The model is then weighted to account for the correlation between the error terms. The general idea of this procedure is summarized as follows:

Sort the y's in ascending order from the smallest to the largest values with their associated $(x_1[i], x_2[i])$, $i = 1, 2, \ldots, n$.

Compute the estimators, $\hat{\beta}_{ijk}$ as:

$$\hat{\beta}_{ijk} = w_k \left(\frac{y_j - y_i}{x_j - x_i} \right), \quad k = 1, 2, \ldots, n, \quad i, j = 1, 2, \ldots, n, \quad i < j, \tag{4}$$

where w_k is the weighted group.

Determine the weight w_k for two cases:

Case One: The weight is computed as:

$$w_k = \text{cov}(x_k, y). \tag{5}$$

Case Two: The weight is computed as:

$$w_k = \frac{\sigma^2_{x_k}}{\sum_{k=1}^{n} \sigma^2_{x_k}}. \tag{6}$$

Theorem 1.*Assuming that the models in eq. (1) and (2) are satisfied, then the estimator based on the Weighted Grouping Method given in eq. (5) is a biased estimator depending on w_k.*

Proof.

$$\hat{\beta}_{ijk} = w_k \left(\frac{y_j - y_i}{x_j - x_i} \right)$$

$$\hat{\alpha} = \bar{y} - \sum_{k=1}^{n} \hat{\beta}_k \bar{x}_k$$

$$E(\hat{\beta}_{ijk}) = E\left(w_k \left(\frac{y_j - y_i}{x_j - x_i} \right) \right) = w_k E(\hat{\beta}_{ijk})$$

$$\text{Then} \quad E(\hat{\beta}_{ijk}) = w_k \beta_{ijk}$$

with associated variance given as:

$$\text{Var}(\hat{\beta}_{ijk}) = \frac{1}{(w_k x_j - w_k x_i)^2} \text{Var}(w_k y_j, w_k y_i)$$

$$= \frac{w_k^2 \text{Var}(y_j) + w_k^2 \text{Var}(y_i) - 2 w_k^2 \text{Cov}(y_j, y_i)}{(w_k x_j - w_k x_i)^2}$$

Also,

$$E(\hat{\alpha}) = E\left(y_i - \sum_{k=1}^{n} \hat{\beta}_k x_{ik} \right)$$

$$= \left(\alpha + \sum_{k=1}^{n} \hat{\beta}_k x_{ik} - \sum_{k=1}^{n} \hat{\beta}_k x_{ik} \right) = \alpha$$

with

$$\text{Var}(\hat{\alpha}) = \text{Var}\left(y_i - \sum_{k=1}^{n} \hat{\beta}_k x_{ik} \right)$$

$$= \text{Var}(y_i) + x_{ik} \text{Var}(\hat{\beta}_k) - 2\text{Cov}(y_j, y_i)$$

This new proposed estimation procedure was then embedded in the Thiel and Siegel estimators (later called modified Theil and modified Siegel estimators) as follows:

3.1.1 Modified Theil Estimator

The estimator is based on the median of the slopes of all possible pairs of observations, and it is less sensitive to outliers than the ordinary least squares (OLS) estimator. Thiel [21] has proposed a free distribution method without relying on the classical assumption of the error terms in MEM by the repeated median method. In this method, the data are ordered in pairs (x_i, y_i) by the y_i's. The modified procedure can be summarized as follows:

1.Sort the y's in ascending order from the smallest to the largest values with their associated $(x_1[i], x_2[i])$, $i = 1, 2, \ldots, n$.
2.Find all pairs of observations by assuming that all $(x_1[i], x_2[i])$ are distinct, and given that:

$$\hat{\beta}_{ijk} = w_k \left(\frac{y_j - y_i}{x_j - x_i} \right), \quad i, j = 1, 2, \ldots, n \quad i < j,$$

which yields $\binom{n}{2}$ slope values.
3.Find the median of the cross medians by:

$$\hat{\beta}_k = \text{med}(\beta_{ijk}). \tag{8}$$

4.Finally, the intercept can be computed as:

$$\alpha = \text{med}\left(y_i - \sum_{k=1}^{n} \hat{\beta}_k x_{ik} \right).$$

3.1.2 Modified Siegel Estimator

As proposed in [1], the method is based on the repeated median for estimating the unknown parameters in MEM. This method can be used to estimate a real parameter β, whenever there is a positive integer k such that every subset of k data point determines an estimate $\hat{\beta}$. The modified procedure can be summarized as follows: Sort the y's in ascending order from the smallest to the largest values with their associated $(x_{1[i]}, x_{2[i]})$, $i = 1, 2, \ldots, n$.

Find all pairs of observations, assuming that all $(x_{1[i]}, x_{2[i]})$ are distinct, and given that:

$$\hat{\beta}_{ijk} = w_k \left(\frac{y_j - y_i}{x_j - x_i} \right), \quad i, j = 1, 2, \ldots, n \quad \text{with} \quad i < j;$$

which yields $\binom{n}{2}$ slope values.
Find the median of the cross medians by:

$$\hat{\beta}_{ik} = \text{med}(\beta_{ijk}).$$

Determine the estimate of the slope by taking:

$$\hat{\beta}_k = \text{med}(\hat{\beta}_{ik}). \quad (9)$$

Finally, the intercept can be computed as:

$$\alpha = \text{med}\left(y_i - \sum_{k=1}^{n} \hat{\beta}_k x_{ik} \right).$$

3.2 The Weighted Grouping Method

The third new procedure proposed in this article involved the Weighted Grouping Method (WGM) where the measurement error is assumed classical and uncorrelated with the true value of the independent variable. Therefore, this section gives a brief introduction of the method. The method involves grouping the data by the values of the independent variable and then estimating the slope of the regression line within each group. The estimates are then combined using weights based on the size of each group. The general procedure of the WGM is summarized as follows: Order the data from the smallest to the largest with their respective associated y_i's, $i = 1, 2, \ldots, n$.

Divide the data into r subgroups of equal size (i.e., the sub-sample size is k) such that $r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Compute the parameters, $\hat{\beta}_i$, as:

$$\hat{\beta}_i = \frac{w_{im}\bar{y}_{im} - w_{i(m-1)}\bar{y}_{i(m-1)}}{w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)}}, \quad m = 1,2,\ldots,r, \quad i = 1,2,\ldots,n. \quad (10)$$

where w_{im} is the weighted group.
Determine the weight w_{im} for two cases:
(i) Case One: weight is computed as:

$$w_{i(m-1)} = \text{cov}(x_{i(m-1)}, y_{(m-1)}). \quad (11)$$

$$w_{im} = \text{cov}(x_{im}, y_m). \quad (12)$$

(ii) Case Two: weight is computed as:

$$w_{im} = \frac{\sigma^2_{x_{im}}}{\sum_{i=1}^{k} \sigma^2_{x_i}}. \quad (13)$$

$$w_{i(m-1)} = 1 - w_{im}. \quad (14)$$

where:

$$\sum (w_{im} + w_{i(m-1)}) = 1.$$

Theorem 2. *Assuming that the model in eq. (1) and (2) are satisfied, then the estimator based on eq. (10) is unbiased if and only if $w_{im} = w_{i(m-1)}$ in the first case and $w_{im} = w_{i(m-1)} = 0.5$ for the second case.*

Proof.

$$\hat{\beta}_i = \frac{w_{im}\bar{y}_{im} - w_{i(m-1)}\bar{y}_{i(m-1)}}{w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)}}.$$

$$\hat{\alpha} = \bar{y} - \sum_{i=1}^{k} \hat{\beta}_i \bar{x}_i.$$

$$E(\hat{\beta}_i) = E\left(\frac{w_{im}\bar{y}_{im} - w_{i(m-1)}\bar{y}_{i(m-1)}}{w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)}}\right)$$

$$= \frac{w_{im}(\alpha + \hat{\beta}_i\bar{x}_{im}) - w_{i(m-1)}(\alpha + \hat{\beta}_i\bar{x}_{i(m-1)})}{w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)}}.$$

By using $w_{im} = w_{i(m-1)}$, then:

$$E(\hat{\beta}_i) = \beta_i.$$

The associated variance is given as:

$$\text{Var}(\hat{\beta}_i) = \frac{1}{(w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)})^2} \text{Var}(w_{im}\bar{y}_{im}, w_{i(m-1)}\bar{y}_{i(m-1)}).$$

$$= \frac{w_{im}^2 \text{Var}(\bar{y}_{im}) + w_{i(m-1)}^2 \text{Var}(\bar{y}_{i(m-1)}) - 2w_{im}w_{i(m-1)}\text{Cov}(\bar{y}_{im}, \bar{y}_{i(m-1)})}{(w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)})^2}.$$

Also,

$$E(\hat{\alpha}) = E\left(\bar{y} - \sum_{i=1}^{k} \hat{\beta}_i \bar{x}_i\right)$$

$$= \left(\alpha + \sum_{i=1}^{k} \hat{\beta}_i \bar{x}_i - \sum_{i=1}^{k} \hat{\beta}_i \bar{x}_i\right) = \alpha.$$

with

$$\mathrm{Var}(\hat{\alpha}) = \mathrm{Var}\left(\bar{y} - \sum_{i=1}^{k} \hat{\beta}_i \bar{x}_i\right)$$
$$= \mathrm{Var}(\bar{y}) + \bar{x}_i \mathrm{Var}(\hat{\beta}_i) - 2\mathrm{Cov}(\bar{y}_{im}, \bar{y}_{i(m-1)}).$$

3.2.1 The Iterative Weighted Method

In this section, the proposed Iterative Weighted Method (IWM) is presented where it involves the modification of the WGM presented earlier. The modification was suggested to account for the fact that the grouping procedure may introduce bias in the estimation of the slope. The iterative procedure proposed is an extension of Walds iterative procedure described in [32]. The general procedure of this new IWM can be summarized as follows: Sort the y's in ascending order from the smallest to the largest values with their associated $\left(x_{1[i]}, x_{2[i]}\right)$, $i = 1, 2, \ldots, n$.

Divide the data into r subgroups of equal size (i.e., the sub-sample size is k) such that $r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Compute the mean for each subgroup $\left(\bar{x}_{1j}, \bar{x}_{2j}, \bar{y}_j\right)$; $j = 1, 2, \ldots, r$.

Compute the pairwise slopes continuously and gradually from each subgroup to another subgroup as illustrated in Figure 1.

The i-th slope can be computed as:

$$\hat{\beta}_{ik} = \frac{w_{im}\bar{y}_{im} - w_{i(m-1)}\bar{y}_{i(m-1)}}{w_{im}\bar{x}_{im} - w_{i(m-1)}\bar{x}_{i(m-1)}}, \quad i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots, r, \quad k = 1, 2, \ldots, (r-1). \quad (15)$$

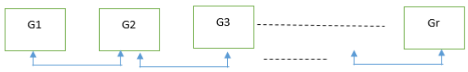

Fig. 1: An Illustration of the Pairwise Slope Between the Subgroups.

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Finally, the unknown parameters of MEM can be estimated as:

$$\hat{\beta}_{ik} = \frac{1}{r-1} \sum_{i=1}^{r-1} \hat{\beta}_{ki}; \quad \text{and} \quad \hat{\alpha} = \bar{y} - \sum_{k=1}^{r-1} \hat{\beta}_k \bar{x}_k. \quad (16)$$

4 Monte Carlo Experiment

Two random samples: inlier and outlier samples, based on 10,000 random samples each of size n were generated from the standard normal MEM of Equation (1). These samples were studied under the following procedures and assumptions.

Order the data from the smallest to the largest with their respective associated Y_i's, $i = 1, 2, \ldots, n$, by using Eq. (1) and (2).

Set the initial values as $\alpha = 1$, $\beta_1 = 2$, $\beta_2 = 3$, $\sigma_\varepsilon^2 = 1$, $\sigma_{\delta_1}^2 = 1$, and $\sigma_{\delta_2}^2 = 1$.

Generate the error terms from a standard normal distribution.

Consider four different data sizes: $n = 50$, 100, 200, and 500.

Contaminate the data with outliers. The last observation was deleted and replaced with the outlier generated according to the following different cases:

–Outliers only in y ($\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$), $\sigma_\varepsilon^2 = 16$.
–Outliers only in x_1 ($\delta_1 \sim N(0, \sigma_{\delta_1}^2)$), $\sigma_{\delta_1}^2 = 16$.
–Outliers only in x_2 ($\delta_2 \sim N(0, \sigma_{\delta_2}^2)$), $\sigma_{\delta_2}^2 = 16$.
–Outliers in both x_1 and x_2 ($\delta_1 \sim N(0, \sigma_{\delta_1}^2)$ and $\delta_2 \sim N(0, \sigma_{\delta_2}^2)$), $(\sigma_{\delta_1}^2, \sigma_{\delta_2}^2) = (16, 16)$.

–Outliers in each of y, x_1, and x_2, $(\sigma_{\varepsilon}^2, \sigma_{\delta_1}^2, \sigma_{\delta_2}^2) = (16, 16, 16)$.

The properties of these estimators were investigated by using the simulated bias and mean square error (MSE) defined as:

$$\text{Bias} = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\mu}_i - \phi);$$

$$\text{MSE} = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\mu}_i - \phi)^2 \quad (17)$$

where $\hat{\mu}_i$ is the estimate given by one of the proposed estimators for the i-th sample.

Tables 1-6 present the bias and MSE values of $\hat{\alpha}$ and $\hat{\beta}$ for each contaminated case with different sample sizes: $n = 50$, 100, 200, and 500. The simulated results indicate that the MSE decreases as the sample size increases.

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	0.0005	-0.0006	-0.0103	0.0001	0.0003	-0.0002	0.0939	-0.006	0.0218	0.2698
		MSE	0.0006	0.0034	0.0202	0.0126	0.001	0.0035	0.0547	0.0232	0.8163	0.4960
	$\hat{\beta}_1$	Bias	-0.0259	-0.0311	0.9631	-0.0569	-0.0001	-0.0291	0.3986	-0.0579	0.6414	0.8068
		MSE	0.0345	0.043	0.2909	0.1301	0.0074	0.0376	0.0353	0.0364	0.9786	0.9125
	$\hat{\beta}_2$	Bias	-0.0529	-0.0657	0.0239	-0.0784	-0.002	-0.0646	0.0265	-0.0784	-0.7697	-0.6759
		MSE	0.1414	0.1768	0.1926	0.1104	0.0041	0.1695	0.0241	0.0221	0.946	0.7497
100	$\hat{\alpha}$	Bias	-0.0002	-0.0001	0.0005	0.0001	0.0001	0.00002	0.0023	-0.001	-0.0014	-0.0495
		MSE	0.0001	0.0004	0.0207	0.013	0.0002	0.0005	0.0518	0.0219	0.2334	0.3387
	$\hat{\beta}_1$	Bias	-0.013	-0.0156	0.0339	-0.02	-0.0009	-0.0136	0.0488	-0.0194	0.327	0.6429
		MSE	0.0171	0.0224	0.201	0.1245	0.0015	0.0176	0.0279	0.0381	0.0782	0.5076
	$\hat{\beta}_2$	Bias	-0.0265	-0.0304	0.008	-0.0296	-0.0012	-0.0289	0.0066	-0.0296	-0.8378	0.2111
		MSE	0.0707	0.0838	0.0915	0.0877	0.0014	0.0757	0.0209	0.0219	0.8187	0.6803
200	$\hat{\alpha}$	Bias	0.0001	-0.0001	-0.032	0.0024	0.0	-.00007	-0.0028	0.0021	0.0049	0.0218
		MSE	0.0	0.0001	0.0092	0.0055	0.0001	.000063	0.0258	0.0356	0.0926	0.3036
	$\hat{\beta}_1$	Bias	-0.0066	-0.0077	0.0188	-0.0078	-0.0003	-0.0065	-0.0017	-0.0098	0.1817	0.3847
		MSE	0.0086	0.0114	0.0707	0.0784	0.0004	0.0084	0.0206	0.0247	0.7759	0.4736
	$\hat{\beta}_2$	Bias	-0.0134	-0.0146	0.0037	-0.0131	-0.0005	-0.0137	0.0028	-0.0159	-0.8608	0.0768
		MSE	0.0359	0.0406	0.0901	0.0743	0.0003	0.036	0.0198	0.0245	0.7925	0.5327
500	$\hat{\alpha}$	Bias	0.0011	-0.0001	0.051	-0.0014	0.0006	-.00006	-0.001	0.0006	0.0009	0.0034
		MSE	0.0004	0.0001	0.0083	0.0028	0.0005	.000008	0.0109	0.0201	0.035	0.589
	$\hat{\beta}_1$	Bias	-0.0027	-0.003	0.0063	-0.0055	-0.0002	-0.0026	0.0067	0.0069	0.1203	0.9089
		MSE	0.0036	0.0044	0.0172	0.0069	0.0001	0.0034	0.0160	0.0184	0.3831	0.3516
	$\hat{\beta}_2$	Bias	-0.0054	-0.0057	0.0026	-0.0013	-0.0003	-0.0053	0.0014	0.002	-0.8712	0.0332
		MSE	0.0147	0.0158	0.0811	0.0579	0.0001	0.014	0.0096	0.0087	0.7791	0.424

Table 1: The Bias and MSE of $\hat{\alpha}$ and $\hat{\beta}$ for samples without outlier.

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	-0.0013	-0.0003	-0.0059	-0.0033	0.0004	0.0002	-0.002	-0.0153	-0.2646	-0.5797
		MSE	0.0032	0.004	0.3974	0.2555	0.0023	0.0030	0.6202	0.5925	0.9593	0.9694
	$\hat{\beta}_1$	Bias	-0.0271	-0.0289	0.0583	-0.0388	-0.0076	-0.0328	-0.0866	-0.0385	-0.7868	0.4156
		MSE	0.0384	0.0381	0.1526	0.137	0.011	0.046	0.1901	0.1899	0.9497	0.8098
	$\hat{\beta}_2$	Bias	-0.0531	-0.0649	0.0304	-0.0581	-0.0114	-0.0673	0.0158	-0.058	0.2848	0.2565
		MSE	0.143	0.1724	0.2508	0.1696	0.0134	0.1832	0.2543	0.2181	0.5711	0.3642
100	$\hat{\alpha}$	Bias	-0.0002	-0.0001	0.0002	0.0047	0.0003	-0.0001	0.0034	0.0005	-0.009	0.1412
		MSE	0.0004	0.0006	0.2862	0.2457	0.0003	0.0004	0.5892	0.5737	0.9498	0.5451
	$\hat{\beta}_1$	Bias	-0.0127	-0.0137	0.0326	-0.0195	-0.0034	-0.0149	0.0352	-0.0197	-0.8448	0.4498
		MSE	0.017	0.0185	0.1179	0.1300	0.0032	0.0205	0.1461	0.1460	0.9722	0.7382
	$\hat{\beta}_2$	Bias	-0.0265	-0.0288	0.0148	-0.0296	-0.0048	-0.0298	0.0152	-0.0296	0.1294	0.1836
		MSE	0.071	0.0758	0.2011	0.1989	0.0036	0.0804	0.1937	0.1901	0.3265	0.2918
200	$\hat{\alpha}$	Bias	-0.0001	-0.0001	0.0311	-0.0018	0.0001	0.0001	0.0006	0.0058	0.055	0.0344
		MSE	0.0001	0.0001	0.2349	0.2050	0.0001	0.0001	0.4022	0.3127	0.349	0.2656
	$\hat{\beta}_1$	Bias	-0.0063	-0.0068	0.0166	-0.0099	-0.0014	-0.0072	0.0249	-0.0099	0.2975	0.3279
		MSE	0.0082	0.0092	0.1081	0.995	0.0007	0.0101	0.0841	0.0711	0.2775	0.583
	$\hat{\beta}_2$	Bias	-0.0132	-0.014	0.0042	-0.0149	-0.002	-0.0143	0.0036	-0.0149	0.0004	0.0733
		MSE	0.0352	0.0376	0.1701	0.1544	0.0012	0.039	0.0914	0.0901	0.2263	0.2811
500	$\hat{\alpha}$	Bias	0.0001	-0.0001	0.0001	0.0021	-0.0001	-0.0001	0.0001	0.0002	0.0147	-0.0567
		MSE	0.001	0.0001	0.2209	0.1998	0.0017	0.0023	0.1292	0.1234	0.2814	0.2213
	$\hat{\beta}_1$	Bias	-0.0027	-0.0027	0.0069	-0.004	-0.0004	-0.0029	0.0075	0.0069	0.2372	0.241
		MSE	0.0036	0.0038	0.0920	0.0900	0.0002	0.0042	0.0332	0.0311	0.2645	0.2955
	$\hat{\beta}_2$	Bias	-0.0054	-0.0055	0.0011	-0.006	-0.0007	-0.0056	0.0011	0.0013	-0.5327	0.0138
		MSE	0.0146	0.015	0.0804	0.0778	0.0003	0.0155	0.0687	0.0581	0.1875	0.1794

Table 2: The Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$ when $\sigma^2_{\delta_1} = 16$ with outliers in x_1.

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	-0.0001	0.0002	-0.019	0.0085	0.0001	-0.0015	0.0959	-0.0379	0.3814	-0.4814
		MSE	0.0036	0.0016	0.1201	0.0959	0.0045	0.0038	0.1022	0.1078	0.9134	0.7786
	$\hat{\beta}_1$	Bias	-0.0146	-0.0194	0.0752	-0.0361	0.0051	-0.0115	0.0671	-0.0401	0.1916	0.1372
		MSE	0.0156	0.0235	0.1231	0.1229	0.0185	0.0116	0.4584	0.1591	0.6823	0.8392
	$\hat{\beta}_2$	Bias	-0.0458	-0.0583	0.0107	-0.0585	0.0057	-0.0536	0.0128	-0.0585	-0.8633	-0.1286
		MSE	0.1109	0.1415	0.1536	0.1521	0.0087	0.1223	0.0861	0.0575	0.2544	0.5925
100	$\hat{\alpha}$	Bias	-0.0001	-0.0002	-0.0662	-0.0040	-0.0004	-0.0004	0.0043	-0.0079	-0.0099	-0.3700
		MSE	0.0004	0.0003	0.1185	0.0936	0.0007	0.0004	0.0714	0.0487	0.9089	0.6376
	$\hat{\beta}_1$	Bias	-0.0095	-0.0093	-0.0602	-0.0193	0.0022	-0.0052	0.0174	-0.0193	0.1614	0.1955
		MSE	0.0108	0.0111	0.1209	0.1200	0.0036	0.0042	0.0415	0.0228	0.5679	0.7781
	$\hat{\beta}_2$	Bias	-0.0245	-0.0255	0.0059	-0.0296	0.0021	-0.0233	0.011	-0.0296	-0.0974	-0.8605
		MSE	0.0612	0.0608	0.0148	0.0879	0.002	0.0503	0.0432	0.0416	0.2595	0.4314

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
200	$\hat{\alpha}$	Bias	0.0029	0.0002	-0.0078	-0.0007	-0.0	0.001	0.0005	-0.0014	-0.0019	-0.2851
		MSE	0.0022	0.0002	0.097	0.0026	0.0001	0.0001	0.0612	0.0324	0.8886	0.6183
	$\hat{\beta}_1$	Bias	-0.0052	-0.006	-0.1023	-0.0098	0.0011	-0.0039	0.0197	-0.0099	0.1596	0.1693
		MSE	0.0061	0.0082	0.096	0.0911	0.0009	0.0036	0.0883	0.0197	0.4712	0.6722
	$\hat{\beta}_2$	Bias	-0.0126	-0.0134	0.0032	-0.0149	0.0033	-0.0116	0.0038	-0.0149	-0.0758	-0.6878
		MSE	0.0321	0.0349	0.0023	0.0025	0.0007	0.0262	0.0213	0.0404	0.2179	0.3466
500	$\hat{\alpha}$	Bias	0.001	-0.0008	0.001	-0.0015	-0.0015	-0.0	0.041	-0.0011	0.0775	-0.1193
		MSE	0.0007	0.0008	0.009	0.0076	0.0024	0.0	0.0302	0.0301	0.5917	0.4216
	$\hat{\beta}_1$	Bias	-0.0025	-0.0027	0.0078	-0.0085	0.0003	-0.0018	0.0071	-0.004	0.2358	0.1679
		MSE	0.0032	0.0038	0.0513	0.0144	0.0019	0.0018	0.0282	0.0178	0.2825	0.6641
	$\hat{\beta}_2$	Bias	-0.0053	-0.0054	0.0014	0.0026	0.0003	-0.0047	0.0014	-0.006	-0.9269	-0.4754
		MSE	0.0139	0.0145	0.0019	0.0013	0.0001	0.0109	0.0201	0.0169	0.1333	0.2971

Table 3: The Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$ when $\sigma^2_{\delta_2} = 16$ with outliers in x_2.

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	0.0011	-0.0022	-0.0062	-0.012	-0.0008	-0.0004	-0.0091	-0.0139	-0.3956	0.5016
		MSE	0.0013	0.0024	0.4685	0.4056	0.0056	0.0018	0.6148	0.6134	0.8555	0.7255
	$\hat{\beta}_1$	Bias	-0.0217	-0.0179	0.0585	-0.0378	-0.0016	-0.0194	0.0139	-0.0145	-0.9812	0.5018
		MSE	0.0277	0.0232	0.4048	0.4021	0.0128	0.0238	0.411	0.3994	0.7596	0.5468
	$\hat{\beta}_2$	Bias	-0.0507	-0.0487	0.0241	-0.0584	-0.0011	-0.048	0.011	0.0994	-0.718	-0.8048
		MSE	0.1311	0.1247	0.2171	0.2099	0.0091	0.1197	0.7022	0.2095	0.7441	0.8049
100	$\hat{\alpha}$	Bias	-0.0001	-0.0002	0.0071	-0.0075	0.0002	0.0001	0.041	-0.346	-0.3903	-0.1765
		MSE	0.0002	0.0002	0.4007	0.3902	0.0006	0.0002	0.5009	0.4515	0.5573	0.4166
	$\hat{\beta}_1$	Bias	-0.0106	-0.0106	0.0615	-0.0195	-0.0012	-0.0095	0.022	-0.0197	-0.176	0.4669
		MSE	0.0129	0.0131	0.2053	0.2008	0.0035	0.0118	0.1478	0.1066	0.6702	0.5008
	$\hat{\beta}_2$	Bias	-0.0252	-0.0249	0.0088	-0.0299	-0.0014	-0.0245	0.2589	-0.0372	0.6054	-0.7889
		MSE	0.0645	0.0637	0.1908	0.1524	0.003	0.0619	0.3124	0.1946	0.6821	0.4622
200	$\hat{\alpha}$	Bias	-0.0001	0.0	0.0007	-0.0097	0.0095	0.001	-0.0022	-0.0052	-0.2256	-0.1529
		MSE	0.0025	0.0	0.2862	0.2481	0.0144	0.0001	0.3808	0.2526	0.377	0.3547
	$\hat{\beta}_1$	Bias	-0.006	-0.006	0.0211	-0.0101	0.0001	-0.0052	0.0244	-0.0099	-0.4022	0.3909
		MSE	0.0076	0.0079	0.1944	0.1085	0.0031	0.0064	0.0955	0.0923	0.208	0.4425
	$\hat{\beta}_2$	Bias	-0.013	-0.0128	0.0038	-0.0149	-0.0004	-0.0121	-0.003	-0.0149	0.1096	-0.4725
		MSE	0.0339	0.0332	0.1007	0.9444	0.0008	0.0302	0.0952	0.0944	0.5781	0.2472
500	$\hat{\alpha}$	Bias	-0.0001	0.002	0.001	-0.0019	0.0001	0.0012	0.001	-0.0012	0.3113	-0.0946
		MSE	0.0011	0.003	0.1992	0.1877	0.0055	0.0014	0.1933	0.1713	0.3353	0.2665
	$\hat{\beta}_1$	Bias	-0.0023	-0.0026	0.0770	-0.046	-0.0002	-0.0023	0.0079	-0.004	0.118	0.1777
		MSE	0.0029	0.0036	0.1356	0.9979	0.0001	0.0029	0.0259	0.0184	0.173	0.2097
	$\hat{\beta}_2$	Bias	-0.0052	-0.0053	0.0026	-0.017	-0.0003	-0.005	0.0014	-0.006	0.1554	-0.2105
		MSE	0.0135	0.0143	0.0622	0.0619	0.0002	0.0128	0.0728	0.0571	0.2385	0.1731

Table 4: The Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$ when $(\sigma^2_{\delta_1}, \sigma^2_{\delta_2}) = (16, 16)$ with outliers in both (x_1, x_2).

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	-0.0002	-0.0001	0.0809	-0.0023	0.0003	0.0012	-0.0069	-0.0346	0.3006	-0.383
		MSE	0.0025	0.0026	0.295	0.2154	0.0031	0.0028	0.3917	0.3331	0.3515	0.9506
	$\hat{\beta}_1$	Bias	-0.0243	-0.0246	0.4935	-0.0363	-0.0018	-0.0233	-0.0325	-0.0425	0.0951	0.8197
		MSE	0.0314	0.0335	0.5981	0.2887	0.007	0.03	0.3355	0.3562	0.4319	0.9823
	$\hat{\beta}_2$	Bias	-0.0519	-0.051	0.0243	-0.0584	-0.003	-0.052	0.0311	-0.0576	0.6224	0.1674
		MSE	0.1366	0.134	0.1598	0.1526	0.0046	0.1373	0.1056	0.1902	0.5843	0.5905

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
100	$\hat{\alpha}$	Bias	-0.0002	-0.0003	0.0004	-0.0012	-0.0004	0.0001	0.0806	-0.3095	-0.0566	-0.3533
		MSE	0.0004	0.0004	0.1921	0.1642	0.0005	0.0004	0.1321	0.1095	0.3392	0.8736
	$\hat{\beta}_1$	Bias	-0.0127	-0.014	0.0447	-0.0195	-0.0007	-0.0123	0.0276	-0.0373	0.2062	0.2733
		MSE	0.0166	0.0202	0.258	0.1897	0.0015	0.0161	0.1709	0.1194	0.4143	0.3996
	$\hat{\beta}_2$	Bias	-0.0262	-0.0275	0.0095	-0.0296	-0.001	-0.0257	0.0123	-0.0296	-0.3901	0.1234
		MSE	0.0691	0.0757	0.1606	0.1078	0.0012	0.0668	0.1004	0.1503	0.4639	0.2641
200	$\hat{\alpha}$	Bias	0.0024	-0.002	0.0201	-0.0303	0.0114	0.0001	-0.0004	-0.0073	-0.0235	0.0061
		MSE	0.0001	0.0001	0.0941	0.0761	0.0051	0.0021	0.0822	0.0788	0.1122	0.1229
	$\hat{\beta}_1$	Bias	-0.0066	-0.0072	0.0201	-0.0099	-0.0004	-0.0061	0.02	-0.0103	0.0883	0.2431
		MSE	0.0088	0.0106	0.123	0.1195	0.0004	0.0079	0.0723	0.0721	0.3886	0.2127
	$\hat{\beta}_2$	Bias	-0.0134	-0.0139	0.0042	-0.0149	-0.0006	-0.0129	0.0033	-0.0149	-0.330	0.0457
		MSE	0.036	0.0385	0.0868	0.0844	0.0003	0.0337	0.0535	0.0445	0.2832	0.1022
500	$\hat{\alpha}$	Bias	0.0020	-0.001	0.011	0.0018	0.0034	-0.002	0.001	0.0028	-0.0182	0.0872
		MSE	0.0001	0.001	0.0781	0.0551	0.0021	0.003	0.0383	0.0297	0.1081	0.1023
	$\hat{\beta}_1$	Bias	-0.0026	-0.003	0.0074	-0.008	-0.0003	-0.0025	0.0045	-0.006	0.0595	0.1686
		MSE	0.0034	0.0044	0.0919	0.0910	0.0001	0.0033	0.0132	0.0119	0.1639	0.2108
	$\hat{\beta}_2$	Bias	-0.0055	-0.0056	0.0015	-0.01	-0.0003	-0.0052	0.001	0.008	-0.2901	0.0365
		MSE	0.0141	0.0155	0.0212	0.0208	0.0011	0.0137	0.0106	0.0110	0.1553	0.0916

Table 5: The Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$ when $\sigma_{\varepsilon}^2 = 16$ with outliers in y.

n	Parameter	Statistic	Weight case 1				Weight case 2				Classical	
			Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	Modified Theil	Modified Siegel	Iterative r = 3	Iterative r = 4	MLE	MOM
50	$\hat{\alpha}$	Bias	-0.0165	-0.0011	0.0133	0.0018	0.0235	-0.0004	0.0112	-0.0005	-0.1943	-0.4650
		MSE	0.0027	0.0026	0.4631	0.3411	0.0059	0.0036	0.5644	0.5294	0.8673	0.8337
	$\hat{\beta}_1$	Bias	-0.0212	-0.0243	0.0568	-0.0375	-0.0003	-0.0196	0.1134	-0.0179	0.1463	0.8529
		MSE	0.0264	0.0316	0.4363	0.2297	0.0119	0.0243	0.3692	0.293	0.9744	0.7704
	$\hat{\beta}_2$	Bias	-0.0522	-0.0517	0.0332	-0.057	-0.0043	-0.0478	0.0489	-0.0581	-0.199	-0.1227
		MSE	0.14	0.1355	0.4058	0.3246	0.0073	0.1198	0.2573	0.187	0.728	0.3386
100	$\hat{\alpha}$	Bias	-0.0003	-0.0002	-0.0042	-0.0001	-0.0002	-0.0003	-0.0242	-0.0001	0.1915	0.3141
		MSE	0.0004	0.0004	0.4113	0.2068	0.0009	0.0005	0.5075	0.4072	0.8117	0.6594
	$\hat{\beta}_1$	Bias	-0.0105	-0.0128	0.0465	-0.0194	0.0002	-0.0097	-0.0532	-0.0192	-0.1953	0.7951
		MSE	0.0126	0.0169	0.3683	0.2004	0.0032	0.0126	0.3336	0.1812	0.8682	0.6918
	$\hat{\beta}_2$	Bias	-0.025	-0.0263	0.022	-0.0296	-0.0007	-0.0239	0.0115	-0.0296	0.1169	-0.1741
		MSE	0.0638	0.0695	0.2462	0.1877	0.0027	0.0592	0.1262	0.1106	0.6714	0.2756
200	$\hat{\alpha}$	Bias	-0.0031	-0.005	0.0324	-0.013	0.0043	0.0034	0.0003	-0.0023	0.061	-0.3119
		MSE	0.0001	0.0001	0.2899	0.1921	0.0001	0.0001	0.4970	0.1569	0.7348	0.5576
	$\hat{\beta}_1$	Bias	-0.0057	-0.0054	0.0215	-0.0148	-0.0006	-0.0052	0.0243	-0.0099	0.1463	0.6997
		MSE	0.007	0.0071	0.2798	0.1442	0.001	0.0066	0.1210	0.1198	0.7709	0.4717
	$\hat{\beta}_2$	Bias	-0.0129	-0.0125	0.0041	-0.0199	-0.0008	-0.0122	0.0049	-0.0149	-0.103	-0.7063
		MSE	0.0337	0.0318	0.1301	0.0992	0.0012	0.0307	0.0913	0.0445	0.415	0.2176
500	$\hat{\alpha}$	Bias	-0.0001	-0.0025	0.0901	-0.0942	0.0001	-0.002	-0.001	0.003	-0.1491	0.2984
		MSE	0.0009	0.0002	0.1093	0.1892	0.0009	0.0001	0.0574	0.0411	0.5395	0.2902
	$\hat{\beta}_1$	Bias	-0.0023	-0.0025	0.0074	-0.006	-0.0002	-0.0023	0.032	-0.054	0.097	0.4496
		MSE	0.0029	0.0034	0.0998	0.0979	0.0003	0.0031	0.0247	0.0183	0.4286	0.3317
	$\hat{\beta}_2$	Bias	-0.0051	-0.0052	0.0014	-0.008	-0.0004	-0.005	0.0015	-0.006	-0.1713	-0.4424
		MSE	0.013	0.0133	0.0938	0.0919	0.0004	0.0129	0.0318	0.0164	0.2774	0.1816

Table 6: The Bias and MSE for $\hat{\alpha}$ and $\hat{\beta}$ when $(\sigma_{\delta_1}^2, \sigma_{\delta_2}^2, \sigma_{\varepsilon}^2) = (16, 16, 16)$ with outliers in all (x_1, x_2, y).

5 Real Data Application

In the past, a nation's overall development levels were determined by its national income because it was believed that the more a nation produced, the more progress it would make both economically and socially. However, we acknowledge that there may be significant differences between societal progress or overall development and GDP growth. Over the past two decades, there has been much discussion about the limitations of using GDP as a gauge of a country's quality of life or social well-being. The fact that a large portion of the population's quality of life has not improved despite a high GDP growth rate has led some people to believe that the GDP measure should be expanded to consider human well-being and life quality. Unemployment is a critical issue for developing countries because it has a direct and significant impact on a country's economy. It is defined as someone who is willing and able to work but does not have a paid job. Meanwhile, the unemployment rate is the most used indicator for assessing labor market conditions. It is the percentage of people in the labor force who are out of work. Understanding the patterns of unemployment rates is critical these days, and it has piqued the interest of researchers from all fields of study all over the world. For policymakers and researchers, unemployment is important when planning a country's monetary progress. An advanced modelling approach is required to efficiently determine the effect of the unemployment rate. Several studies have recently relied on traditional testing methods to estimate the effect of the unemployment rate. Furthermore, unemployment is typically non-stationary in nature. As a result, using traditional methods to demonstrate them will yield unpredictable results. To address the issue associated with traditional techniques, a better approach is required to deal with the effect of the unemployment rate [28]. The Human Development Index (HDI), a multidimensional indicator of development, has proven to be more reasonable in comparison to the measure of GDP growth, which is one-dimensional in income. This is in line with the general belief that well-being is a multidimensional concept that cannot be measured by market production or GDP alone [16], so that the value of all goods produced in a nation during a fiscal year is used to define its GDP. It is discovered to be one of the economic growth and production indicators, and to play a crucial strategic role in employment, development, and the balance of payments [37]. In this article, the new procedures were applied to determine relationships between GDP and HDI. Data were collected from the yearly Jordans economic report (19902021) [45,46] and are presented in Table 7.

Year	HDI	GDP	Unemployment Rate	Year	HDI	GDP	Unemployment Rate
1990	0.625	1166.611	16.810	2006	0.741	2513.029	14.000
1991	0.636	1155.234	19.513	2007	0.744	2735.379	13.100
1992	0.657	1335.288	19.274	2008	0.745	3455.770	12.700
1993	0.668	1334.229	19.700	2009	0.743	3559.692	12.900
1994	0.679	1414.339	17.171	2010	0.737	3736.645	12.500
1995	0.693	1466.045	14.600	2011	0.734	3852.890	12.900
1996	0.695	1463.888	13.700	2012	0.735	3910.347	12.200
1997	0.699	1494.511	13.686	2013	0.729	4044.427	12.600
1998	0.702	1600.398	13.703	2014	0.729	4131.447	11.900
1999	0.706	1619.536	13.707	2015	0.730	4164.109	13.080
2000	0.711	1651.622	13.700	2016	0.729	4175.357	15.280
2001	0.717	1720.361	14.700	2017	0.726	4231.518	18.140
2002	0.715	1802.055	15.300	2018	0.728	4308.151	18.270
2003	0.720	1876.259	14.500	2019	0.729	4405.487	16.810
2004	0.726	2044.964	14.580	2020	0.729	4282.766	19.026
2005	0.738	2183.395	14.800	2021	0.730	4405.839	19.252

Table 7: Yearly Dataset of HDI, GDP and Unemployment Rate of Jordan (19902021)

A descriptive analysis of the data is tabulated in Table 8. It could also be noted that there is a strong positive and significant correlation between GDP and HDI ($r = 0.739$, $p < 0.001$) and a strong negative and significant correlation between the unemployment rate and HDI ($r = -0.538$, $p < 0.001$).

Variable	Min	Max	Mean	STDEV
Unemployment Rate	11.9	19.7	15.1	2.5
GDP	1155.2	4405.8	2726.3	1242.6
HDI	.63	.75	.71	.03

Table 8: Descriptive Statistics

The trend of the variables within the study period are given in Figure 2, 3 and 4.

Fig. 2: The trend of the HDI within 1990-2021.

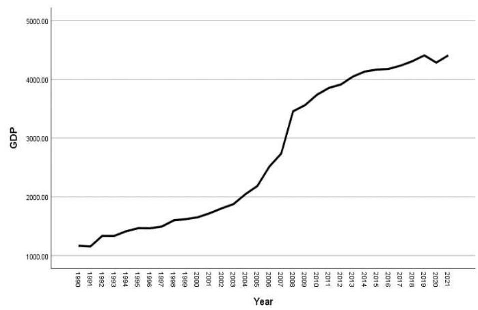

Fig. 3: The trend of the national GDP within 1990-2021.

Fig. 4: The trend of the unemployment rate within 1990-2021.

Moreover, the scatter plots in Figure 5 and 6 suggests that there is almost a linear relationship between the variables.

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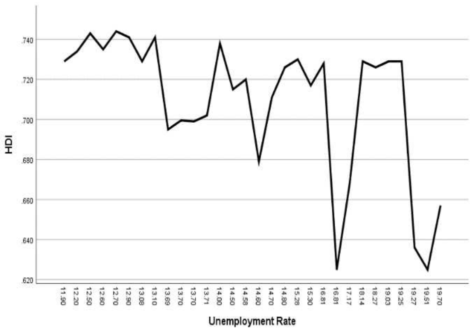

Fig. 5: The line plot of HDI and Unemployment rate.

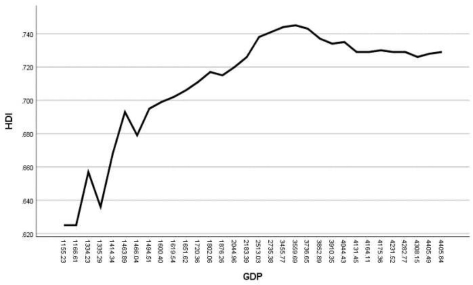

Fig. 6: The line plot of HDI and GDP.

Also, the scatter plots in Figure 7 indicate that there is heteroscedasticity problem in fitting the model.

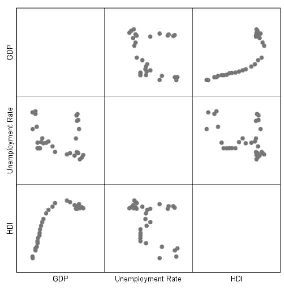

Fig. 7: The line plot of HDI and GDP.

These analyses suggest that the GDP, unemployment rate, and HDI can be modeled as linear relationships; however, it is believed that all variables are subject to error because its value is affected by several other factors. As a result, it is suggested to consider MEM for studying the relationship between HDI, unemployment rate, and GDP. The model under consideration can therefore be reformulated as follows:

$$\text{HDI} = \alpha + \beta_1 \times (\text{GDP} - \delta_1) + \beta_2 \times (\text{Unemployment Rate} - \delta_2) + \varepsilon. \quad (18)$$

Table 9 displays the outcomes of the estimation methods for each the Weighted Latent Variables, Iterative Weighted, MLE and MOM. The results indicate that based on mean square error (MSE), the proposed Weighted Latent Variables (the modified Theil and Siegel estimators) procedures produced more accurate estimators for each weight case than the other estimation methods. Meanwhile, results from the iterative weighted procedure show that the weight from case 2 produces a more accurate estimator compared to the weight from case 1. Also, from the residual plots in Figure 8 and 9 suggests that the proposed Weighted Latent Variables and the Iterative Weighted procedures are more efficient than classical procedures (MLE and MOM) for fitting the model.

Weight case	Method	Criterion	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}$	MSE
1	Modified Thiel		2.7e-05	-0.0051	0.6979	0.0003
	Modified Siegel		4e-06	-0.0013	0.7024	0.0009
	Iterative weighted	r = 3	0.0002	0.0459	-0.4868	0.0527
		r = 4	0.0003	0.0461	-0.6882	0.0949
2	Modified Thiel		2.7e-05	-0.0051	0.6967	0.0003
	Modified Siegel		4e-06	-0.0013	0.7024	0.0009
	Iterative weighted	r = 3	0.0002	0.0244	-0.2951	0.0801
		r = 4	0.0001	0.0433	-0.3008	0.0334
Classical	MLE		1.6e-05	0.0026	0.651	0.1926
	MOM		1.15e-05	0.00624	0.3481	0.1083

Table 9: Parameter Estimation of HDI vs GDP and Unemployment rate.

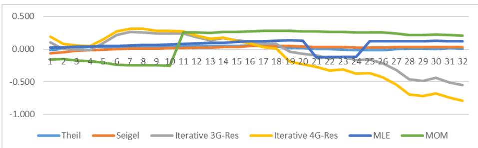

Fig. 8: Residual of each estimation method for Case 1.

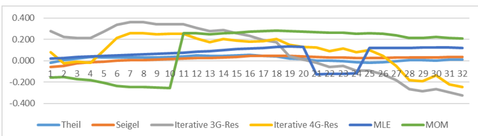

Fig. 9: Residual of each estimation method for Case 1.

6 Concluding Remarks

To fit the multiple structural MEM, this study proposed three new nonparametric estimation procedures: The Iterative Weighted Grouping and the modifications of Theil and Siegel. Monte Carlo simulations illustrate the superiority of the proposed estimation procedures over the classical methods (MLE and MOM) for each sample size. Furthermore, the results for the Iterative Weighted procedure in weight case 2 are better than the results for weight case 1, suggesting that the new proposed procedures are more efficient for fitting multiple SMEM, and they are better than MLE and MOM results. Furthermore, real data were used to investigate the effect of GDP and the unemployment rate on HDI. Results suggested that the GDP and HDI have a strong positive relationship, while there is a strong negative relationship between the unemployment rate and HDI. It is recommended that further research be undertaken to consider other sources of measurement error such as ultra-structural and functional MEM and dividing the data into four groups or more.

Declarations

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