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Geometrical Approach for Construction of Balanced Incomplete Block Design

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Abstract: Construction of balanced incomplete block design has been a major concern in the field of combinatorial design. Different techniques have been proposed by several authors to verify the existence and non-existence of balanced incomplete block designs (BIBDs); however, there is still no general method or algorithm to solve this challenge. In this paper, we propose a projective geometry approach of order $\text{PG}(N,p)$ to construct balanced incomplete block designs with different N-dimensions. We establish $\text{PG}(2,2)$ and $\text{PG}(3,2)$ orders to construct two different sets of BIBDs. It is observed that the constructed balanced incomplete block designs (BIBDs) are also symmetry balanced incomplete block designs (SBIBDs), but the reverse is not true in some cases. Hence, this approach is easier and more straightforward for constructing BIBDs and has proven to be effective in determining the existence and non-existence of balanced incomplete block designs.

Keywords: Geometry;Projective Geometry; block design; balanced design; Incomplete balanced design .

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1 Introduction

Block design has a non-void set $X = \{x_1, x_2, \ldots, x_v\}$, where the elements are known as treatments (varieties), and a collection of non-void subsets of X is called blocks. A design with an equal number of block sizes is referred to as a uniform block, and a simple design with no two identical blocks is also a block design. There are numerous applications of these designs, especially in the areas of scientific investigation and beyond.

When a complete randomized design (CRD) or completely randomized block design (CRBD) cannot handle an experimental situation, an incomplete block design becomes an appropriate choice for the experiment. This occurs when there is a limited number of experimental units or plots to accommodate all the treatments in a block [13]. A block design whose block size is fewer than the number of varieties/treatments is called an incomplete block design.

It is recorded that [17] was the pioneer to propose the concept of a balanced incomplete block design (BIBD). However, history also recorded that [10] contributed immensely in the area of incomplete block design, showing concern for the structure and construction of balanced incomplete block designs.

A design in which the number of treatments (t) is equal to the same number of block sizes (k) is known as a balanced complete design. A balanced incomplete block design (BIBD) is an incomplete block design where all pairs of treatments occur together within a block an equal number of times (λ).

For example, consider the design $V = \{v_1, v_2, \ldots, v_7\}$ with the following blocks:

$$\overset{b_1}{\begin{pmatrix}1\\2\\3\end{pmatrix}}, \overset{b_2}{\begin{pmatrix}1\\4\\5\end{pmatrix}}, \overset{b_3}{\begin{pmatrix}1\\6\\7\end{pmatrix}}, \overset{b_4}{\begin{pmatrix}2\\4\\6\end{pmatrix}}, \overset{b_5}{\begin{pmatrix}2\\5\\7\end{pmatrix}}, \overset{b_6}{\begin{pmatrix}3\\4\\7\end{pmatrix}}, \overset{b_7}{\begin{pmatrix}3\\5\\6\end{pmatrix}}$$

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This is a balanced incomplete block design with 7 treatments/varieties (v), 7 blocks (b), 3 treatments per block (block size $k = 3$), 3 appearances of each treatment in a block ($r = 3$), and each pair of treatments appearing together exactly once ($\lambda = 1$). Algebraically, this BIBD has parameters $(v,b,r,k,\lambda) \Rightarrow (7,7,3,3,1)$, which is also a symmetry balanced incomplete block design (SBIBD) with parameters $(v,k,\lambda) \Rightarrow (7,3,1)$.

Various approaches and techniques have been mentioned in the literature such as [1] and [6] for constructing the design. The construction of mutually orthogonal Latin squares was discussed by [14] to elucidate parameter correlation in balanced incomplete block designs. [15] opined that variance balancing, efficiency balancing, and neighbor balancing are key components of balancing in incomplete block designs.

Balanced incomplete block designs, as well as lattice designs, were extended by using balanced incomplete block designs to manage heterogeneity in various directions, resulting in several generic designs superior to partially balanced incomplete block designs [11]. [16] explored the importance of BIBDs in the following areas of statistics: controlled and finite sample support balanced incomplete cross-validation, Box-Behnken designs, randomized response procedures, intercropping experiments, group testing, validation studies, tournaments, fractional plans, balanced half-samples, and lotto designs.

[9] revealed that the generation of block designs remains an unsolved problem in combinatorial design. [2] applied the concept of Euclidean geometry to construct balanced incomplete block designs as a better method for building design structures. Orthogonal balanced incomplete block designs were constructed by [5], while pairwise balanced designs were constructed by [4].

[12] applied the concept of a Galois field of order seven (7) as a prime number to construct mutually orthogonal Latin squares, which were later used to develop a balanced incomplete block design (BIBD). [3] evaluated some algebraic structures such as groups, rings, and finite fields using a balanced incomplete block design approach.

Some unique construction techniques of dispersed balanced block designs with iterated blocks were developed based on the special effect of the incidence matrices of the subject designs [8]. [9] established a new strategy for constructing efficiency-balanced block designs with iterated blocks for varieties and explored directions for generating novel designs for different numbers of varieties.

The objective of this paper is to construct a balanced incomplete block design using a finite projective geometry approach. Since the existence and non-existence of BIBDs for given parameters have been a major concern in the area of blocking, this work contributes to solving this problem.

2 Basic Rudiments of Balanced Incomplete Block Design

The five parameters that make up a balanced incomplete block design are (v,b,r,k,λ). The parameter v stands for the number of varieties, b stands for the number of blocks, r is the number of replications per variety, k represents the block size, and λ is the number of times pairs of varieties occur together in a block. The following conditions are necessary but not sufficient for the existence of a BIBD:

$$(i)\quad rv = bk \tag{1}$$

$$(ii)\quad r(k-1) = \lambda(v-1) \tag{2}$$

$$(iii)\quad \lambda < r \tag{3}$$

$$(iv)\quad b = v \tag{4}$$

A BIBD cannot exist unless conditions (1) and (2) are satisfied. However, this does not mean that whenever (1) and (2) occur, a BIBD exists. Additionally, λ must be a positive integer.

Proposition 1.*In all BIBDs, the number of varieties is less than or equal to the number of blocks (i.e., $v \leq b$).*

*Proof.*By definition, let H be the incidence matrix of the BIBD. Then:

$$HH^T = \begin{pmatrix} r & \lambda & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \lambda & \dots & \lambda \\ \lambda & \lambda & r & \lambda & \dots & \lambda \\ \vdots & \vdots & \ddots & \vdots & & \\ \lambda & \lambda & \lambda & \lambda & \dots & r \end{pmatrix}.$$

The determinant of the incidence matrix does not change even when the first column is subtracted from the other columns. Therefore, the above incidence matrix becomes:

$$|HH^T| = \begin{pmatrix} r & \lambda-r & \lambda-r & \lambda-r & \dots & \lambda-r \\ \lambda & r-\lambda & 0 & 0 & \dots & 0 \\ \lambda & 0 & r-\lambda & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \\ \lambda & 0 & 0 & 0 & \dots & r-\lambda \end{pmatrix}.$$

When other rows of the matrix $|HH^T|$ are added to the first row, the determinant remains unchanged.
Therefore,

$$|HH^T| = \begin{pmatrix} r+(v-1)\lambda & 0 & 0 & 0 & \cdots & 0 \\ \lambda & r-\lambda & 0 & 0 & \cdots & 0 \\ \lambda & 0 & r-\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & r-\lambda & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda & 0 & 0 & 0 & \cdots & r-\lambda \end{pmatrix}$$

Since the upper triangle of this matrix is all zeros, the determinant is the product of the diagonal entries. Thus;

$$|THH| = [r+(v-1)\lambda]\,(r-\lambda)^{v-1}$$

Since $0 < r-\lambda$, then $r+(v-1)\lambda$ is non-negative and $|(HH)^T|$ is not trivial. Then, for the matrix $(v \times v)$ of HH^T, the rank is v. This implies that the rank of $v \times b$ incidence matrix H is near b. Also, the rank (v) of the matrix HH^T is less than or equal to the rank of matrix H, follows that $v \leq b$.

Theorem 1.*Let $H_{v\times b}$ be an incidence matrix and $v \geq k \geq 2$. Suppose $H_{v\times b}$ is another incidence matrix of a (v,b,k,r,λ)-BIBD if and only if $HH^T = \lambda J_v + (r-\lambda)I_v$*
and $U_vH = kU_b$.

*Proof.*Assume I_v is the identity matrix of size $v \times v$, and J_v is a $v \times v$ matrix where all entries are 1. Suppose U_v is a vector of length v where each coordinate is 1. Let (Y,W) be a (v,b,r,k,λ)-BIBD, where $Y = (y_1,y_2,\dots,y_t)$ and $W = (w_1,w_2,\dots,w_b)$. If $HH^T = B$ and H is an incidence matrix, then the $(b_{i,j})$ component of matrix B is the inner product of the i^{th} and j^{th} rows of matrix H.

Every component on the principal diagonal of B (b_{ii}) represents 1's in the i-th row of matrix H, which corresponds to the number of blocks that each component occurs. Since $i \neq j$, this implies that the i^{th} and j^{th} rows have 1 in the same column if and only if both x_i and x_j belong to the same column.

Therefore, both the upper and lower triangular matrices of HH^T is λ. Hence:

$$HH^T = \begin{pmatrix} r & \lambda & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \lambda & \cdots & \lambda \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ \lambda & \lambda & \lambda & r & \cdots & r \end{pmatrix} = \lambda J_v + (r-\lambda)I_v$$

Also, U_vH is a column matrix of size $(1 \times b)$ where each entry in the i-th entry is 1, which shows that $vU_b = kUH$.

On the other hand, let H be a $v \times b$ matrix such that: $HH^T = \lambda J_v + (r-\lambda)I_v$ and $U_vH = kU_b$.
Let H be the incidence matrix of the design (Y,W), since H is a $v \times b$ matrix, then $|Y| = v$ and $|W| = b$. Recall from condition one (1), every variety appears in r blocks, and each pair of varieties appears in λ blocks. Secondly, it follows from condition two (2) that each block of W contains K varieties. Therefore, a design (Y,W) is a BIBD.

3 Methodology

3.1 Finite Projective Geometry Design

A finite projective geometry of N dimensions consists of ordered sets $(x_1,x_2,\dots,x_N)$, called points, where not all are simultaneously zero (i.e., not all $x_i = 0$). The order is denoted by $PG(N,p)$. Then $x's$ are homogeneous coordinates of the points that is for any $m \neq 0$, the point $(mx_0,mx_1,mx_2,\dots,mx_N)$ is considered the same point as $(x_1,x_2,\dots,x_N)$.

It can easily be shown that the number of varieties (points) in $PG(N,p)$ is given by:

$$P^N + P^{N+1} + \ldots + P + 1 = \frac{P^{N+1}-1}{P-1} \quad (5)$$

The number of varieties that satisfy $N-m$ independent linear homogeneous equations form an m-dimensional subspace given below:

$$\begin{cases} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = 0 \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = 0 \\ \vdots \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \vdots \\ a_{N-m,0}x_0 + a_{N-m,1}x_1 + a_{N-m,2}x_2 + \cdots + a_{N-m,N}x_N = 0 \end{cases} \quad (6)$$

Equation (6) represents the flat. For $N-m$ independent equations, a set can be obtained using linear combinations of the equations in (6) having a similar set of solutions, which can be represented as m-flat. Therefore, the m-flat of $PG(N,p)$ is given as:

$$\Omega(N,m,p) = \frac{(P^{N+1}-1)(P^N-1)\ldots(P^{N-m-1}-1)}{(P^{m+1})(P^m-1)\ldots(P-1)} \quad (7)$$

To every variety in $PG(N,p)$, there is a corresponding block containing all those varieties that occur in the m-flat. Every m-flat lies on $\Omega(m,0,p) = \frac{(P^{N+1}-1)}{P-1}$ points. Based on this flat, we have the following designs:

Equation (8) is used to obtain the numbers of points and lines, also known as varieties and blocks, in $PG(N,p)$, where N is the dimension and p is the side of the design:

$$v = b = \Omega(N,m,p) = \frac{P^{N+1}-1}{P-1} \quad (8)$$

Equation (9) gives the number of points (varieties) in each line (block), known as the block size due to the m-flat:

$$k = \Omega(m,0,p) = \frac{P^{m+1}-1}{P-1} \quad (9)$$

Equation (10) is obtained by subtracting one from the m-flat in Equation (8), which gives the number of times each treatment (variety) appears in a line (block):

$$r = \Omega(N-1,m-1,p) = \frac{P^N-1}{P-1} \quad (10)$$

Equation (11) is obtained by subtracting two from the m-flat in Equation (8), which gives the number of blocks through any two points connected by exactly one line or the number of times each pair of treatments (varieties) appears together:

$$\lambda = \Omega(N-2,m-2,p) = \frac{P^{N-1}-1}{P-1} \quad (11)$$

3.2 Projective Planes as BIBDs

Lemma 1: For any two different lines L and L' of a projective plane P, there exists a point x such that $x \in L \cup L'$.

Proof: Let N be the intersection of the lines L and L'. Let $0,1 \in L-N$ and $2,3 \in L'-N$, as shown in Figure 1.

.

Here, points are varieties and lines are blocks. Let x intersect blocks 02 and 13. Then, 0 is uniquely intersected by varieties 02 and L, hence $x \notin L$. Also, 3 is uniquely intersected by variety 13 and L', which implies that $x \notin L'$.

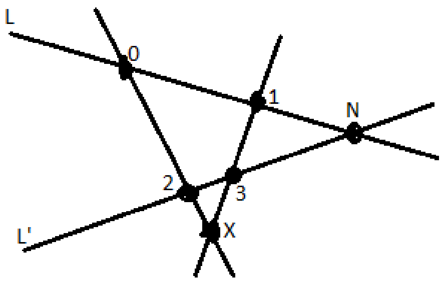

Fig. 1: The point x does not belong to the two lines.

4 Results

4.1 Construction of BIBD with $PG(2,2)$

In a $PG(2,2)$, the first 2 indicates the dimension of the projective space (in this case, it is a 2-D projective plane), and the second 2 indicates the size of the finite field, which consists of 2 elements. To construct $PG(2,2)$, the coordinates of the points (varieties) of the geometry are constructed using the binary elements 0 and 1, and the blocks are obtained as follows:

$$\begin{aligned} x_i &= 0, \forall i \in [0,2] \\ x_i + x_j &= 0, \forall i,j = [0,2] : i \neq j \\ x_0 + x_1 + x_2 &= 0 \end{aligned}$$

Here N= 2 and p=1 in 1- flat (m=1) . The parameters (v,b,kv,λ) - BIBD are obtained as follows;

$$b = t = \frac{P^{N+1}-1}{p-1} = 7, k = \frac{P^{m+1}-1}{P-1} = 3, r = \frac{P^{N}-1}{P-1} = 3, \lambda = \frac{P^{N-1}-1}{P-1} = 1.$$

The blocks are generated from the equation $4x_0 + 2x_1 + x_2$, where the coefficients of the equation are obtained from the expression $P^N + P^{N-1} + \ldots + P^{N-N}$.

$x_0 : x_1 = 010, x_2 = 001, x_1 + x_2 = 011$
$x_1 : x_0 = 100, x_2 = 001, x_0 + x_2 = 101$
$x_2 : x_0 = 100, x_1 = 010, x_0 + x_1 = 110$
$x_0 + x_1 : x_2 = 001, x_0 + x_1 = 110, x_0 + x_1 + x_2 = 111$
$x_0 + x_2 : x_1 = 010, x_0 + x_2 = 101, x_0 + x_1 + x_2 = 111$
$x_1 + x_2 : x_0 = 100, x_1 + x_2 = 011, x_0 + x_1 + x_2 = 111$
$x_0 + x_1 + x_2 : x_1 + x_2 = 011, x_0 + x_2 = 101, x_0 + x_1 = 110$

To summarize PG (2,2) in a tabular form, the coordinates correspond to the number x_0, x_1, x_2 except 0's. The number corresponding to x_0, x_1, x_2 is in harmony with the equation $4x_0 + 2x_1 + x_2$ and the results are shown in Table 1. Therefore, $PG(2,2)$ is a BIBD with parameters (v,b,k,r,λ) which corresponds to $(7,7,3,3,1)$, which is also a symmetry balanced incomplete block design (SBIBD) of parameters (v,k,λ) where $(7,3,1)$ given as:

$$X = \{1,2,3,4,5,6,7\}$$

and

$$B = \{(213),(415),(426),(167),(257),(437),(356)\}$$

Points	Co-ordinates			**Blocks**
t	x_0	x_1	x_2	x_0, x_1, x_2
1	0	0	1	$x_0 = 0 : 2, 1, 3$
2	0	1	0	$x_1 = 0 : 4, 1, 5$
3	0	1	1	$x_2 = 0 : 4, 2, 6$
4	1	0	0	$x_0 + x_1 = 0 : 1, 6, 7$
5	1	0	1	$x_0 + x_2 = 0 : 2, 5, 7$
6	1	1	0	$x_1 + x_2 = 0 : 4, 3, 7$
7	1	1	1	$x_0 + x_1 + x_2 = 0 : 3, 5, 6$

Table 1: The number of treatments and blocks for $(7, 7, 3, 3, 1)$ BIBD using PG(2,2)

4.2 Construction of BIBD with PG(3, 2)

In a $PG(3,2)$, the first 3 indicates the dimension of the projective space (in this case, it is a 3-D projective plane), and the second 2 indicates the size of the finite field which consists of 2 elements. To construct $PG(3,2)$, the coordinates of the points (varieties) of the geometry are made up of the binary elements with two elements, 0 and 1. The blocks are obtained using the following equations:

$$\begin{aligned} x_i &= 0, i = 0 \leq x \leq 3 \\ x_i + x_j &= 0, i, j = 0 \leq x \leq 3 : i \neq j \\ x_i + x_j + x_k &= 0, i, j, k = 0 \leq x \leq 3 : i \neq j, j \neq k, k \neq i \\ x_0 + x_1 + x_2 + x_3 &= 0 \end{aligned}$$

For $N = 3$ and $p = 2$ in 2-flats ($m = 2$), the total number of varieties in $PG(N, p)$ implies

$$PG(3,2) = \frac{P^{N+1} - 1}{P - 1}, v = b = \frac{P^{N+1} - 1}{P - 1} = 15 \text{ points (variables)}, k = \frac{P^{m+1} - 1}{P - 1} = 7, \lambda = \frac{P^{N-1} - 1}{P - 1} = 3, r = \frac{P^N - 1}{P - 1} = 7.$$

According to equation (6), the blocks are generated from the following equation: $8x_0 + 4x_1 + 2x_2 + x_3$ where the coefficients of the equation are obtained from the expression:

$$P^N + P^{N-1} + \ldots + P^{N-N}.$$

$x_0 : x_3 = 0001, x_2 = 0010, x_2 + x_3 = 0011, x_1 = 0100, x_1 + x_3 = 0101, x_1 + x_2 = 0110, x_1 + x_2 + x_3 = 0111$
$x_1 : x_3 = 0001, x_2 = 0010, x_2 + x_3 = 0011, x_0 = 1000, x_0 + x_3 = 1001, x_0 + x_2 = 1010, x_0 + x_2 + x_3 = 1011$
$x_2 : x_3 = 0001, x_1 = 0100, x_1 + x_3 = 0101, x_0 = 1000, x_0 + x - 3 = 1001, x_0 + x_1 = 1100, x_0 + x_1 + x_2 = 1101$
$x_3 : x_2 = 0010, x_1 = 0100, x_1 + x_2 = 0110, x_0 = 1000, x_0 + x_2 = 1010, x_0 + x_1 = 1100, x_0 + x_1 + x_2 = 1110$
$x_0 + x_1 : x_3 = 0001, x - 2 = 0010, x_2 + x_3 = 0011, x_0 + x_1 = 1100, x_0 + x_1 + x_3 = 1101, x_0 + x_1 + x_2, x_0 + x_1 + x_2 + x_3$
$x_0 + x_2 : x_3 = 0001, x_1 = 0100, x_1 + x_3 = 0101, x_0 + x_2 = 1010, x_0 + x_2 = x_3 = 1011, x_0 + x_1 + x_2 = 1110,$
$x_0 + x_1 + x_2 + x_3 = 1111$
$x_0 + x_3 : x_2 = 0010, x - 1 + x_2 = 0110, x_0 + x_3 = 1001, x_0 + x_2 + x_3 = 1011, x_0 + x_1 + x_3 = 1101, x_0 + x_1 + x_2 = 1111$
$x_1 + x_2 : x_3 = 0001, x_0 = 1001, x_0 + x_3 = 1001, x_1 + x_2 = 0110, , x_1 + x_2 + x_3 = 0111, x_0 + x_1 + x_2 = 1100,$
$x_0 + x_1 + x_2 + x_3 = 1111$
$x_1 + x_3 : x_2 = 0010, x_0 = 1000, x_0 + x_2 = 1010, x_1 + x_3 = 0101, x_1 = x_2 + x_3 = 0111, x_0 + x_1 + x_3 = 1101,$
$x_0 + x_1 + x_2 + x_3 = 1111$
$x_2 + x_3 : x_1 = 0100, x_0 = 1000, x_0 + x_1 = 1100, x_2 + x_3 = 0011, x_1 + x_2 + x_3 = 0111, x_0 + x_2 + x_3 = 1011,$
$x_0 + x_1 + x_2 + x_3 = 1111$
$x_0 + x_1 + x_2 : x_2 + x_3 = 0011, x_1 = x_3 = 0101, x_1 + x_2 = 0110, x_0 = 1000, x_0 + x_2 + x_3 = 1011, x_0 + x_1 + x_3 = 1101,$
$x_0 + x_1 + x_2 = 1110$
$x_0 + x_1 + x_3 : x_3 + x_2 = 0011, x_0 + x_3 = 1001, x_0 + x_2 = 1010, x_1 = 0100, x_1 + x_2 + x_3 = 0111, x_0 + x_1 + x_3 = 1101,$
$x_0 + x_1 + x_2 = 1110$
$x_0 + x_2 + x_3 : x_1 + x_3 = 0101, x_0 + x_3 = 1001, x_0 + x_1 = 1100, x_2 = 0010, x_1 = x_2 + x_3 = 0111, x_0 + x_2 + x_3 = 1011,$
$x_0 + x_1 + x_2 = 1110$
$x_1 + x_2 + x_3 : x_1 + x_2 = 0110, x_0 + x_2 = 1010, x_0 + x_1 = 1100, x_3 = 0001, x_1 + x_2 + x_3 = 0111, x_0 + x_2 + x_3 = 1011,$
$x_0 + x_1 + x_3 = 1101$
$x_0 + x_1 + x_2 + x_3 : x_2 + x_3 = 0011, x_1 + x_3 = 0100, x_1 + x_2 = 0100, x_0 + x_3 = 1001, x_0 + x_2 = 1010, x_0 + x_1 = 1100,$
$x_0 + x_1 + x_2 + x_3 = 1111$

The above constructed $PG(3,2)$ can be summarized in the table. The coordinates correspond to the numbers x_0, x_1, x_2, x_3, except the zero points. The numbers correspond to x_0, x_1, x_2, x_3 in consonance with the equation: $8x_0 + 4x_1 + 2x_2 + x_3$ and the result is displayed in Table 2.

Points	Co-ordinates				**Blocks**
t	x_0	x_1	x_2	x_3	x_0, x_1, x_2, x_3
1	0	0	0	1	$x_0 = 0 : 1,2,3,4,5,6,7$
2	0	0	1	0	$x_1 = 0 : 1,2,3,8,9,10,11$
3	0	0	1	1	$x_2 = 0 : 1,4,5,8,9,12,13$
4	0	1	0	0	$x_3 = 0 : 2,4,6,8,10,12,14$
5	0	1	0	1	$x_0 + x_1 = 0 : 1,2,3,12,13,14,15$
6	0	1	1	0	$x_0 + x_2 = 0 : 1,4,5,10,11,14,15$
7	0	1	1	1	$x_0 + x_3 = 0 : 2,4,6,9,11,13,15$
8	1	0	0	0	$x_1 + x_2 = 0 : 1,8,9,6,7,14,15$
9	1	0	0	1	$x_1 + x_3 = 0 : 2,8,10,5,7,11,15$
10	1	0	1	0	$x_2 + x_3 = 0 : 4,8,12,3,7,11,15$
11	1	0	1	1	$x_0 + x_1 + x_2 = 0 : 3,5,6,8,11,13,14$
12	1	1	0	0	$x_0 + x_1 + x_3 = 0 : 3,9,10,4,7,13,14$
13	1	1	0	1	$x_0 + x_2 + x_3 = 0 : 5,9,12,2,7,11,14$
14	1	1	1	0	$x_1 + x_2 + x_3 = 0 : 6,10,12,1,7,11,13$
15	1	1	1	1	$x_0 + x_1 + x_2 + x_3 = 0 : 3,5,6,9,10,12,15$

Table 2: Shows the number of treatments and blocks for $(15,15,7,7,3)$ BIBD using PG(3,2)

Hence, $PG(3,2)$ gives the result $(v,b,r,k,\lambda) \Rightarrow (15,15,7,7,3)$ parameters to be a balanced incomplete block design (BIBD) and also a symmetric balanced incomplete block design (BIBD) as shown below:

$$X = \{1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$$

$$B = \left\{ \begin{array}{l} (123457),(12389AB),(14589CD),(2468ACE),(123CDEF), \\ (145ABEF),(2469BDF),(18967EF),(28457DF),(48C37BF), \\ (3568BDE),(39A47DE),(59C27BE),(64C17BD),(3659ACF) \end{array} \right\}$$

where $A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$.

5 Discussion of Results

The projective geometry of two- and three-dimensional space with two finite elements, $PG(2,2)$ and $PG(3,2)$, were used to construct $(7,7,3,3,1)$ and $(15,15,7,7,3)$ – BIBDs where the coordinates of the points are built up of two binary elements 0 and 1. The finite $PG(2,2)$ and $PG(3,2)$ give the two equations: $4x_0 + 2x_1 + x_2$ and $8x_0 + 4x_1 + 2x_2 + x_3$ correspond to the numbers: x_0, x_1, x_2 and x_0, x_1, x_2, x_3 without zero's point. The new equations obtained are used to generate each block of 7 blocks and 15 blocks. In this design, X represents the variety, and B represents the blocks, which are designated as the (X,B) design.

6 Conclusion

Based on the results found in section (iv), it is clearly shown that the projective geometry (PG) approach for constructing balanced incomplete block designs (BIBDs) proved to be an effective way in determining the reality and non-existence of the concerned block design. The two constructed designs also formed a symmetric balanced incomplete block design (SBIBD).

The advantages ("pros") of this method are:

1. **Simplicity and Efficiency:** This approach is a straightforward and direct method for constructing BIBDs, which can simplify the design process compared to more complex methods.
2. **Symmetric Structures:** This method also allows for the construction of symmetric balanced incomplete block designs (SBIBDs), providing greater flexibility in the application of these designs in experiments.
3. **Existence Determination:** The projective approach facilitates the determination of the existence and non-existence of certain BIBDs, which is a crucial aspect in experimental design.

The most specified *con* among others in this article leads to the recommendation that higher dimensions of PG should be explored. This implies that this method might be limited to certain configurations of BIBDs and not universally applicable.

Declarations

Competing interests: Authors declared no competing interests.
Authors' contributions: This article was carried out in collaboration among all authors. Author U.P. Akra contributed to the study conceptualization, writing – original draft and methodology. Author E. E. Bassey wrote review - editing and formal analysis. The authors U. P. Akra and A. C. Etim did the geometrical construction of BIBDs and its visualization. The validation and supervision of the work was done by S. S. Akpan and O. E. Ntekim. All authors read and approved the final manuscript
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