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Development of Conformable Fractional Numerical Methods of Constant Order using Fractional Power Series Theorem

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Abstract: This study aims to employ novel numerical approaches for the constant-order conformable fractional derivative. By utilizing the fractional power series theorem, two innovative numerical techniques have been devised: the constant-order conformable Euler method and the constant-order conformable Runge Kutta 2-stage method. Furthermore, these techniques account for various fractional constant-order derivatives. Different models have been analyzed to demonstrate their behavior under varying constant orders, and their agreement and validation with standard Runge-Kutta and Euler methods have been confirmed. Notably, both methods hold promise for application in fractional financial models. The study includes a comparative analysis of these methods against classical derivatives, supported by tabular data showcasing the numerical outcomes.

Keywords: Conformable constant order derivative; Fractional financial model; Numerical technique; Classical derivative.

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1 Introduction

Extensive research into fractional calculus was conducted throughout the 20th century and documented in scientific and engineering literature. Fractional calculus has found applications in diverse fields such as fluid mechanics, integral equations, electrochemistry, biological modeling, and viscoelasticity models [4], [6], [8]. It serves as a valuable mathematical tool for addressing a wide range of problems across science, engineering, and mathematics. This article aims to highlight the relevance of fractional calculus by showcasing new and contemporary applications in biological and engineering sciences, with the intent to garner increased interest in the subject and demonstrate its practical utility [3], [7]. Moreover, recent studies have employed fractional calculus as a method to analyze the nonlinear dynamics of various problems [1], [2].

Analytical solutions are often unattainable for fractional differential equations with constant order. Therefore, comprehending the impacts of nonlinear problem solutions necessitates the utilization of semi-analytical and numerical methods [9]. Over recent decades, numerous techniques for addressing both linear and nonlinear dynamical systems have been devised by various researchers [13], [12], [20], [5]. These advancements have led to the emergence of two novel numerical approaches: the α-differentiable Euler and α-differentiable Runge-Kutta 2-stage methods of constant orders. Various models have been examined to illustrate the behavior across different constant orders, and their conformity and validation with standard Runge-Kutta and Euler methods have been established.

The analytical solutions for constant-Order fractional differential equations are very difficult to find because of the constant fractional derivative. As a result, it is necessitating the use of numerical approaches to comprehend the results of solutions to linear and nonlinear problems. There are a number of numerical approaches that have been presented to date, like: Polynomial methods [16], Lagrange multipliers technique [17], collocation method [18], Galerkin finite element method [19], wavelet methods [20], Homotopy Analysis Method [21]-[22], Homotopy Sumudu approach method[23]

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and another method using artificial neural network [24].
The following is the research strategy. In section-2 based on the definition as well as a few characteristics of the conformable fractional derivative of constant order introduced in [9], and also we give conformable constant orders Taylors theorem. Derive the conformable constant orders Euler method and conformable constant orders Rung-Kutta 2-stage method by using the conformable constant orders Taylors theorem in section -3 and 4. Afterwards, applied the proposed numerical methods on different constant orders of fractional differential equations and present the numerical results in section - 5. In section-6 the fractional financial model of constant orders chaotic oscillator are presented.

2 Basic definition and Tools

Definition 1. *If $f:[0,\infty)\to R$ is a differentiable function,the α- differentiable fractional derivative of constant order α is defined as [9]*

$$D^{\alpha}\mathrm{f}(\mathrm{t})=\frac{d^{\alpha}\mathrm{f}(\mathrm{t})}{d\mathrm{t}^{\alpha}}=\lim_{\varepsilon\to 0}\frac{\mathrm{f}\left(\mathrm{t}+\varepsilon\mathrm{t}^{1-\alpha}\right)-\mathrm{f}(\mathrm{t})}{\varepsilon},\forall\mathrm{t}>0,0<\alpha\leq 1$$

Theorem 1.Fractional power series theorem
At a point t_0, if $0<\alpha\leq 1$ and f is an α-differentiable function . The fractional power series expansion of f is then as follows [15]:

$$\mathrm{f}(\mathrm{t})=\sum_{i=0}^{\infty}\frac{(D^{\alpha}\mathrm{f})^{i}(\mathrm{t}_0)(\mathrm{t}-\mathrm{t}_0)^{i\alpha}}{\alpha^{i}i!},\ \mathrm{t}_0<\mathrm{t}<\mathrm{t}_0+\mathrm{R}^{\frac{1}{\alpha}},\mathrm{R}>0.$$

Here $(D^{\alpha}\mathrm{f})^{i}(\mathrm{t}_0)$ means applying the fractional derivative i multiple times.

3 α-differentiable Euler methods constant order

In order to obtain the α- differentiable Euler methods constant orders to determine the numerical solution of an IVP with a constant orders fractional derivative in α-differentiable sense, we take a look at the problem of initial value in the form

$$\begin{aligned}&\tfrac{d^{\alpha}\mathrm{y}(\mathrm{t})}{d\mathrm{t}^{\alpha}}=\mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t})),\quad (o<\alpha\leq 1)\\&\mathrm{y}(0)=\mathrm{y}_0.\end{aligned}\tag{1}$$

where $D^{\alpha}\mathrm{y}(\mathrm{t})$ represented derivative operator. We are trying to solve the problem in Eq.(1) for an interval denoted by $[0,\mathrm{b}]$. A set of points $[\mathrm{t_i},\mathrm{y}(\mathrm{t_i})]$ is used to determine an approximation. The interval $[0,\mathrm{b}]$ is subdivided into n sub intervals $[\mathrm{t_i},\mathrm{t_{i+1}}]$ of equal step size $\mathrm{h}=\frac{\mathrm{b}}{\mathrm{n}}$ using the nodal point $\mathrm{t_i}=ih$ for $i=0,1,2,...n$. Assume that $\mathrm{y}(\mathrm{t})$, $D^{\alpha}\mathrm{y}(\mathrm{t})$ and $D^{2\alpha}\mathrm{y}(\mathrm{t})$ are continuous functions on the interval $[0,b]$ and further applying fractional power series theorem involving fractional constant order derivatives leads to

$$\mathrm{y}(\mathrm{t}+1)=\mathrm{y}(\mathrm{t})+\frac{h^{\alpha}}{\alpha}D^{\alpha}\mathrm{y}(\mathrm{t})+\frac{h^{2\alpha}}{2!\alpha^{2}}D^{2\alpha}\mathrm{y}(\mathrm{t})+...+\frac{h^{n\alpha}}{n!\alpha^{n}}D^{n\alpha}\mathrm{y}(\mathrm{t}).\tag{2}$$

We ignoring the higher terms involving h^{α} or higher for small step sizes, and by replacing the value of $D^{\alpha}\mathrm{y}(\mathrm{t})$ in Eq.(2), we obtain

$$\mathrm{y}(\mathrm{t}+h)=\mathrm{y}(\mathrm{t})+\frac{h^{\alpha}}{\alpha}D^{\alpha}\mathrm{y}(\mathrm{t}),$$

$$\mathrm{y}(\mathrm{t}+h)=\mathrm{y}(\mathrm{t})+\frac{h^{\alpha}}{\alpha}\mathrm{f}(\mathrm{t},\mathrm{y}(\mathrm{t})).$$

obtain the following iterative formula

$$\mathrm{y_{n+1}}=\mathrm{y_n}+\frac{h^{\alpha}}{\alpha}\mathrm{f}(\mathrm{t_n},\mathrm{y_n}).\tag{3}$$

If $\alpha=1$, α- differentiable Euler methods constant order reduced to standard Euler method

4 α-differentiable Runge-Kutta 2-stage method of constant order

Consider constant orders fractional derivative with initial value problem in Eq. (1)

$$\begin{array}{l}\frac{d^{\alpha}\mathrm{y(t)}}{d\mathrm{t}^{\alpha}} = \mathrm{f}(\mathrm{t},\, \mathrm{y(t)}), \quad (o < \alpha \leq 1)\\ \mathrm{y}(0) = \mathrm{y}_0.\end{array}$$

the fractional power series formula is

$$\mathrm{y}(\mathrm{t}+h) = \mathrm{y(t)} + \frac{h^{\alpha}}{\alpha} D^{\alpha}\mathrm{y(t)} + \frac{h^{2\alpha}}{2!\alpha^2} D^{2\alpha}\mathrm{y(t)} + \dots + \frac{h^{\mathrm{n}\alpha}}{\mathrm{n}!\alpha^{\mathrm{n}}} D^{\mathrm{n}\alpha}\mathrm{y(t)}, \tag{4}$$

where

$$D^{2\alpha}\mathrm{y(t)} = D_{\mathrm{t}}^{\alpha}\mathrm{f(t,y)} + \mathrm{f(t,y)} D_{\mathrm{y}}^{\alpha}\mathrm{f(t,y)}.$$

Therefor Eq.(4) becomes

$$\mathrm{y}(\mathrm{t}+h) = \mathrm{y(t)} + \frac{h^{\alpha}}{\alpha} D^{\alpha}\mathrm{y(t)} + \frac{h^{2\alpha}}{2!\alpha^2}[D_{\mathrm{t}}^{\alpha}\mathrm{f(t,y)} + \mathrm{f(t,y)} D_{\mathrm{y}}^{\alpha}\mathrm{f(t,y)}], \tag{5}$$

Rearranging the previous Eq.(5)

$$\mathrm{y}(\mathrm{t}+h) = \mathrm{y(t)} + \frac{h^{\alpha}}{\alpha} D^{\alpha}\mathrm{y(t)} + \frac{h^{\alpha}}{2\alpha}\left\{\mathrm{f(t,y)} + \frac{h^{\alpha}}{\alpha} D_{\mathrm{t}}^{\alpha}\mathrm{f(t,y)} + \frac{h^{\alpha}}{\alpha}\mathrm{f(t,y)} D_{\mathrm{y}}^{\alpha}\mathrm{f(t,y)}\right\}, \tag{6}$$

It can also be written

$$\mathrm{y}(\mathrm{t}+h) = \mathrm{y(t)} + \frac{h^{\alpha}}{\alpha} D^{\alpha}\mathrm{y(t)} + \frac{h^{\alpha}}{2\alpha}\mathrm{f}\left(\mathrm{t} + \frac{h^{\alpha}}{\alpha},\, \mathrm{y(t)} + \frac{h^{\alpha}}{\alpha}\mathrm{f(t,y)}\right). \tag{7}$$

The following formula defines the α-differentiable Runge-Kutta 2-stage method of constant order, It depends on the expression given above

$$\begin{array}{l}\mathrm{y}_{\mathrm{n}+1} = \mathrm{y}_{\mathrm{n}} + \frac{h^{\alpha}}{2\alpha}\{k_1 + k_2\},\\ k_1 = \mathrm{f(t,y)},\\ k_2 = \mathrm{f}\left(\mathrm{t}_{\mathrm{n}} + \frac{h^{\alpha}}{\alpha},\, \mathrm{y}_{\mathrm{n}} + \frac{h^{\alpha}}{\alpha}\mathrm{f}(\mathrm{t}_{\mathrm{n}}, \mathrm{y}_{\mathrm{n}})\right).\end{array}$$

If α=1, then the conformable Runge-Kutta 2-stage method of constant order changed to standard Runge-Kutta second order method.

5 Examples

Two distinct models are analyzed, and the validation outcomes of the exact solution and constant order are compared, in order to introduce the α-differentiable Euler method of constant orders and α-differentiable RungeKutta 2-stage method of constant order.

*Example 1.*Solve the following constant order of fractional differential equation using α-differentiable Euler methods constant order:

$$\begin{array}{l}\frac{d^{\alpha}\mathrm{y(t)}}{d\mathrm{t}^{\alpha}} = -\mathrm{y(t)}, \quad (o < \alpha \leq 1)\\ \mathrm{y}(0) = 1.\end{array} \tag{8}$$

where $t \in (0,1)$ and step $h = 0.1$
Using the α-differentiable Euler methods constant order; the iterative relation for Eq. (8) to obtain

$$\mathrm{y}_{\mathrm{n}+1} = \mathrm{y}_{\mathrm{n}} + \frac{h^{\alpha}}{\alpha}\mathrm{f}(\mathrm{t}_{\mathrm{n}}, \mathrm{y}_{\mathrm{n}})$$

$$\mathrm{f}(\mathrm{t}_{\mathrm{n}}, \mathrm{y}_{\mathrm{n}}) = -\mathrm{y}_{\mathrm{n}}.$$

This solution given the following table-1
The exact solution in this problem is

$$\begin{array}{l}\frac{d^{\alpha}\mathrm{y(t)}}{d\mathrm{t}^{\alpha}} = -\mathrm{y(t)}, \quad (o < \alpha \leq 1)\\ \mathrm{y}(0) = 1.\end{array}$$

using the conformable fractional derivative formula is

$$t^{1-\alpha}\frac{dy}{dt} = -y$$

$$I.F = e^{\int \frac{1}{t^{1-\alpha}}dt} = e^{\frac{t^{\alpha}}{\alpha}}$$

The solution is

$$e^{\frac{t^{\alpha}}{\alpha}}y(t) = c$$

using the intial condition $c = 1$
Therefore

$$y(t) = e^{\frac{-t^{\alpha}}{\alpha}}$$

If $\alpha = 1$ then the exact solution is

$$y(t) = e^{-t}$$

This exact solution given in the following table-2

Table 1: Results of Example-1 for $\alpha = 0.86$ and step $h = 0.1$.

t	Exact y	Approximate y	Error
0	1	1	0
0.1	0.8517094	0.83949	0.0122194
0.2	0.7472679	0.704744	0.0425239
0.3	0.6617404	0.591626	0.0701144
0.4	0.589324	0.496664	0.09266
0.5	0.52695374	0.416944	0.18999026
0.6	0.4726517	0.350021	0.1226307
0.7	0.425015	0.293839	0.131176
0.8	0.3829879	0.246675	0.1363129
0.9	0.345741	0.207081	0.13866
1	0.31261255	0.173843	0.138769

*Example 2.*Solve the following constant order of fractional differential equation using α-differentiable Runge-Kutta 2-stage method of constant order:

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} = -y(t), \quad (o < \alpha \leq 1) \qquad y(0) = 1. \tag{9}$$

Table 2: Results of Example-1 for $\alpha = 1$ and step $h = 0.1$.

t	Exact y	Approximate y	Error
0	1	1	0
0.1	0.904837	0.9	0.00837418
0.2	0.818730753	0.81	0.00873075
0.3	0.74081822	0.729	0.011818
0.4	0.6703200046	0.6567	0.01422004
0.5	0.60653065	0.59049	0.01604065
0.6	0.5488116361	0.531441	0.0173706
0.7	0.49658530	0.478297	0.01828830
0.8	0.4493289641	0.430467	0.01886196
0.9	0.40656965	0.38742	0.0191496
1	0.367879441	0.348678	0.01920144

where $t \in (0,1)$ and step $h = 0.1$.
Using the α-differentiable Runge-Kutta 2-stage method of constant order; The iterative relationship corresponding to Eq. (9) is

$$y_{n+1} = y_n + \frac{h^\alpha}{2\alpha}\{k_1 + k_2\},$$
$$k_1 = f(t,y),$$
$$k_2 = f\left(t_n + \frac{h^\alpha}{\alpha}, y_n + \frac{h^\alpha}{\alpha} f(t_n, y_n)\right).$$

This solution given the following table-3
The exact solution in this problem is

$$\frac{d^\alpha y(t)}{dt^\alpha} = -y(t), \quad (o < \alpha \le 1)$$
$$y(0) = 1.$$

using the conformable fractional derivative formula is

$$t^{1-\alpha}\frac{dy}{dt} = -y$$

$$I.F = e^{\int \frac{1}{t^{1-\alpha}} dt} = e^{\frac{t^\alpha}{\alpha}}$$

The solution is

$$e^{\frac{t^\alpha}{\alpha}} y(t) = c$$

using the intial condition $c = 1$
Therefore

$$y(t) = e^{\frac{-t^\alpha}{\alpha}}$$

If $\alpha = 1$ then the exact solution is

$$y(t) = e^{-t}$$

This exact solution given in the following table-4

Table 3: Results of Example-2 for $\alpha = 0.85$ and step $h = 0.1$.

t	Exact y	Approximate y	Error
0	1	1	0
0.1	0.8517094	0.847627	0.00824
0.2	0.7472679	0.718472	0.028795
0.3	0.6617404	0.608996	0.0527444
0.4	0.589324	0.516202	0.073122
0.5	0.5269537	0.437547	0.0894067
0.6	0.4726517	0.370876	0.1017757
0.7	0.425015	0.314365	0.11065
0.8	0.3829879	0.266464	0.1165239
0.9	0.345741	0.225862	0.119879
1	0.31261255	0.191447	0.1211655

6 Application

Numerical method for constant orders Fractional Financial system
Consider the constant orders Fractional Financial model expressed by the equations, by Mohammed, et.al [14]

$$\begin{cases} {}_0D_t^\alpha x(t) = z(t) + (y(t) - a)x(t) \\ {}_0D_t^\alpha y(t) = 1 - by(t) - x^2(t) \\ {}_0D_t^\alpha z(t) = -x(t) - cz(t) \end{cases} \tag{10}$$

Table 4: Results of Example-2 for $\alpha = 1$ and step $h = 0.1$.

t	Exact y	Approximate y	Error
0	1	1	0
0.1	0.904837	0.905	0.00916
0.2	0.818730753	0.819025	0.00294247
0.3	0.74081822	0.741218	0.0039978
0.4	0.670320046	0.670802	0.00481954
0.5	0.60653065	0.607076	0.005923639
0.6	0.5488116361	0.549404	0.005923639
0.7	0.49658530	0.49721	0.006247
0.8	0.4493289641	0.449975	0.006460359
0.9	0.40656965	0.407288	0.0065835
1	0.3678794412	0.368541	0.006615588

conditions are

$$x(0) = 2,\ y(0) = -1,\ \text{and}\ z(0) = 1 \tag{11}$$

and values are $a = 1$, $b = 0.1$, and $c = 1$. where
$x(t)$: Interest rate.
$y(t)$: Demand for investment.
$z(t)$: Price index.

Solution: The numerical solution of the constant orders fractional financial system using the α-differentiable Euler methods constant order with $h = 0.01$; the iterative Eq. (10) reflected as

$$\begin{aligned} x_{n+1} &= x_n + \frac{h^\alpha}{\alpha}(z_n + (y_n - a)x_n), \\ y_{n+1} &= y_n + \frac{h^\alpha}{\alpha}(1 - by_n - x_n^2), \\ z_{n+1} &= z_n + \frac{h^\alpha}{\alpha}(-x_n - cz_n). \end{aligned} \tag{12}$$

The numerical solution to Eq.(12) for various values of fractional constant order and standard order using the α-differentiable Euler methods constant order is expressed in the following figures.

The numerical solution of the constant order fractional financial system using the α-differentiable Runge-Kutta 2-stage

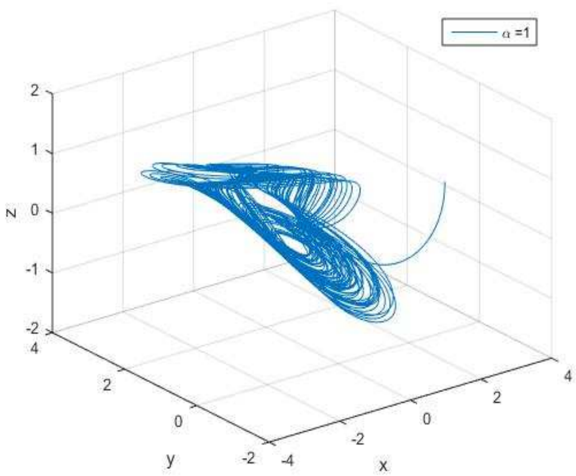

Fig. 1: The α-differentiable Euler methods constant order for fractional financial system of exact solutions $\alpha = 1$.

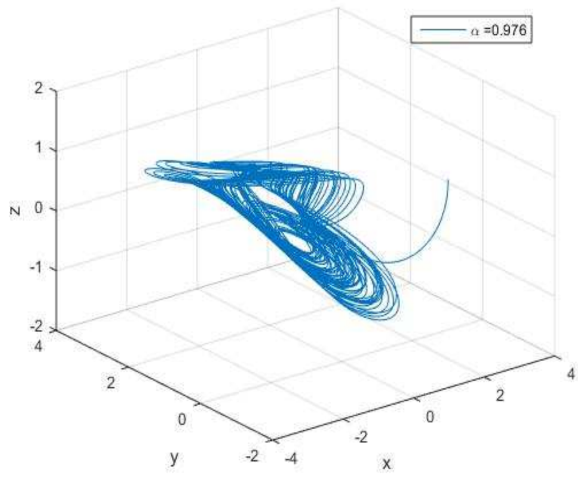

Fig. 2: The Euler method of the constant order fractional financial system of $\alpha = 0.976$.

method of constant order with $h = 0.01$; the iterative Eq.(10) represented as

$$\begin{array}{l} x_{n+1} = x_n + \frac{h^\alpha}{2\alpha}(k_1 + k_2), \\ y_{n+1} = y_n + \frac{h^\alpha}{2\alpha}(l_1 + l_2), \\ z_{n+1} = z_n + \frac{h^\alpha}{2\alpha}(m_1 + m_2). \end{array} \tag{13}$$

where

$$\begin{array}{l} k_1 = f(t_n, x_n, y_n, z_n), \\ l_1 = g(t_n, x_n, y_n, z_n), \\ m_1 = h(t_n, x_n, y_n, z_n), \\ k_2 = f\left(t_n + \frac{h^\alpha}{\alpha}, x_n + \frac{h^\alpha}{\alpha}k_1, y_n + \frac{h^\alpha}{\alpha}l_1, z_n + \frac{h^\alpha}{\alpha}m_1\right), \\ l_2 = g\left(t_n + \frac{h^\alpha}{\alpha}, x_n + \frac{h^\alpha}{\alpha}k_1, y_n + \frac{h^\alpha}{\alpha}l_1, z_n + \frac{h^\alpha}{\alpha}m_1\right), \\ m_2 = h\left(t_n + \frac{h^\alpha}{\alpha}, x_n + \frac{h^\alpha}{\alpha}k_1, y_n + \frac{h^\alpha}{\alpha}l_1, z_n + \frac{h^\alpha}{\alpha}m_1\right). \end{array} \tag{14}$$

here

$$\begin{array}{l} f(t_n, x_n, y_n, z_n) = (z_n + (y_n - a)\,x_n), \\ g(t_n, x_n, y_n, z_n) = \left(1 - by_n - x_n^2\right), \\ h(t_n, x_n, y_n, z_n) = (-x_n - cz_n). \end{array} \tag{15}$$

The numerical solution to Eq.(14) for various value of fractional constant order, and standard order using the α-differentiable Runge-Kutta 2-stage method of constant order is expressed in the following figures.

7 Results and conclusion

This research's main goal is to develop the numerical scheme needed to solve fractional differential equation of constant order. The fractional numerical method was used to achieved the objective. The two new constant order fractional numerical methods are applied to achieve the goal, one is α-differentiable Rung-kutta 2-stage methods constant order and other is α-differentiable Euler methods constant order. The fractional power series theorem is applied to present the derivation of these two methods. Example-1 shows the constant order of a fractional differential equation. The solution is determined by comparing it with the approximate solutions of fractional differential equations. Table-1 and 2, however, show these results along with the computation of error. Example-2 goes on to illustrate the constant order of a fractional differential equation and how it is handled by α-differentiable Runge-kutta 2-stage constant order method. Furthermore examined are the chaotic dynamics of a constant order fractional financial system. Figures 1-4 shows that the constant order derivation is a valuable tool for explaining chaotic phenomena, as demonstrated by the outcomes of the Euler and Runge-kutta 2-stage method.

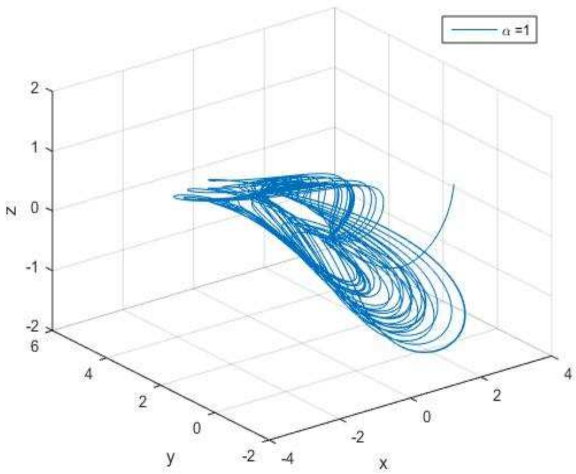

Fig. 3: The α-differentiable Runge-Kutta 2-stage method for fractional financial system of standard order $\alpha = 1$.

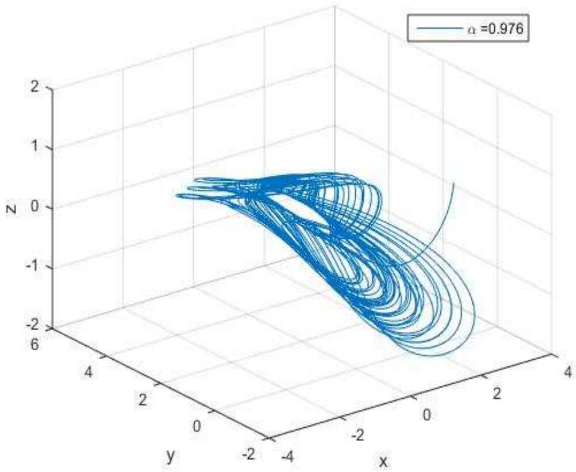

Fig. 4: The α-differentiable Runge-Kutta 2-stage method for fractional financial system of constant order $\alpha = 0.976$.

Declarations

Competing interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Authors' contributions: All the authors have equally contributed to complete the manuscript, i.e. PJ has formulated the problem, simulated the numerical results and completed the draft, AKP wrote and finalized the manuscript, making necessary correction.
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