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Wrapped Generalized Akash Distribution: Properties and Applications

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Abstract: Circular probability distributions are the most appropriate statistical tools to model directional data. In this paper, we introduce a new circular distribution based on the wrapping method and it is named as wrapped generalized Akash distribution. We also studied some important properties of the proposed probability distribution such as characteristic function, trigonometric moments, invariance properties etc. Further we have applied maximum likelihood procedure for the estimation of the parameters of the distribution. The proposed model is applied to real- life dataset and studied its suitability to compare with other similar circular probability distributions in the literature.

Keywords: Circular Statistics; Wrapped distribution; Generalized Akash distribution; Trigonometric moments.

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

In many scientific domains, observations take the form of 'directions'. Such data are used to describe the orientation or movement of an object or phenomenon in space. For instance, ecologists would be interested in examining a bird's flight path, an animal's orientation or the orientation of plant growth, while a geologist might be more interested in determining the Earth's magnetic pole's direction[6]. In epidemiology, angular data reveal the spread patterns of certain diseases which in turn gives us the trend of infection outbreaks. In medicine, directional data can be used to assess variation in the onset of certain diseases, such as leukemia. Directional data can be used to analyze the movements of ocean currents in Oceanography and to investigate wind directions in meteorology[6]. Given the circular shape of the data makes linear analytic approaches inadequate. To effectively examine and assess circular data, specialized statistical techniques have been designed. Circular distributions have been produced using a variety of techniques, involving the application of circular distributions tailored to encapsulate the periodicity inherent in angular data. One of the most widely used technique is wrapping a known distribution around the unit circle. This process ensures that the distribution remains circular. Any linear random variable X on the real line may be remodeled to a circular random variable θ by reducing its modulo 2π i.e., we define

$$\theta = X(mod\ 2\pi) \tag{1}$$

Taking the real line and wrapping it around the circle of unit radius corresponds to this operation. Then by taking the probability over all the overlapping points $X = \theta,\ \theta \pm 2\pi,\ \theta \pm 4\pi, \ldots$ Since this is clearly a many-to-one mapping, the probability density function and cumulative distribution function for $\theta \in [0, 2\pi)$ are given by, respectively, if g(θ) represents the circular density and $f(X)$ the density of the real-valued random variable.

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2m\pi) \tag{2}$$

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$$G(\theta)=\sum_{m=0}^{\infty}[F(\theta+2m\pi)-F(2m\pi)] \tag{3}$$

2 Wrapped Generalized Akash Distribution.

There are numerous continuous distributions that have been proposed for analyzing lifetime data. Because of its mathematical properties, the most widely used distributions is the Lindley distribution [9]. Shanker [12] proposed a novel distribution that is more flexible than the Lindley distribution. There are also many wrapped distributions in the literature. For example, Al-khazaleh and Alkhazaleh [1] proposed a new wrapped Quasi-Lindley distribution. The Akash distribution is a mixture of two distributions: an exponential distribution with scale parameter λ and a gamma distribution with shape parameter 3 and scale parameter λ and their mixing proportion is $\frac{\lambda^2}{\lambda^2+2}$ .The probability density function and cumulative distribution function of Akash distribution are given by

$$f(x,\theta)=\frac{\theta^3}{(\theta^2+2)}(1+x^2)e^{-\theta x}\;;x>0,\,\theta>0 \tag{4}$$

$$F(x,\theta)=1-\left[1+\frac{(\theta x+2)\theta x}{\theta^2+2}\right]e^{-\theta x}\;;x>0,\;\theta>0 \tag{5}$$

The two-parameter generalized Akash distribution with parameters θ and α which is a blend of exponential (θ) and gamma (3, θ) distributions was first described by Shankar et.al [13]. The Generalized Akash distribution's probability density function and cumulative distribution function are provided by

$$f(x,\theta,\alpha)=\frac{\theta^3}{(\theta^2+2\alpha)}(1+\alpha x^2)e^{-\theta x};x>0,\;\theta>0,\;\alpha>0 \tag{6}$$

$$F(x,\theta,\alpha)=1-\left[1+\frac{(\theta x+2)\theta\alpha x}{\theta^2+2\alpha}\right]e^{-\theta x};x>0,\;\theta>0,\;\alpha>0 \tag{7}$$

Assuming X to be a generalized Akash random variable, $\theta = X \pmod{2\pi}$ is the circular random variable produced by X and its probability density function g (θ; λ) provided by

$$g(\theta,\lambda,\alpha)=\sum_{m=0}^{\infty}f(\theta+2m\pi)$$

$$=\sum_{m=0}^{\infty}\frac{\lambda^3}{\lambda^2+2\alpha}(1+\alpha(\theta+2m\pi))^2)e^{-\lambda(\theta+2m\pi)}$$

$$=\frac{\lambda^3}{\lambda^2+2\alpha}e^{-\lambda\theta}\sum_{m=0}^{\infty}(1+\alpha(\theta+2m\pi))^2)e^{-\lambda 2m\pi}$$

$$g(\theta,\lambda,\alpha)=\frac{\lambda^3}{\lambda^2+2\alpha}\frac{e^{-\lambda(\theta-2\pi)}\left[(1+\alpha\theta^2)\Lambda^2+4\pi\alpha(\pi(\Lambda+2)+\theta\Lambda)\right]}{\Lambda^3} \tag{8}$$

where $\Lambda=(e^{2\pi\lambda}-1)$ and $\lambda,\alpha>0, 0\leq\theta\leq 2\pi$

Now the random variable θ is said to be wrapped generalized Akash distribution denoted by WGA(θ,λ,α) i.e; $\theta\sim$ WGA(θ,λ,α).

Let g (θ) be the pdf of WGA distribution. The cdf of the WGA distribution is obtained as follows

$$G(\theta)=\int_0^{\theta}g(t)\,dt \tag{9}$$

$$G(\theta,\lambda,\alpha)=\frac{\lambda^3e^{2\pi\lambda}}{(\lambda^2+2\alpha)\Lambda^3}\left[\Lambda^2\left(\frac{e^{-\lambda\theta}(\lambda^2-\alpha\theta^2-2\alpha\lambda\theta-2\alpha)-\lambda^2+2}{\lambda^3}\right)+4\pi\alpha\Lambda\left(\frac{1-e^{-\lambda\theta}}{\lambda}\right)+4\pi\alpha\Lambda\left(\frac{1-e^{-\lambda\theta}(\lambda\theta+1)}{\lambda^2}\right)\right]$$

$$=\frac{\lambda^3e^{2\pi\lambda}}{(\lambda^2+2\alpha)\Lambda^3\lambda^3}\left[\Lambda^2\left(e^{-\lambda\theta}(\lambda^2-\alpha\theta^2-2\alpha\lambda\theta-2\alpha)-\lambda^2+2\right)+4\pi\alpha\Lambda\lambda^2\left(1-e^{-\lambda\theta}\right)+4\pi\alpha\Lambda\lambda\left(1-e^{-\lambda\theta}(\lambda\theta+1)\right)\right] \tag{10}$$

The following figure represents plots of the PDF of the wrapped Generalized Akash distribution for different parameters of λ and α.

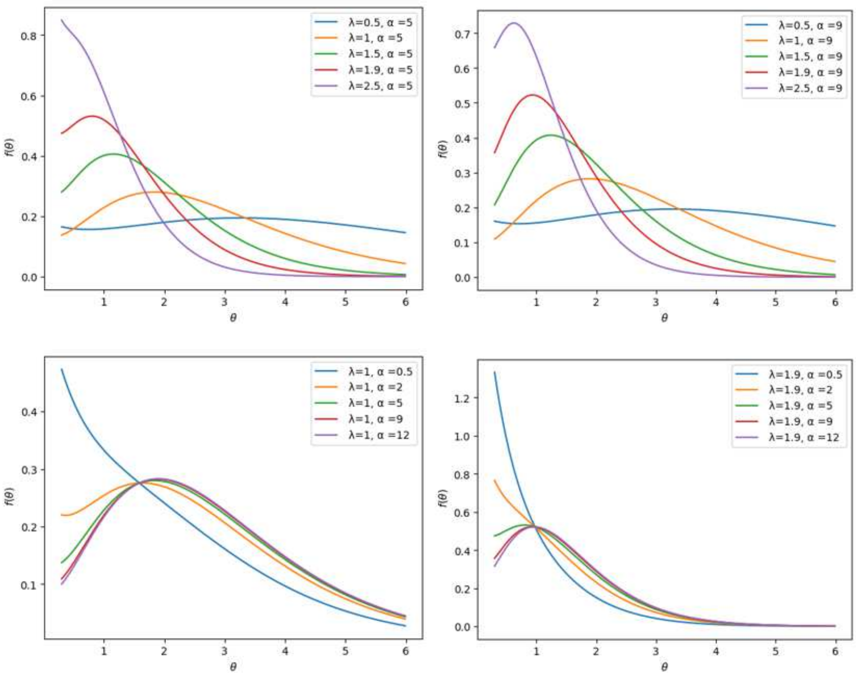

Fig. 1: Graphical representations of the pdf of the Wrapped Generalized Akash distribution with varying parameter values .

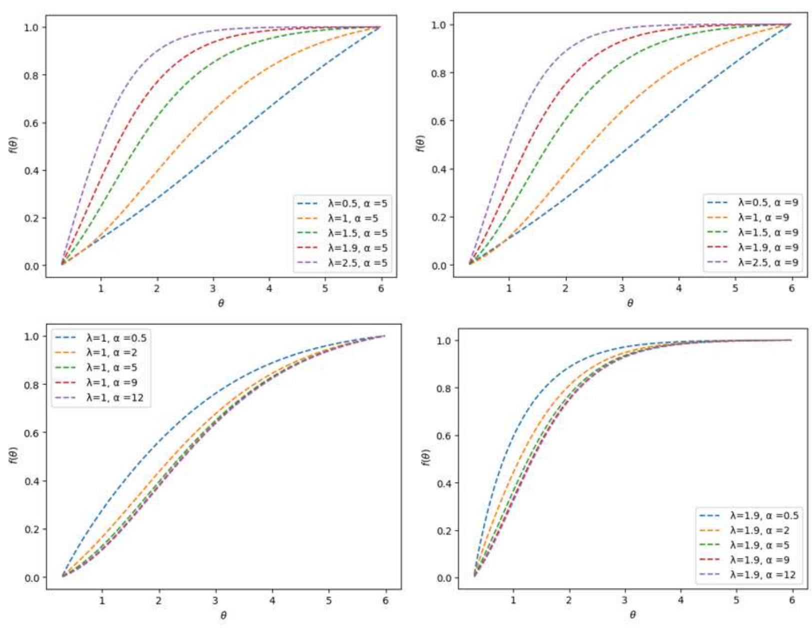

Fig. 2: Graphical representations of the cdf of the Wrapped Generalized Akash distribution with varying parameter values .

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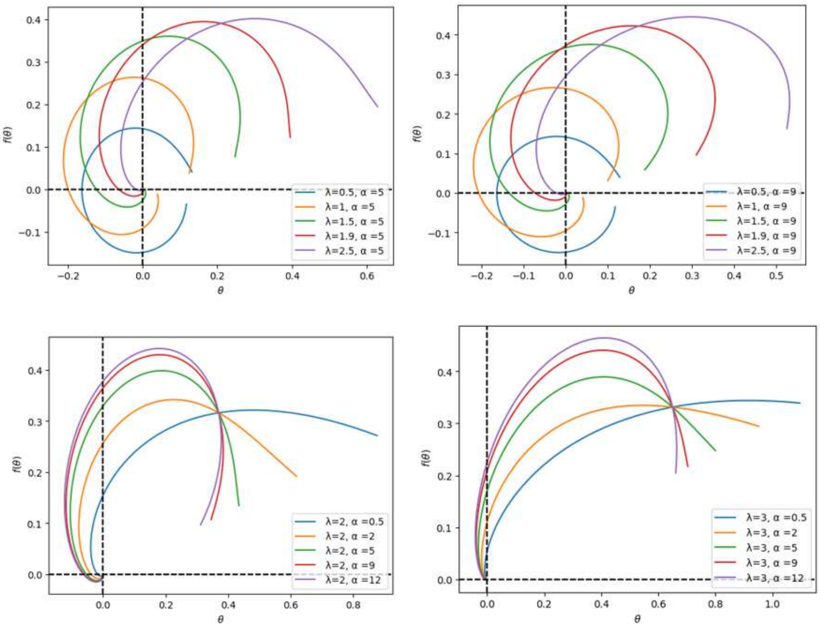

Fig. 3: Circular representations of the pdf of the Wrapped Generalized Akash distribution with varying parameter values .

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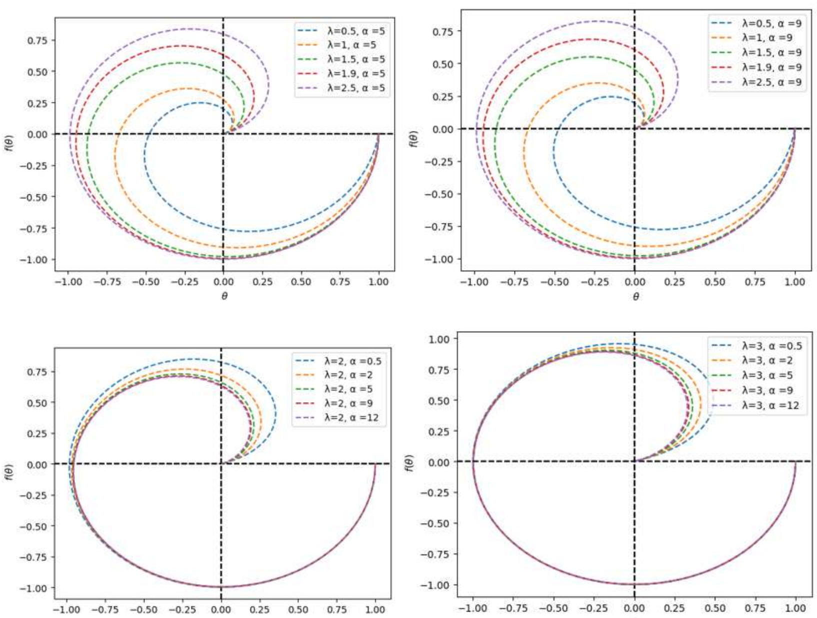

Fig. 4: Circular representations of the cdf of the Wrapped Generalized Akash distribution with varying parameter values .

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3 Mathematical Properties

3.1 Characteristic Function

The characteristic function of the wrapped generalized Akash distribution can be obtained as follows

$$\Phi_X(t) = E(e^{itX}) = \int_0^\infty e^{itx} f(x)\,dx \tag{11}$$

$$\Phi_X(t) = \frac{\lambda^3\left[\lambda^3(1+2\alpha)+t^2(2\lambda^3-6\alpha\lambda)+t^4\lambda\right]}{(\lambda^2+2\alpha)(\lambda^2+t^2)^3} + i\frac{\lambda^3\left[t(\lambda^2+6\alpha\lambda^2)+t^3(1-2\alpha)\right]}{(\lambda^2+2\alpha)(\lambda^2+t^2)^3} \tag{12}$$

where $i=\sqrt{-1}$.This is the characteristic function of the wrapped generalized Akash distribution.

The wrapped probability distribution's characteristic function was articulated by Jammalamadaka and Sengupta [6]. This function can be obtained by utilizing the characteristic function of the appropriate base probability distribution. i.e; $\Phi_X(t) = \Phi_\theta(p)$
Therefore, the characteristic of wrapped generalized Akash distribution can be obtained by using the above equation as follows.

$$\Phi_\Theta(p) = E(e^{ip\Theta}) = E[\cos(p\Theta)+i\sin(p\Theta)] \tag{13}$$

$$\Phi_\Theta(p) = \frac{\lambda^3\left[\lambda^3(1+2\alpha)+p^2(2\lambda^3-6\alpha\lambda)+p^4\lambda\right]}{(\lambda^2+2\alpha)(\lambda^2+p^2)^3} + i\frac{\lambda^3\left[p(\lambda^2+6\alpha\lambda^2)+p^3(1-2\alpha)\right]}{(\lambda^2+2\alpha)(\lambda^2+p)^3} \tag{14}$$

where p must be restricted to integer values, i.e., $p=\pm1,\pm2,\ldots$.

3.2 Trigonometric Moments and other parameters

In the statistical analysis of circular data, trigonometric moments are crucial tools that offer important critical metrics for describing measures like mean directions and angular concentration, skewness, kurtosis, etc. These moments are fundamental building blocks for analyzing the distributional characteristics that circular datasets inherently possess.
Let $\Theta \sim \text{WGA}(\theta,\lambda,\alpha)$. Consider the p^{th} trigonometric moment (Φ_p), which is the value of the characteristic function at an integer value p. This can also be defined in terms of the p^{th} cosine moment, i.e., $\alpha_p = E(\cos(p\Theta))$, and the p^{th} sine moment, i.e., $\beta_p = E(\sin(p\Theta))$. Therefore, we have

$$\Phi_p = \Phi_\Theta(p) = \alpha_p + i\beta_p, \quad p = 0,\pm1,\pm2,\ldots \tag{15}$$

Hence the p^{th} cosine moment of the $\text{WGA}(\theta,\lambda,\alpha)$ distribution is

$$\alpha_p = \frac{\lambda^3\left[\lambda^3(1+2\alpha)+p^2(2\lambda^3-6\alpha\lambda)+p^4\lambda\right]}{(\lambda^2+2\alpha)(\lambda^2+p^2)^3} \tag{16}$$

The p^{th} sine moment is given by

$$\beta_p = \frac{\lambda^3\left[p(\lambda^2+6\alpha\lambda^2)+p^3(1-2\alpha)\right]}{(\lambda^2+2\alpha)(\lambda^2+p)^3} \tag{17}$$

where $p=0,\pm1,\pm2,\ldots$.

The p^{th} trigonometric moment of the $\text{WGA}(\theta,\lambda,\alpha)$ can be represented in

$$\Phi_p = \rho_p e^{i\mu_p}$$

where $\rho_p = \sqrt{\alpha_p^2+\beta_p^2}$ and $\mu_p = \text{atan}\left(\alpha_p\beta_p^{-1}\right)$

In particular, μ_1 and ρ_1 are called the mean direction and angular concentration, respectively. Here, $\text{atan}(\cdot)$ is the quadrant inverse tangent function and is defined as:

$$\text{atan}\left(\alpha_p\beta_p^{-1}\right)\left(\frac{y}{x}\right)=\begin{cases}\tan^{-1}\left(\frac{x}{y}\right), & \text{if } y>0 \text{ and } x\geq 0\\ \frac{\pi}{2}, & \text{if } y=0 \text{ and } x>0\\ \tan^{-1}\left(\frac{x}{y}\right)+\pi, & \text{if } y<0\\ \tan^{-1}\left(\frac{x}{y}\right)+2\pi, & \text{if } y\geq 0 \text{ and } x<0\\ \text{undefined}, & \text{if } y=0 \text{ and } x=0\end{cases}$$

Then the central trigonometric moments are :

$$\overline{\alpha}_p=\rho_p\cos(\mu_p-p\mu_1)$$

$$\overline{\beta}_p=\rho_p\sin(\mu_p-p\mu_1)$$

Table 1 shows the values of non-central and central trigonometric moments for $p=1,2$ and $\lambda=0.5,1,1.5,2,2.5$ when $\alpha=1,2$, respectively.

Parameter		**Non-central Trigonometric Moments**				**Central Trigonometric Moments**			
λ	α	α_1	α_2	β_1	β_2	$\overline{\alpha_1}$	$\overline{\alpha_2}$	$\overline{\beta_1}$	$\overline{\beta_2}$
0.5	1	-0.5013	-0.672	-0.017	-0.0271	0.0627	0.00174	0	-0.0014
	2	-0.6230	-0.0457	-0.0772	-0.0437	0.0778	0.00584	0	0.0007
1	1	0	0.125	0.008	0.0061	0.25	0.0077	0	0.0042
	2	-0.1	-0.0304	0.0666	0	0.269	-0.0294	0	-0.026
1.5	1	0.2290	0.3370	0.0473	0.0570	1.3870	0.1960	0	-0.04016
	2	0.2200	0.0235	0.2203	0.0738	1.5790	0.0764	0	-0.0596
2	1	0.264	0.32	0.0781	0.1223	3.1255	0.6445	0	-0.5678
	2	0.314	0.0781	0.381	0.1816	4.0180	0.5550	0	-0.6821
2.5	1	0.2389	0.2663	0.0942	0.1935	4.9958	0.8089	0	-1.85607
	2	0.3134	0.1146	0.5279	0.3104	6.9218	0.9092	0	-2.3643

Table 1: Trigonometric moments for wrapped Generalized Akash distribution for $p=1,2$ and $\lambda=0.5,1,1.5,2,2.5$ for $(\alpha=1,2)$

3.3 Mean Direction and Mean Resultant Length

In circular distribution, the mean direction of the distribution is given by μ_p at $p=1$. The mean resultant length is used to quantify the circular data's spread around the mean and is given by ρ_p at $p=1$. More data are concentrated towards the mean, the closer the mean resultant is to 1.

The mean direction and the mean resultant length for wrapped generalized Akash distribution are given by

$$\mu_1=\text{atan}\left(\alpha_1\beta_1^{-1}\right)=\text{atan}\left(\frac{\lambda^3(1+2\alpha)+2\lambda^3-6\alpha\lambda}{\lambda^2(1+6\alpha)+(1-2\alpha)}\right)$$

The angular concentration of WGA distribution is

$$\rho_1=\sqrt{\alpha_1^2+\beta_1^2}$$

$$=\frac{\lambda^3\sqrt{((\lambda^3(1+2\alpha)+(2\lambda^3-6\alpha\lambda)+\lambda)^2+(\lambda^3((\lambda^2+6\alpha\lambda^2)+(1-2\alpha))^2}}{(\lambda^2+2\alpha)(\lambda^2+1)^3} \tag{18}$$

Means	**Parameter**					
	λ	0.5	1	1.5	2	2.5
Direction μ	$\alpha = 1$	0.1515	0.1682	0.3628	0.5581	0.6587
	$\alpha = 2$	1.3266	1.0926	0.2769	0.6557	0.8376
Resultant Length ρ	$\alpha = 1$	0.0627	0.25	1.3870	3.1255	4.9958
	$\alpha = 2$	0.0778	0.2692	1.5790	4.0180	6.9218

Table 2: Means for wrapped Generalized Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$ and $\alpha = 1, 2$

The circular variance can be obtained from the above equation and is given by

$$V = 1 - \rho_1$$

$$= 1 - \left(\frac{\lambda^3 \sqrt{\left((\lambda^3(1+2\alpha) + (2\lambda^3 - 6\alpha\lambda) + \lambda)^2 + (\lambda^3 (\lambda^2 + 6\alpha\lambda^2) + (1-2\alpha))^2 \right)}}{(\lambda^2 + 2\alpha)(\lambda^2+1)^3} \right) \tag{19}$$

Using the above equation, we can obtain circular standard deviation as

$$\sigma = \sqrt{-2\ln\rho_1}$$

Measure of Variation	α	λ				
		0.5	1	1.5	2	2.5
Circular Variance V	1	0.9372	0.75	-0.3870	-2.1255	-3.9958
	2	0.9221	0.7307	-0.5790	-3.0180	-5.9218
Circular Standard Deviation σ	1	2.3533	1.6651	-0.6542	-2.2791	-3.2171
	2	2.2594	1.6199	-0.9135	-2.7815	-3.8693

Table 3: Measures of variation for wrapped Generalized Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$ and $\alpha = 1, 2$

3.4 Hazard Function

Mathematically, the reliability function can be obtained as the CDF's complement. The reliability function of wrapped generalized Akash distribution is provided by

$$R(\theta;\lambda;\alpha) = 1 - G(\theta;\lambda;\alpha)$$

$$= 1 - \frac{\lambda^3 e^{2\pi\lambda}}{(\lambda^2+2\alpha)\Lambda^3\lambda^3} \left[\Lambda^2 \left(e^{-\lambda\theta}(\lambda^2 - \alpha\theta^2 - 2\alpha\lambda\theta - 2\alpha) - \lambda^2 + 2\right) + 4\pi\alpha\Lambda\lambda^2(1-e^{-\lambda\theta}) + 4\pi\alpha\Lambda\lambda\left(1 - e^{-\lambda\theta}(\lambda\theta+1)\right)\right]$$

$$= \frac{(\lambda^2+2\alpha)\Lambda^3\lambda^3 - \lambda^3 e^{2\pi\lambda}\left[\Lambda^2\left(e^{-\lambda\theta}(\lambda^2-\alpha\theta^2-2\alpha\lambda\theta-2\alpha)-\lambda^2+2\right)+4\pi\alpha\Lambda\lambda^2(1-e^{-\lambda\theta})+4\pi\alpha\Lambda\lambda(1-e^{-\lambda\theta}(\lambda\theta+1))\right]}{(\lambda^2+2\alpha)\Lambda^3\lambda^3} \tag{20}$$

The wrapped generalized Akash distribution's hazard function is provided by

$$H(\theta;\lambda;\alpha) = \frac{g(\theta;\lambda;\alpha)}{R(\theta;\lambda;\alpha)}$$

$$= \frac{\lambda^6 e^{-\lambda(\theta-2\pi\lambda)}\left[(1+\alpha\theta^2)\Lambda^2+4\pi\alpha(\pi(\Lambda+2)+\theta\Lambda)\right]}{(\lambda^2+2\alpha)\Lambda^3\lambda^3-\lambda^3e^{2\pi\lambda}\left[\Lambda^2\left(e^{-\lambda\theta}(\lambda^2-\alpha\theta^2-2\alpha\lambda\theta-2\alpha)-\lambda^2+2\right)+4\pi\alpha\Lambda\lambda^2(1-e^{-\lambda\theta})+4\pi\alpha\Lambda\lambda(1-e^{-\lambda\theta}(\lambda\theta+1))\right]} \tag{21}$$

4 Invariance Properties

With a closed-form density and the unit circle serving as support, probability distributions for circular data frequently presuppose a generic structure. However, analyzing circular data requires considering specific features that may be overlooked in traditional analysis methods. Since circular data lacks a planned zero or end point, identifying the natural orientation is arbitrary, which is one of its main characteristics. Although tractable forms and well-known circular distributions are available, employing these distributions without taking into account the concerns of beginning direction and orientation can result in deceptive inference. Therefore, it is essential to address these specific features of circular data in any analysis. By doing so, researchers can ensure accurate and meaningful interpretation of the data. Circular distributions must possess two properties: invariance under changes of initial direction (ICID) and invariance under changes of system orientation (ICO). These properties are essential for inference that is independent of the reference system.

Every circular distribution with probability density function g (θ) is ICID and ICO if and only if for $\delta = \{-1,1\}$, $\xi \in \{2\pi j\}_{j=0}^{l-1}$ and $\theta^* = \delta(\theta+\xi)$ the probability densities functions g (θ) and g (θ^*) belong to the same parametric family (i.e., their functional form of their characteristic function must coincide). The value of the characteristic function that produces the wrapped distribution equals the p^{th} trigonometric moment for the wrapped distribution. Hence, another way to express the p^{th} trigonometric moment for the wrapped Generalized Akash distribution is as follows:

$$\varphi_\theta(p) = \varphi_X(p)$$

$$= \frac{\lambda^3}{\lambda^2+2\alpha}\left[\frac{1}{\lambda-ip}+\frac{2\alpha}{(\lambda-ip)^3}\right]$$

$$= \frac{\lambda^3}{\lambda^2+2\alpha}\left[\frac{\lambda^2-p^2-2ip\lambda+2\alpha}{(\lambda-ip)^3}\right]; \quad i=\sqrt{-1}, \quad p=\pm1,\pm2,\ldots \tag{22}$$

The characteristic function of the wrapped Generalized Akash distribution, denoted as θ^* is given by

$$\varphi_{\theta^*}(p) = e^{ip\delta\xi}\varphi_\theta(p\delta)$$

$$= \frac{\lambda^3(\cos(p\delta\xi)-i\sin(p\delta\xi))}{\lambda^2+2\alpha}\left[\frac{\lambda^2-p^2-2ip\delta\lambda+2\alpha}{(\lambda-ip\delta)^3}\right] \tag{23}$$

Since only when $\lambda = 1$ and $\xi = 0$ do the real and imaginary components of equations (22) and (23) equalize, the wrapped Generalized Akash distribution is not both ICID and ICO. The wrapped probability distribution's invariant form is provided by

$$g(\theta,\lambda,\alpha,\delta,\xi) = \frac{\lambda^3}{\lambda^2+2\alpha}\left[\frac{e^{2\pi\lambda}}{(e^{2\pi\lambda}-1)^3}\right]\left[(1+\alpha(\delta\theta^*-\xi)^2)(e^{2\pi\lambda}-1)^2+4\pi\alpha(\pi(e^{2\pi\lambda}-1+2)+(\delta\theta^*-\xi)(e^{2\pi\lambda}-1))\right] \tag{24}$$

$$= \frac{\lambda^3}{\lambda^2+2\alpha}\left[\frac{e^{2\pi\lambda}}{(e^{2\pi\lambda}-1)^3}\right]\left[(1+\alpha(\delta^2\theta^{*2}-2\delta\theta^*+\xi^2))(e^{2\pi\lambda}-1)^2+4\pi\alpha(\pi(e^{2\pi\lambda}+1)+(\delta\theta^*-\xi)(e^{2\pi\lambda}-1))\right] \tag{25}$$

5 Maximum Likelihood Estimate

In this section, we discuss the method of maximum likelihood estimation to estimate the parameters. Let $\theta_1,\theta_2,\ldots,\theta_n$ be a random sample of size n drawn from the WGA distribution with PDF (θ,λ,α) , then the log likelihood function denoted by "L" is given by

$$L = \prod_{i=1}^{n} g(\theta,\lambda,\alpha)$$

$$= \prod_{i=1}^{n}\left[\frac{\lambda^3}{\lambda^2+2\alpha}\frac{e^{-\lambda(\theta-2\pi)}}{\Lambda^3}\left[(1+\alpha\theta^2)\Lambda^2+4\pi\alpha(\pi(\Lambda+2)+\theta\Lambda)\right]\right]$$

$$= 3n\ln\lambda - n\ln(\lambda^2+2\alpha) - \lambda\sum_{i=1}^{n}\theta_i + 2n\pi\lambda - 3n\ln(e^{2\pi\lambda}-1) + \sum_{i=1}^{n}\ln\left[(1+\alpha\theta_i^2)(e^{2\pi\lambda}-1)^2 + 4\pi\alpha(\pi(e^{2\pi\lambda}+1)+\theta_i(e^{2\pi\lambda}-1))\right] \quad (26)$$

Regarding the parameter λ, the derivative of the log-likelihood function is

$$\frac{\partial \log L}{\partial\lambda} = \frac{3n}{\lambda} - \frac{2n\lambda}{\lambda^2+2\alpha} - \sum_{i=1}^{n}\theta_i + 2n\pi - \frac{3ne^{2\pi\lambda}}{e^{2\pi\lambda}-1} + \frac{\sum_{i=1}^{n}\left[8\pi^3 e^{2\pi\lambda}(e^{2\pi\lambda}+1)+8\pi^2\alpha e^{2\pi\lambda}(e^{2\pi\lambda}-1)\right]}{(1+\alpha\theta_i^2)(e^{2\pi\lambda}-1)^2+4\pi\alpha(\pi(e^{2\pi\lambda}+1)+\theta_i(e^{2\pi\lambda}-1))} \quad (27)$$

The log-likelihood function's partial derivative with respect to α is provided by

$$\frac{\partial \log L}{\partial\alpha} = \frac{2n}{2\alpha+\lambda^2} + \frac{\sum_{i=1}^{n}\left[\theta_i^2(e^{2\pi\lambda}-1)^2+4\pi\theta_i(e^{2\pi\lambda}-1)\right]}{(1+\alpha\theta_i^2)(e^{2\pi\lambda}-1)^2+4\pi\alpha(\pi(e^{2\pi\lambda}+1)+\theta_i(e^{2\pi\lambda}-1))} \quad (28)$$

Taking the above partial derivative and equating it to zero in terms of λ and α

$$\frac{\partial \log L}{\partial\lambda} = 0$$

$$\frac{\partial \log L}{\partial\alpha} = 0$$

Equating equation (27) and (28) to zero and solving with respect to λ and α gives the MLEs of λ and α.There is no explicit solution so we can use numerical methods to solve them.

6 Application

In this section, we fit the wrapped Generalized Akash distribution to the Fisher-B16 data set [5] and compare its performance with that of wrapped Akash distribution [2], wrapped Lindley distribution [7] and wrapped exponential distribution [5] by fitting these distributions. The data set is described as the Cross-bed measurements from Pakistani Himalayan molasse. The dataset is shown as the rose diagram that is given below.

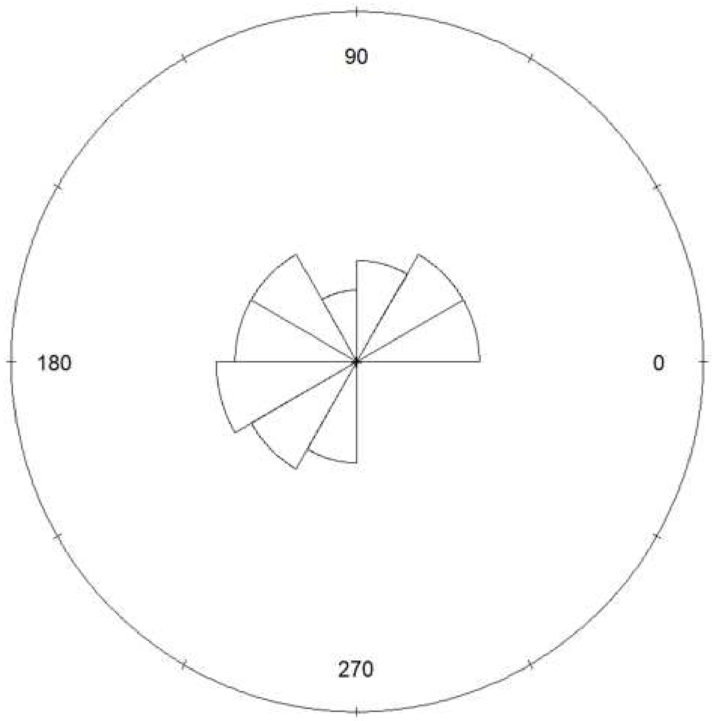

Fig. 5: Rose Diagram of Fisher B16 dataset

Table 4 provides the parameter's maximum likelihood estimates (MLEs) and accompanying standard errors (SE) for the Fisher-B16 data set.

Distribution	MLE	SE
Wrapped Generalized Akash	0.7542	0.1016
Wrapped Akash	0.6688	0.1301
Wrapped Lindley	0.4849	0.1415
Wrapped exponential	0.1995	0.1169

Table 4: MLEs and their SE for the selected distributions.

Based on -2 loglikelihood (-2L), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (K-S) and the corresponding p-values for the models, the goodness of fit of the wrapped Generalized Akash distribution to the Fisher-B16 data set is compared with wrapped Akash distribution, wrapped Lindley distribution and wrapped exponential distribution. Table 5 presents the findings. The wrapped Generalized Akash distribution is clearly the one with the lowest -2L, AIC, BIC, K-S (stat) values. Thus, the most appropriate distribution for fitting the Fisher-B16 dataset is the wrapped Generalized Akash distribution.

Distribution	-2L	AIC	BIC	AICC	K-S(stat)
Wrapped Generalized Akash	24.4219	28.422	30.7781	28.9934	0.84379
Wrapped Akash	85.3144	84.936	86.1140	85.1178	0.86209
Wrapped Lindley	83.5763	85.576	86.7540	85.7578	0.88624
Wrapped exponential	82.9367	87.192	88.3700	87.3738	0.88308

Table 5: -2L, AIC, BIC, K-S (stat) and K-S (p value) statistics for the fitted distributions.

7 Conclusion

A novel circular distribution derived from the Generalized Akash distribution is proposed. The wrapped Generalized Akash distribution's probability density function and cumulative density function were found to have clear expressions. Also, we have obtained the reliability function and hazard function. Moreover, we conducted a study on the trigonometric moments and the measures of variation of the new distribution to understand it better. The invariance properties of the distribution are investigated, invariance under change of initial direction, and the reference system orientation. The applicability of the new model is illustrated by the dataset and in contrast to other appropriate circular distributions, it provides a more flexible and well-fitting model. Crossbedding in Himalayan molasses and related sedimentary sequences must be measured and analyzed in order to comprehend Earth's past, previous environmental circumstances, and the processes that have sculpted the terrain over geological time. Therefore, this model has practical uses like resource exploration and hazard assessment as well as consequences for environmental studies and geological study.

Declarations

Competing interests: The authors declare that they have no competing interests.
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