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Strongly Topologically Transitive, Supermixing, And Hypermixing Maps On General Topological Spaces

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Abstract: We give some basic properties of strongly topologically transitive, supermixing, and hypermixing maps on general topological spaces. Then we present some other results for which our mappings need to be continuous.

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1 Introduction

Let X be a topological space. A map $f : X \to X$ is said to be *hypercyclic* (or orbit-transitive) if there is some $x \in X$ for which

$$\operatorname{orb}(x, f) = \{f^n x : n = 0, 1, 2. \cdots\}$$

is dense in X. Here $f^0 = id_X$, the identity map on X, and $f^n = f \circ f^{n-1}$ $(n \geq 1)$. Thus, it is clear that non-separable topological spaces cannot support hypercyclic maps. If $\overline{\operatorname{orb}(x, f)} = X$ then x is called a *hypercyclic point* for f. The set of all hypercyclic points for f is denoted by $HC(f)$ and it is easy to see that, if X has no isolated point and f is hypercyclic, then $HC(f)$ is dense in X. If $HC(f) = X$ then f is called *hypertransitive* (or minimal). For a map $f : X \to X$, a set $\emptyset \neq E \subsetneq X$ is said to be f-invariant (or invariant under f) whenever $f(E) \subseteq E$. It is easy to verify that if f is hypertransitive then f lacks closed invariant subsets. The converse is true if f is continuous.

A map $f : X \to X$ is called *topologically transitive* if, for each pair of nonempty open sets $U, V \subseteq X$, there is some $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. One can easily verify that if X has no isolated point then any hypercyclic map is topologically transitive. If, other than being empty of isolated points, X is a second countable Baire space then every continuous topologically transitive map is hypercyclic (see Theorem 1.2 of [6] and its following remark).

In [5,2,4], the concept of *strong topological transitivity* has been introduced and investigated for continuous linear operators on topological vector spaces. We can consider it for general maps on general topological spaces as well. The authors in [1, Definition 2.7] have defined the notion of *strong transitivity* as follows: A map $f : X \to X$ is called *strongly transitive* if $\bigcup_{n=0}^{\infty} f^n(U) = X$ for all nonempty open subsets U of X. However, in view of the following remark, we had better use the title "strong topological transitivity" for that notion. Since a map $f : X \to X$ is topologically transitive if and only if $\overline{\bigcup_{n=0}^{\infty} f^n(U)} = X$ for every nonempty open subset U of X, we see that strong topological transitivity implies topological transitivity.

Remark. In [9, Definition 1], the finite union is considered to define the notion of strong transitivity, i.e., a map $f : X \to X$ is called strongly transitive if, for any nonempty open set $U \subseteq X$, there exists a positive integer s such that $\bigcup_{n=0}^{s} f^n(U) = X$. This notion is called *very strongly transitive* in [1, Definition 2.7].

A map f is said to be *mixing* if, for each pair of nonempty open subsets U, V of X, there exists some $N \geq 0$ such that $V \cap f^n(U) \neq \emptyset$ for all integers $n \geq N$. Obviously, a mixing map is topologically transitive. It is clear that, in the definition

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of a mixing map, there is no guarantee that V intersects $A_N(U) = \bigcap_{n=N}^{\infty} f^n(U)$. Clearly, if V intersects $A_N(U)$, then it also intersects each $T^n(U)$ $(n \geq N)$.

So, the property that, for any pair U, V of nonempty open subsets of X, there exists some $N \geq 0$ such that $V \cap A_N(U) \neq \emptyset$ would be stronger than the mixing property. It is equivalent to say that $\bigcup_{i=0}^{\infty} \bigcap_{n=i}^{\infty} f^n(U)$ is dense in X for each nonempty open subset U of X.

Recently, the authors in [3] have introduced and investigated the notions of supermixing and hypermixing for continuous linear operators on topological vector spaces. We decided to consider these concepts for general maps supported by general topological spaces.

Definition 1. *Let X be a topological space. A map $f : X \to X$ is called supermixing if, for each nonempty open set $U \subseteq X$,*

$$\overline{\bigcup_{i=0}^{\infty} \bigcap_{n=i}^{\infty} f^n(U)} = X.$$

We say that f is hypermixing if, for every nonempty open subset U of X, we have

$$\bigcup_{i=0}^{\infty} \bigcap_{n=i}^{\infty} f^n(U) = X.$$

It is clear that any hypermixing map is supermixing and all supermixing maps are mixing. Meanwhile, every hypermixing map is obviously strongly topologically transitive. The following example may be helpful when we want to think about the possible implications between the mentioned dynamical properties.

Example 1. (A) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$ be a topology on X.
(A_1) If $f : X \to X$ is the constant map $f(X) = \{c\}$ then it is easy to see that f is strongly topologically transitive but not mixing.

(A_2) Let f be the function on X for which $f(a) = f(b) = b$ and $f(c) = a$. Then f is mixing but not strongly topologically transitive.

(A_3) If we define $f : X \to X$ by $f(a) = b, f(b) = c, f(c) = a$, then it is not difficult to show that f is mixing but not supermixing.

(A_4) Define the map f on X by $f(a) = a, f(b) = c, f(c) = b$. Then it can be easily seen that f is supermixing but not hypermixing.

(B) Let $X = \mathbb{Z}$ be equipped with the topology $\tau = \{\emptyset, X, X \backslash \{0\}\}$ and the map $f : X \to X$ be defined by $f(m) = m + 1$ $(m \in \mathbb{Z})$. One can readily verify that f is hypermixing.

In Section 2, we give some basic results concerning strong topological transitivity, supermixing, and hypermixing properties. In Section 3, we present some results involving other well-known dynamical properties for continuous maps.

2 Supermixing and hypermixing maps

The reader may have thought about the existence of a hypermixing map on the topological space given in Example 1 (A), but, while trying to find some, we felt that it is impossible. Then we proved the following statement.

Theorem 1. *Let X be a topological space. If there is a nontrivial finite open set U in X, then there is no hypermixing map on X. In particular, if X is a finite set, then, equipped with any nontrivial topology, X cannot support a hypermixing map.*

Proof. Assume that U is a nontrivial finite open subset of X. If there is some $p \geq 1$ for which $U \subseteq f^p(U)$ then we must have $U = f^p(U)$ since U is a finite set. Then

$$\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} f^n(U) = \bigcap_{n=0}^{p-1} f^n(U) \subseteq U \neq X.$$

If for all $p \geq 1$, there is some $x(p) \in U \backslash f^p(U)$ then $x(p) \notin \bigcap_{n=p}^{\infty} f^n(U)$. Since U is a finite set, there must be some $x \in U$ and a strictly increasing sequence of positive integers $(p_s)_s$ such that $x = x(p_s)$ for all $s = 1, 2, 3, \cdots$. Hence $x \notin \bigcap_{n=p_s}^{\infty} f^n(U)$ for all $s \geq 1$. Now, if $x \in \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} f^n(U)$ then there is some $q \geq 0$ such that $x \in \bigcap_{n=q}^{\infty} f^n(U)$. On the other hand, there is some $s_0 \geq 1$ for which $p_{s_0} > q$. Hence $\bigcap_{n=q}^{\infty} f^n(U) \subseteq \bigcap_{n=p_{s_0}}^{\infty} f^n(U)$ and so $x \in \bigcap_{n=p_{s_0}}^{\infty} f^n(U)$, a contradiction.

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A natural question which may often be asked whenever a new property concerning a dynamical behavior of a map f is introduced, is that whether or not the maps f^p $(p \geq 2)$ too have that property. The proof of the next result is exactly the same as that of Proposition 2.11 of [3], but, for the sake of the reader's convenience, we bring it.

Proposition 1. *If $f : X \to X$ is a supermixing (hypermixing) map then f^p is also supermixing (hypermixing) for all integers $p \geq 2$.*

Proof. Fix an integer $p \geq 2$. Suppose that U is any nonempty open subset of X. Then, for any $i \geq 0$, we have

$$\{i, i+1, i+2, \cdots\} \supseteq \{ip, (i+1)p, (i+2)p, \cdots\}.$$

Therefore,

$$\bigcap_{n=i}^{\infty} f^n(U) \subseteq \bigcap_{n=i}^{\infty} (f^p)^n(U),$$

for each $i \geq 0$. Hence

$$\bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty} f^n(U) \subseteq \bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty} (f^p)^n(U).$$

The following result shows that strongly topologically transitive maps are nearly always onto.

Proposition 2. *Let X be a topological space in which there is some nonempty open set U such that $\overline{U} \neq X$. Then every strongly topologically transitive map on X is onto.*

Proof. Suppose f is a strongly topologically transitive map on X. Let $x \in X$. If $x \notin U$ then $x \in f(X)$ since we have $\bigcup_{n=0}^{\infty} f^n(U) = X$. Now, suppose that $x \in U$. Since $\overline{U} \neq X$ there must be some nonempty open set V in X such that $U \cap V = \emptyset$. Thus, $x \notin V$ and the equality $\bigcup_{n=0}^{\infty} f^n(V) = X$ shows that $x \in f(X)$.

If f is a strongly topologically transitive *open* map then we can omit the condition of having a nondense nonempty open set in the above proposition. Indeed, for a given $x \in X$ and a fixed nonempty open set U, we replace U by the open set $f(U)$ in the equality $\bigcup_{n=0}^{\infty} f^n(U) = X$ to obtain $\bigcup_{n=1}^{\infty} f^n(U) = X$. Then it is clear that $f(X) = X$. Briefly, we can give the following result.

Proposition 3. *Every strongly topologically transitive open map is onto.*

Regrading the definitions of hypermixing and supermixing maps and the fact that $\bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty} f^n(X) \subseteq f(X)$, the following result looks obvious.

Proposition 4. *Every hypermixing map is onto and all supermixing maps have dense range.*

The following result says that nearly all of the hypermixing maps fail to be one-to-one.

Proposition 5. *Let X be a topological space in which there is some nonempty open set U such that $\overline{U} \neq X$. Then no hypermixing map on X is one-to-one.*

Proof. Let f be a hypermixing map on X and $V = X \setminus \overline{U}$. Then we have $\bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty} f^n(U) = X = \bigcup_{i=0}^{\infty}\bigcap_{n=i}^{\infty} f^n(V)$. Pick a point $x \in X$. Then there is a positive integer N, some $u \in U$, and some $v \in V$ such that $f^N(u) = x = f^N(v)$. Thus, f is not one-to-one since $u \neq v$.

3 Continuous maps

By proposition 4, every supermixing map has dense range. For continuous maps on compact Hausdorff spaces, the range is the whole space.

Proposition 6. *Every continuous supermixing map on a compact Hausdorff space is onto.*

Proof. Let f be a continuous supermixing map on a compact Hausdorff space X. Then, by Proposition 4, f has dense range. But $f(X)$ is compact and hence closed. Thus $f(X) = X$.

We devote the rest of our note to give some connections between the dynamical properties of continuous maps. Fix a continuous map $f : X \to X$. For a point $x \in X$, the set $J_f^{mix}(x) = J^{mix}(x)$ is defined by

$$J^{mix}(x) = \{y \in X : \exists (x_n)_n \text{ in } X \text{ such that } x_n \to x \text{ and } f^n x_n \to y\}.$$

It is known that $J^{mix}(x)$ is a closed f-invariant set, and that f is mixing if and only if $J^{mix}(x) = X$ for all $x \in X$, or equivalently, for all x in a dense subset of X [8, Exercise 1.4.4].

If, for a set $B \subseteq X$, we put $J^{mix}(B) = \bigcup_{x\in B} J^{mix}(x)$, then we give the following necessary and sufficient condition for a continuous hypercyclic map to be mixing.

Proposition 7. *Let X be a topological space without isolated points and $f : X \to X$ be a continuous hypercyclic map. Then the following are equivalent.*
(1) *f is mixing*
(2) $J^{mix}(HC(f)) \cap HC(f) \neq \emptyset$

Proof. (1) $\Rightarrow$ (2). Suppose f is mixing. Then $J^{mix}(x) = X$ for all $x \in X$. Thus $J^{mix}(HC(f)) \cap HC(f) = X \cap HC(f) = HC(f) \neq \emptyset$.

(2) $\Rightarrow$ (1). Let $y \in J^{mix}(HC(f)) \cap HC(f)$. Then there is a hypercyclic point x such that $y \in J^{mix}(x)$. Since $J^{mix}(x)$ is a closed f-invariant set we have $X = \overline{\text{orb}(y, f)} \subseteq \overline{J^{mix}(x)} = J^{mix}(x)$. If we show that $J^{mix}(f^k x) = X$ for all $k \geq 1$, then for all points z in the dense set $\text{orb}(x, f)$ we have $J^{mix}(z) = X$ which shows that f is mixing. To this end, fix $k \geq 1$. Since $y \in J^{mix}(x)$, it is easy to see that $f^k y \in J^{mix}(f^k x)$. Then, since $J^{mix}(f^k x)$ is f-invariant, we have $X = \overline{\text{orb}(f^k y, f)} \subseteq \overline{J^{mix}(f^k x)} = J^{mix}(f^k x)$ and we are done.

Corollary 1. *Let X be a topological space without isolated points and $f : X \to X$ be a continuous hypertransitive map. Then the following are equivalent.*
(1) *f is mixing*
(2) $J^{mix}(X) \neq \emptyset$
(3) $\overline{J^{mix}(X)} = X$.

Proof. (1) $\Rightarrow$ (2). If f is mixing then, by Proposition 7,

$$\emptyset \neq J^{mix}(HC(f)) \cap HC(f) = J^{mix}(X) \cap X = J^{mix}(X).$$

(2) $\Rightarrow$ (3). Suppose $J^{mix}(X) \neq \emptyset$. Since $\overline{J^{mix}(X)}$ is a closed f-invariant set and, meanwhile, f lacks closed invariant subsets (since f is hypertransitive), we must have $\overline{J^{mix}(X)} = X$.

(3) $\Rightarrow$ (1). Assume that $\overline{J^{mix}(X)} = X$. Then

$$J^{mix}(HC(f)) \cap HC(f) = J^{mix}(X) \cap X \neq \emptyset.$$

Hence f is mixing by Proposition 7.

Recall from the introduction that a continuous map is hypertransitive if and only if it does not admit closed invariant subsets.

Proposition 8. *Let X be a topological space and $f : X \to X$ be a continuous hypertransitive map. Then the following are equivalent.*
(1) *f is supermixing*
(2) *$\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} f^n(U) \neq \emptyset$, for any nonempty open set $U \subseteq X$.*

Proof. (1) $\Rightarrow$ (2). If f is supermixing then $\overline{\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} f^n(U)} = X$, for any nonempty open set $U \subseteq X$.

(2) $\Rightarrow$ (1). Let U be a nonempty open subset of X and put $A_k = \bigcap\limits_{n=k}^{\infty} f^n(U)$. We need to show that $\overline{\bigcup_{k=0}^{\infty} A_k} = X$. It is clear that $f(A_k) \subseteq A_{k+1}$ and meanwhile $\bigcup_{k=0}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k$ (since $A_0 \subseteq A_1$). Hence

$$f(\bigcup_{k=0}^{\infty} A_k) = \bigcup_{k=0}^{\infty} f(A_k) \subseteq \bigcup_{k=0}^{\infty} A_{k+1} = \bigcup_{k=0}^{\infty} A_k.$$

Since f is continuous, we have $f(\overline{\bigcup_{k=0}^{\infty} A_k}) \subseteq \overline{\bigcup_{k=0}^{\infty} A_k}$ which shows that $\overline{\bigcup_{k=0}^{\infty} A_k} = X$ because f is hypertransitive (and $\bigcup_{k=0}^{\infty} A_k \neq \emptyset$ by our assumption).

Regarding Proposition 7, Corollary 1, and Proposition 8, let us clarify that, for a continuous map, hypercyclicity (resp. hypertransitivity) by itself does not imply the mixing (resp. supermixing) property. To this end, let $X = \{a,b,c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$. Define $f : X \to X$ by $f(a) = b$, $f(b) = f(c) = a$. Then it is easy to verify that f is a continuous hypertransitive (and hence hypercyclic) map which is not supermixing (since it is not mixing).

If the curios reader is thinking about the possibility of the implication (supermixing)⇒ (hypertransitive) for continuous maps, this example would be helpful: let $X = \{a,b,c\}$ and $\tau_1 = \{\emptyset, X, \{a\}, \{a,b\}\}$ be a topology on X. Define $f : X \to X$ by $f(a) = a$, $f(b) = f(c) = c$. Then f is a continuous supermixing map which is not hypertransitive.

We finish this note by mentioning that the authors in [7] have given some results concerning strong topological transitivity, supermixing, and hypermixing properties for continuous maps on metric spaces. They in particular have offered a sufficient condition for the hypermixing property [7, Theorem 2.1] and have used it to show that some well-known maps are hypermixing, for example the "tent map" $f : [0,1] \to [0,1]$ defined by $f(x) = 2x$ if $x \in [0,1/2]$, and $f(x) = 2 - 2x$ if $x \in (1/2,1]$.

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