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On The Solvability Of The (SSIE) $\left(\mathbf{s}_R^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$, Involving The Infinite Triple Band Matrix $B(r,s,t)$

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Abstract: In this article, we consider the infinite triple band matrix $B(r,s,t)$, with r, s, $t \neq 0$. Then, under the condition $\Delta = s^2 - 4rt$, t, $-s$ and $r > 0$, we state an interesting characterization of the set $\mathscr{I}_R^{(c)}(r,s,t)$ of all positive sequences $x = (x_n)_{n\in\mathbb{N}}$, such that $\left(\mathbf{s}_R^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$ for $R > 0$. Then, we obtain some numerical applications, and results associated with the fine spectrum theory. Finally, we consider the triple band matrix $B\left(1,2s,as^2\right)$ and we solve the (SSIE) $\left(\mathbf{s}_R^{(c)}\right)_{B(1,2s,as^2)} \subset \mathbf{s}_x^{(c)}$ and we state some tauberian results, using the Silverman-Toeplitz theorem. These results extend those stated in [37, 8, 9].

Keywords: Matrix transformations; Silverman-Toeplitz theorem; Tauberian theorem; (SSIE); band matrix $B(r,s,t)$.

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1 Introduction.

The set of all complex sequences is denoted by ω, and c is the set of all convergent sequences. Also U^+ denotes the set of all positive sequences, and for $R > 0$, $\mathbf{s}_R^{(c)}$ denotes the set of all sequences $y \in \omega$ such that $(y_n/R^n)_{n\in\mathbb{N}} \in c$. In the same way, for $a \in U^+$, we denote by $\mathbf{s}_a^{(c)}$ the set of all sequences $y \in \omega$ such that $y/a \in c$. Troughout this article, we use the notations and definitions for the classical sequence spaces stated in [8, p. 3959] and [9] and the characterization of (c,c) stated in [9, pp. 23-24]. We also refer the reader to the paper [11] on the spectrum of linear operators represented by triangle matrices, and the recent text [1] devoted to summability theory with applications that contains the chapter titled, *Spectrum of Some Particular Matrices*.

Then we use the known infinite tridiagonal matrix $B(r,s,t)$, (cf. [2-5, 8]). This matrix is associated with the equation

$$rv^2 + sv + t = 0, \tag{1}$$

where v is the unknown, and we write $\Delta = s^2 - 4rt$. Then, for given $R > 0$ we state some general results on the solvability of the (SSIE)

$$\left(\mathbf{s}_R^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}, \tag{2}$$

(cf. [9, Chapter 5, p. 229]), and we denote by $\mathscr{I}_R^{(c)}(r,s,t)$ the set of all positive sequences $x = (x_n)_{n\in\mathbb{N}}$, such that the (SSIE) in (2) holds. The solvability of this (SSIE) consists in determining the set of all sequences $x \in U^+$ for which

$$\lim_{n\to\infty} \frac{ry_n + sy_{n-1} + ty_{n-2}}{R^n} = l_1 \text{ implies } \lim_{n\to\infty} \frac{y_n}{x_n} = l_2,$$

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for some scalars l_1, l_2 and for all y. Then, for practical reasons we will denote by (δ), the condition Δ, t, $-s$ and $r > 0$.

This paper is organized as follows. In Section 2, we state a characterization of the set $\mathscr{I}_R^{(c)}(r,s,t)$ in the case $\Delta > 0$. In Section 3, we give a characterization of the set $\mathscr{I}_R^{(c)}(r,s,t)$ under condition (δ). Finally, in Section 4, we determine the set $\mathscr{I}_R^{(c)}(1,2s,as^2)$ and we consider an application of the Silverman-Toeplitz theorem involving the triple band matrix $B(1,2s,as^2)$.

2 Characterization of the set $\mathscr{I}_R^{(c)}(r,s,t)$ in the case $\Delta > 0$

In this section, we assume that r, s and t are nonzero real numbers and we associate with the matrix $B(r,s,t)$, the equation in (1), whose the real or complex roots are $v_1 = \left(-s-\sqrt{\Delta}\right)/2r$ and $v_2 = \left(-s+\sqrt{\Delta}\right)/2r$, if $\Delta \neq 0$, and $v = -s/2r$, if $\Delta = 0$. Note that all the roots of the equation in (1), are distinct from zero. For any given real $R > 0$ and r, s, $t \neq 0$, we determine the set $\mathscr{I}_R^{(c)}(r,s,t)$. Using the inverse of $B(r,s,t)$ stated in [8, Lemma 4.2, p. 3961], and the notation

$$v'_{nk} = v_2^{n-k+1} - v_1^{n-k+1} \text{ for all } k \leq n \text{ and all } n,$$

we obtain the following lemma.

Lemma 1. *Assume that Δ, $R > 0$ and let v_1 and v_2 be the roots of (1). Then we have $x \in \mathscr{I}_R^{(c)}(r,s,t)$ if and only if the next conditions hold,*

(a)

$$\sup_n \left(\frac{1}{x_n} \sum_{k=1}^{n} \left| v'_{nk} \right| R^k \right) < \infty, \tag{3}$$

(b)

$$\lim_{n\to\infty} \frac{1}{x_n} \sum_{k=1}^{n} v'_{nk} R^k = l, \text{ for some scalar } l, \tag{4}$$

(c)

$$\lim_{n\to\infty} \frac{1}{x_n} v'_{nk} R^k = l_k, \text{ for some scalars } l_k \text{ and } k = 1,2,\ldots. \tag{5}$$

Proof. We have $x \in \mathscr{I}_R^{(c)}(r,s,t)$ if and only if $D_{1/x} B^{-1}(r,s,t) D_R \in (c,c)$, where

$$\left[D_{1/x} B^{-1}(r,s,t) D_{(R^n)_n} \right]_{nk} = \frac{1}{\sqrt{\Delta} x_n} v'_{nk} R^k, \text{ for } k \leq n, \text{ and for all } n.$$

From the characterization of (c,c), and by [8, Lemma 4.2, p. 3961], we obtain $x \in \mathscr{I}_R^{(c)}(r,s,t)$ if and only if each of the conditions in (3), (4) and (5) hold. This completes the proof.

3 Determination of the set $\mathscr{I}_R^{(c)}(r,s,t)$, under condition (δ)

In this section, our aim is to give a simple determination of the set $\mathscr{I}_R^{(c)}(r,s,t)$, under the condition in (δ). So we have $0 < v_1 < v_2$ and we use the notation,

$$\sigma_n(x,R) = \frac{1}{\sqrt{\Delta} x_n} \sum_{k=1}^{n} v'_{nk} R^k.$$

3.1 Characterization of the sequence $(\sigma_n(x,R))_{n\in\mathbb{N}}$, under the condition in (δ)

In this part, we determine the sequence $(\sigma_n(x,R))_{n\in\mathbb{N}}$ in each of the cases, $v_2 > R$, $v_2 < R$ and $v_2 = R$. We can state the following lemma.

Lemma 2. *Let $x \in U^+$ and $R > 0$ and assume that the condition in (δ) holds. Then we have:*

$$\sigma_n(x,R) \sim \frac{1}{\sqrt{\Delta x_n}} v_2^{n+1} \frac{R}{v_2 - R} \ (n \to \infty) \ (n \to \infty), \text{ for } v_2 > R, \tag{6}$$

$$\sigma_n(x,R) \sim \frac{R^{n+1}}{\sqrt{\Delta x_n}} \left(\frac{v_2}{R - v_2} - \frac{v_1}{R - v_1} \right) \ (n \to \infty), \text{ for } v_2 < R, \tag{7}$$

and

$$\sigma_n(x,R) \sim \frac{nR^{n+1}}{\sqrt{\Delta x_n}} \ (n \to \infty), \text{ for } v_2 = R. \tag{8}$$

Proof. The condition in (δ) implies $0 < v_1 < v_2$, and we have

$$\sigma_n(x,R) = \frac{1}{\sqrt{\Delta x_n}} \left[v_2^{n+1} \sum_{k=1}^{n} \left(\frac{R}{v_2} \right)^k - v_1^{n+1} \sum_{k=1}^{n} \left(\frac{R}{v_1} \right)^k \right] \text{ for all } n.$$

So we are led to deal with the next cases,

(I) $v_2 > R$, with (a) $0 < v_1 < R < v_2$, (b) $0 < R < v_1 < v_2$ and (c) $0 < R = v_1 < v_2$.

(II) $0 < v_1 < v_2 < R$ and

(III) $0 < v_1 < v_2 = R$.

Case (I) (a). We have $0 < R/v_2 < 1$ and

$$\sum_{k=1}^{n} \left(\frac{R}{v_2} \right)^k \sim \frac{R}{v_2 - R} \ (n \to \infty) \ .$$

Since $R/v_1 > 1$ we have

$$\sum_{k=1}^{n} \left(\frac{R}{v_1} \right)^k \sim \frac{\left(\frac{R}{v_1} \right)^{n+1}}{\frac{R}{v_1} - 1} \ (n \to \infty),$$

and

$$v_1^{n+1} \sum_{k=1}^{n} \left(\frac{R}{v_1} \right)^k \sim R^{n+1} \frac{v_1}{R - v_1} \ (n \to \infty).$$

We deduce

$$\sigma_n(x,R) = \frac{1}{\sqrt{\Delta x_n}} \left(v_2^{n+1} \frac{R}{v_2 - R} \xi_n - R^{n+1} \frac{v_1}{R - v_1} \eta_n \right) \text{ for all } n,$$

where $\xi = (\xi_n)_{n\in\mathbb{N}}$ and $\eta = (\eta_n)_{n\in\mathbb{N}}$ are two sequences tending to 1. Since $v_2 > R$ the statement in (6) holds.

Case (b). We have $v_1 > R$ and there are two sequences ξ and η tending to 1, such that

$$\sigma_n(x,R) = \frac{1}{\sqrt{\Delta x_n}} \left(v_2^{n+1} \frac{R}{v_2 - R} \xi_n - v_1^{n+1} \frac{R}{v_1 - R} \eta_n \right) \text{ for all } n,$$

and again the condition in (6) holds.

Case (c). $0 < R = v_1 < v_2$. By similar arguments as above, we have

$$\sigma_n(x,R) = \frac{1}{\sqrt{\Delta x_n}} \left(v_2^{n+1} \frac{R}{v_2 - R} \xi_n - nR^{n+1} \right) \text{ for all } n,$$

and again, the condition in (6) holds. This concludes the proof of Case (I).

(II) Case when $v_2 < R$. We have $v_1 < R$ and

$$v_i^{n+1}\sum_{k=1}^{n}\left(\frac{R}{v_i}\right)^k \sim \frac{v_i}{R-v_i}R^{n+1}\ (n\to\infty) \text{ for } i=1,2. \tag{9}$$

Thus, there are two sequences ξ and η tending to 1 such that,

$$\sigma_n(x,R)=\frac{R^{n+1}}{\sqrt{\Delta}x_n}\left(\frac{v_2}{R-v_2}\xi_n-\frac{v_1}{R-v_1}\eta_n\right) \text{ for all } n.$$

So the statement in (7) holds. This concludes the proof of Case (II).

(III) Case when $v_2 = R$. We have $v_1 < R$ and

$$\sum_{k=1}^{n} v_2^{n-k+1}R^k = nR^{n+1}.$$

Then, using the statement in (9), we deduce that there is a sequence $\eta=(\eta_n)_{n\in\mathbb{N}}$ tending to 1, such that

$$\sigma_n(x,R)=\frac{R^{n+1}}{\sqrt{\Delta}x_n}\left(n-\frac{v_1}{R-v_1}\eta_n\right) \text{ for all } n,$$

and the statement in (8) holds. This concludes the proof.

3.2 The main result. Determination of the set $\mathscr{I}_R^{(c)}(r,s,t)$ under the condition in (δ).

In this part, we determine the set $\mathscr{I}_R^{(c)}(r,s,t)$ using the notation $\overline{\mathbf{s}_a^{(c)}}=\left\{x\in U^+ : 1/x\in \mathbf{s}_a^{(c)}\right\}$, where $a\in U^+$. From Lemmas 1 and 2, we obtain the following theorem.

Theorem 1.*Let $R>0$. Under the condition in (δ), we have:*

$$\mathscr{I}_R^{(c)}(r,s,t)=\begin{cases}\overline{\mathbf{s}_{1/v_2}^{(c)}} & \text{if } v_2>R,\\ \overline{\mathbf{s}_{1/R}^{(c)}} & \text{if } v_2<R,\\ \overline{\mathbf{s}_{(1/nR^n)_{n\in\mathbb{N}}}^{(c)}} & \text{if } v_2=R.\end{cases} \tag{10}$$

*Proof.*Since $t/r=v_1v_2$ and $-s/r=v_1+v_2>0$, we have $0<v_1<v_2$. Then, by Lemma 1, the condition $x\in\mathscr{I}_R^{(c)}(r,s,t)$ is equivalent to the statements in (3), (4) and (5). Then, the condition in (4) implies (3) since $v'_{nk}>0$, for $k=1,2,...,n$. We conclude that $x\in\mathscr{I}_R^{(c)}(r,s,t)$ if and only if the conditions in (3) and (4) hold. Now we are led to deal with each of the cases (a) $v_2>R$, (b) $v_2<R$ and (c) $v_2=R$.

(a) Case $v_2>R$. Then, the statement in (6) of Lemma 2 holds. So the condition $\lim_{n\to\infty}\sigma_n(x,R)=l$ for some scalar l holds if and only if

$$\lim_{n\to\infty}\frac{v_2^n}{x_n}=l\frac{v_2-R}{Rv_2}\sqrt{\Delta},$$

that is, $x\in\overline{\mathbf{s}_{1/v_2}^{(c)}}$. Then, the condition in (5) holds, since

$$\lim_{n\to\infty}\frac{v_1^n}{x_n}=\lim_{n\to\infty}\left(\frac{v_1}{v_2}\right)^n\frac{v_2^n}{x_n}=0.$$

(b) Case $v_2<R$. Then, the statement in (7) of Lemma 2 holds. Then we have $\lim_{n\to\infty}\sigma_n(x,R)=l$ if and only if

$$\lim_{n\to\infty}\frac{R^n}{x_n}=l\rho\sqrt{\Delta}$$

for some scalar l, where

$$\frac{1}{\rho} = R\left(\frac{v_2}{R-v_2} - \frac{v_1}{R-v_1}\right) > 0,$$

and we have shown $x \in \overline{\mathbf{s}_{1/R}^{(c)}}$. Then, the condition in (5) holds, since

$$\frac{v_i^n}{x_n} = \left(\frac{v_i}{R}\right)^n \frac{R^n}{x_n} \to 0 \ (n \to \infty) \text{ for } i = 1, 2.$$

(c) Case when $v_2 = R$. Then, the statement in (8) of Lemma 2 holds. Then, the condition $\lim_{n\to\infty} \sigma_n(x,R) = l$ for some scalar l, holds if and only if

$$\lim_{n\to\infty} \frac{nR^n}{x_n} = l\frac{\sqrt{\Delta}}{R}, \tag{11}$$

and we have $x \in \overline{\mathbf{s}_{(1/nR^n)_{n\in\mathbb{N}}}^{(c)}}$. Then we have $v_i^n/x_n \le R^n/x_n$ for all n, and since $\lim_{n\to\infty} R^n/x_n = 0$, by the condition in (11), we obtain $\lim_{n\to\infty} v_i^n/x_n = 0$ $(n\to\infty)$, for $i = 1, 2$. So, the statement in (5) holds. This concludes the proof.

The set $\mathscr{I}_1^{(c)}(r,s,t)$ may be rewritten using the results on the fine spectrum theory, stated in [4, 1]. Recall that in [1], there is a chapter titled *Spectrum of Some Particular Matrices*. There is also a recent paper [11], on the spectra of triangles over sequence spaces. In this way, we can connect the solvability of the (SSIE) $c_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$ with the fine spectrum theory, (cf. [6]), considering the polynomial $(r-\lambda)X^2 + sX + t = 0$ associated with the operator $B(r,s,t) - \lambda I$, whose roots depend on the complex number $\sqrt{s^2 - 4(r-\lambda)t}$. Then, under the condition in (δ), we may determine the solutions of the (SSIE) $c_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$, using the point, the continuous and the residual spectra of the operator $B(r,s,t)$ over c, (cf. [4, Theorem 2.10, p. 997]), as follows. We have $0 \notin \sigma(B(r,s,t),c)$ if and only if

$$v_2 = \left(-s + \sqrt{s^2 - 4rt}\right)/2r < 1.$$

Then we have $v_2 = 1$ if and only if either of the conditions $0 \in \sigma_c(B(r,s,t),c)$ or $r+s+t = 0$ holds. Finally, the condition $v_2 > 1$ holds if and only if $0 \in \sigma_r(B(r,s,t),c)$ and $r+s+t \neq 0$. So we can state the following corollary.

Corollary 1.*Let r, s, $t \in R$. Under the condition in (δ), we have*

$$\mathscr{I}_1^{(c)}(r,s,t) = \begin{cases} \overline{\mathbf{s}_{(1/n)_{n\in\mathbb{N}}}^{(c)}} & \text{if } 0 \notin \sigma(B(r,s,t),c), \\ \overline{c} & \text{if } 0 \in \sigma_c(B(r,s,t),c) \text{ and } r+s+t \neq 0, \\ \overline{\mathbf{s}_{1/v_2}^{(c)}} & \text{if } 0 \in \sigma_r(B(r,s,t),c) \text{ or } r+s+t = 0. \end{cases}$$

Now, if we denote by (δ') the condition, Δ, t, s and $r > 0$, we can state the next remark.

*Remark.*Under the condition in (δ'), we have $v_1v_2 = t/r > 0$ and $v_1 + v_2 = -s/r < 0$ which imply $v_1 < v_2 < 0$. Then, writing $w_1 = -v_2 > 0$ and $w_2 = -v_1 > 0$, we obtain $0 < w_1 < w_2$ and by Theorem 1, where $R > 0$, we obtain the following result,

$$\mathscr{I}_R^{(c)}(r,s,t) = \begin{cases} \overline{\mathbf{s}_{-1/v_1}^{(c)}} & \text{if } v_1 < -R, \\ \overline{\mathbf{s}_{1/R}^{(c)}} & \text{if } v_1 > -R, \\ \overline{\mathbf{s}_{(1/nR^n)_{n\in\mathbb{N}}}^{(c)}} & \text{if } v_1 = -R. \end{cases}$$

We can give some examples, that are direct consequences of Theorem 1.

*Example 1.*Let $R > 0$. We have $(v - R/2)(v - R) = v^2 - 3/2Rv + R^2/2$, and by Theorem 1 with $v_2 = R$ we obtain the next statement. The set of all $x = (x_n)_{n\in\mathbb{N}} \in U^+$ that satisfy the condition

$$\lim_{n\to\infty} \left(2y_n - 3Ry_{n-1} + R^2 y_{n-2}\right)/R^n = l_1 \text{ implies } \lim_{n\to\infty} y_n/x_n = l_2,$$

for some scalars l_1, l_2 and for all y, is determined by $nR^n/x_n \to L$ $(n\to\infty)$ for some scalar L.

Example 2. Assume that the condition in (δ) holds, with $t=1$. Then, the condition $v_2<1$ is associated with the inequality, $r+s+1>0$, and we have $\Delta>0$ if and only $s<-2\sqrt{r}$. So, by Theorem 1 we have $\mathscr{I}_1^{(c)}(r,s,t)=\overline{c}$, and since $e\in\overline{c}$ we obtain the next tauberian result,

$$\lim_{n\to\infty}(ry_n+sy_{n-1}+y_{n-2})=l_1 \text{ implies } \lim_{n\to\infty}y_n=l_2,$$

for some scalars l_1, l_2 and for all y. For instance, if $r=6$ and $s=-5$ we obtain the following statement,

$$\lim_{n\to\infty}(6y_n-5y_{n-1}+y_{n-2})=l_1 \text{ implies } \lim_{n\to\infty}y_n=l_2,$$

for some scalars l_1, l_2 and for all y.

Example 3. We may determine the set $\mathscr{I}_1^{(c)}(1,-3,2)$ of all sequences $x\in U^+$ such that $c_{B(1,-3,2)}\subset\mathbf{s}_x^{(c)}$. Since the roots of the equation $v^2-3v+2=0$ are $v_1=1$ and $v_2=2$, by Theorem 1, we obtain $\mathscr{I}_1^{(c)}(1,-3,2)=\overline{\mathbf{s}_2^{(c)}}$. So, the set of all sequences $x\in U^+$ that satisfy the statement

$$\lim_{n\to\infty}(y_n-3y_{n-1}+2y_{n-2})=l_1, \text{ implies } \lim_{n\to\infty}y_n/x_n=l_2,$$

for some scalars l_1, l_2 and for all y, is determined by $(2^n/x_n)_{n\in\mathbb{N}}\in c$.

Example 4. The roots of the equation $2v^2-3v+1=0$, are $v_1=1/2$, $v_2=1$, and by Theorem 1 we obtain $\mathscr{I}_1^{(c)}(2,-3,1)=\overline{\mathbf{s}_{(1/n)_{n\in\mathbb{N}}}^{(c)}}$. So, the set of all sequences $x\in U^+$ that satisfy the statement

$$\lim_{n\to\infty}(2y_n-3y_{n-1}+y_{n-2})=l_1, \text{ implies } \lim_{n\to\infty}y_n/x_n=l_2,$$

for some scalars l_1, l_2 and for all y, is determined by $(n/x_n)_{n\in\mathbb{N}}\in c$.

3.3 On the solvability of the (SSIE) $c_{B(r,s,t)}\subset\mathbf{s}_x^{(c)}$, where only one real among r, s, t, is zero

The previous results can be extended to the (SSIE) $c_{B(r,s,t)}\subset\mathbf{s}_x^{(c)}$ where only one real among r, s, t, is zero. We are led to consider each of the (SSIE), $c_{B(r,s,0)}\subset\mathbf{s}_x^{(c)}$, $c_{B(0,s,t)}\subset\mathbf{s}_x^{(c)}$ and $c_{B(r,0,t)}\subset\mathbf{s}_x^{(c)}$, and we assume that the roots of each of the equations $rv+s=0$ and $sv+t=0$ are positive, that is, $v_0=-s/r>0$, $v_0'=-t/s>0$. Then, for the (SSIE) $c_{B(r,0,t)}\subset\mathbf{s}_x^{(c)}$ we have $rv^2+t=0$ and, as in Theorem 1, we consider the upper root $v_0''=\sqrt{-t/r}$ with $-t/r>0$ of this equation.

We can state the following results, that can be shown using similar arguments as given in the proof of Theorem 1.

Proposition 1. *Let r, s and t be real numbers. Then, the sets $\mathscr{I}_1^{(c)}(r,s,0)$, $\mathscr{I}_1^{(c)}(0,s,t)$ and $\mathscr{I}_1^{(c)}(r,0,t)$ are determined by (10) in Theorem 1, where v_2 is successively replaced by $v_0=-s/r>0$, $v_0'=-t/s>0$ and $v_0''=\sqrt{-t/r}$, with $-t/r>0$.*

Example 5. We have $\mathscr{I}_1^{(c)}(1,0,-1)=\overline{\mathbf{s}_{(1/n)_{n\in\mathbb{N}}}^{(c)}}$ since $v_0''=1$. In a similar way, we have $\mathscr{I}_1^{(c)}(0,-1,2)=\overline{\mathbf{s}_{1/2}^{(c)}}$, and $\mathscr{I}_1^{(c)}(1,-2,0)=\overline{c}$.

4 Some applications involving the triple band matrix $B\left(1,2s,as^2\right)$

In this section, we apply the previous results to the solvability of the particular (SSIE)

$$\left(\mathbf{s}_R^{(c)}\right)_{B\left(1,2s,as^2\right)}\subset\mathbf{s}_x^{(c)},$$

for $R>0$ and $s<0<a<1$ and we give some examples. Then we state an application of the Silverman-Toeplitz theorem.

4.1 Solvability of the (SSIE) $\left(\mathbf{s}_R^{(c)}\right)_{B\left(1,2s,as^2\right)} \subset \mathbf{s}_x^{(c)}$

In this part, we consider the triple band matrix,

$$B\left(1,2s,as^2\right)=\begin{pmatrix} 1 & & & \\ 2s & 1 & & 0 \\ as^2 & 2s & 1 & \\ 0 & as^2 & 2s & 1 \\ . & . & . & . & . \end{pmatrix}.$$

Then, we study the solvability of the (SSIE) $\left(\mathbf{s}_R^{(c)}\right)_{B\left(1,2s,as^2\right)} \subset \mathbf{s}_x^{(c)}$, for $R>0$, under the condition in (δ), which is equivalent to the condition, $s<0<a<1$. Then, as above, this study consists in determining the set of all $x\in U^+$ for which

$$\lim_{n\to\infty}\frac{y_n+2sy_{n-1}+as^2y_{n-2}}{R^n}=l_1 \text{ implies } \lim_{n\to\infty}\frac{y_n}{x_n}=l_2, \tag{12}$$

for all y and for some scalars l_1 and l_2. We denote by $\mathscr{I}_R^{(c)}(a,s)$, the set of all $x\in U^+$ for which the statement in (12) holds. Here, we have $b(v)=v^2+2sv+as^2=0$, with $s<0<a<1$ and $\Delta=4s^2(1-a)>0$. So, the equation $b(v)=0$ has two roots that satisfy the inequalities,

$$0<v_1=-s\left(1-\sqrt{1-a}\right)<v_2=-s\left(1+\sqrt{1-a}\right).$$

From Theorem 1 and Lemma 2, we can state the following result, where we write $s_2(R)=-R\left(1-\sqrt{1-a}\right)/a$.

Corollary 2. *Let $R>0$ and $s<0<a<1$. The set $\mathscr{I}_R^{(c)}(a,s)$ is determined in the following way,*

$$\mathscr{I}_R^{(c)}(a,s)=\begin{cases}\overline{\mathbf{s}_{1/v_2}^{(c)}} & \text{for } s<s_2(R),\\ \overline{\mathbf{s}_{1/R}^{(c)}} & \text{for } s>s_2(R),\\ \overline{\mathbf{s}_{(1/nR^n)_{n\in\mathbb{N}}}^{(c)}} & \text{for } s=s_2(R).\end{cases}$$

In particular, if $R=1$, then we let $s_2=s_2(1)=-\left(1-\sqrt{1-a}\right)/a$ and the set $\mathscr{I}^{(c)}(a,s)=\mathscr{I}_1^{(c)}(a,s)$ is determined by

$$\mathscr{I}^{(c)}(a,s)=\begin{cases}\overline{\mathbf{s}_{1/v_2}^{(c)}} & \text{for } s<s_2,\\ \overline{c} & \text{for } s>s_2,\\ \overline{\mathbf{s}_{(1/n)_{n\in\mathbb{N}}}^{(c)}} & \text{for } s=s_2.\end{cases}$$

Proof. We have $v_2>R$ if and only if

$$v_2=-s\left(1+\sqrt{1-a}\right)>R$$

that is, $s<s_2(R)$. By Theorem 1, the condition $s<s_2(R)$ implies $\mathscr{I}_R^{(c)}(a,s)=\overline{\mathbf{s}_{1/v_2}^{(c)}}$. In a similar way, the condition $s>s_2(R)$ is equivalent to $v_2<R$ and implies $\mathscr{I}_R^{(c)}(a,s)=\overline{\mathbf{s}_{1/R}^{(c)}}$, and the condition $s=s_2(R)$ implies $\mathscr{I}_R^{(c)}(a,s)=\overline{\mathbf{s}_{(1/nR^n)_{n\in\mathbb{N}}}^{(c)}}$. This concludes the proof.

In the next examples, we apply Corollary 2, with $a=3/4$, which implies $s_2=-2/3$. So we are led to state the following examples, involving the next triple band matrices $B(1,-2,3/4)$, $B(1,-4/3,1/3)$ and $B(1,-1,3/16)$.

Example 6. Case of the matrix $B(1,-2,3/4)$. We have $s=-1<s_2$, then we obtain $v_1=1/2$, $v_2=3/2$ and $as^2=3/4$. So, by Corollary 2, the set of all $x\in U^+$, that satisfy the statement,

$$\lim_{n\to\infty}(4y_n-8y_{n-1}+3y_{n-2})=l_1 \Longrightarrow \lim_{n\to\infty}y_n/x_n=l_2,$$

for all y and for some scalars l_1 and l_2, is determined by $1/x\in\mathbf{s}_{2/3}^{(c)}$ that is, $\left((3/2)^n/x_n\right)_{n\in\mathbb{N}}\in c$.

*Example 7.*Case of the matrix $B(1,-4/3,1/3)$. Here, we have $\mathrm{s} = s_2 = -2/3$ and $a\mathrm{s}^2 = 1/3$. By Corollary 2, the set of all $x \in U^+$, that satisfy the statement,

$$\lim_{n\to\infty}(3y_n - 4y_{n-1} + y_{n-2}) = l_1 \Longrightarrow \lim_{n\to\infty} y_n/x_n = l_2,$$

for all y and for some scalars l_1 and l_2, is determined by $(n/x_n)_{n\in\mathbb{N}} \in c$.

*Example 8.*Case of the matrix $B(1,-1,3/16)$. We have $\mathrm{s} = -1/2 > s_2$ and by Corollary 2, the statement

$$\lim_{n\to\infty}\left(y_n - y_{n-1} + \frac{3}{16}y_{n-2}\right) = l_1 \Longrightarrow \lim_{n\to\infty} y_n/x_n = l_2,$$

for all y and for some scalars l_1 and l_2 holds if and only if $1/x \in c$.

4.2 An application using the Silverman-Toeplitz theorem

In this section, we apply Corollary 2, to obtain a result which is a consequence of the Silverman-Toeplitz theorem, stated in [10, Theorem 1.3.8] as follows.

Lemma 3.*[10, Theorem 1.3.8] Let $A \in (c,c)$ and $z \in c$. If $\lim_{k\to\infty} a_{nk} = 0$ for all $k \geq 1$, then we have*

$$\lim_{n\to\infty} z_n = L \Longrightarrow \lim_{n\to\infty} A_n z = lL,$$

where $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = l$.

We obtain the following result.

Proposition 2.*Let a and s be real numbers, with $\mathrm{s} < 0 < a < 1$ and assume*

$$\mathrm{s} > s_2 = -\left(1 - \sqrt{1-a}\right)/a.$$

Then we have

$$\lim_{n\to\infty}\left(y_n + 2\mathrm{s}y_{n-1} + a\mathrm{s}^2 y_{n-2}\right) = L \Longrightarrow \lim_{n\to\infty} y_n = \frac{1}{1+2\mathrm{s}+a\mathrm{s}^2}L,$$

for some scalar L and for all y.

*Proof.*Let $A = B^{-1}\left(1,2\mathrm{s},a\mathrm{s}^2\right) = (a_{nk})_{n,k\in\mathbb{N}}$ and let $z_n = y_n + 2\mathrm{s}y_{n-1} + a\mathrm{s}^2 y_{n-2}$ for any sequence $y \in \omega$, in Lemma 3. Then we have $y = Az$. Since $\mathrm{s} > s_2$, by Corollary 2 we have $\mathscr{I}^{(c)}(a,\mathrm{s}) = \overline{c}$, which implies $e \in \mathscr{I}^{(c)}(a,\mathrm{s}) = \overline{c}$. Then, as we have seen in the proof of Corollary 2, the condition $\mathrm{s} > s_2$ is equivalent to $v_2 < 1$. So, by Lemma 2 with $R = 1$, we have

$$l = \lim_{n\to\infty}\sum_{k=1}^{n} a_{nk} = \lim_{n\to\infty}\sigma_n(e,1),$$

and

$$l = \frac{1}{\sqrt{\Delta}}\left(\frac{v_2}{1-v_2} - \frac{v_1}{1-v_1}\right) = \frac{1}{\sqrt{\Delta}}\frac{v_2 - v_1}{(1-v_2)(1-v_1)}.$$

By elementary calculations, we have

$$(1-v_2)(1-v_1) = 1 + a\mathrm{s}^2 + 2\mathrm{s},$$

and since $\sqrt{\Delta} = -2\mathrm{s}\sqrt{1-a} = v_2 - v_1$, we obtain

$$l = \frac{1}{1+2\mathrm{s}+a\mathrm{s}^2}.$$

Since $0 < v_1 < v_2 < 1$ we have $A \in S_1$, (cf. [8, p. 3959]) and $\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} v'_{nk} = 0$ for all $k \geq 1$. We obtain $A \in (c,c)$ and we conclude by Lemma 3. This completes the proof.

This result can be illustrated by the next example.

*Example 9.*In Example 8, where $a = 3/4$, $\mathrm{s} = -1/2$, $s_2 = -2/3$, we have $s_2 < \mathrm{s} < 0 < a < 1$ and $1 + 2\mathrm{s} + a\mathrm{s}^2 = 3/16$. So we can apply Proposition 2, which gives

$$\lim_{n\to\infty}\left(y_n - y_{n-1} + \frac{3}{16}y_{n-2}\right) = L \Longrightarrow \lim_{n\to\infty} y_n = \frac{16}{3}L,$$

for some scalar L and for all y.

5 conclusion

In this article, we have stated some general results on the solvability of the (SSIE) $\left(\mathbf{s}_R^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_x^{(c)}$. Then, under the condition in (δ), we have stated some practical results on the solvability of this (SSIE), that have been illustrated by some examples. In future, it should be interesting to state similar results on this solvability, replacing the condition in (δ), by another condition on r, s and t, that may belong to $\mathbb{C}$, and considering each of the case $\Delta = 0$ and $\Delta \neq 0$. Then, as it has been suggested in Corollary 1, some results on the fine spectrum of the operator $B(r,s,t)$ over the spaces c and c_0, could be associated with the solvability of each of the (SSIE) of the form

$$D_x * X_{B(r,s,t)-\lambda I} \subset Y, \text{ with } \lambda \in \mathbb{C},$$

on ω, where X and Y are any of the sets c_0, c or ℓ_∞.

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References

[1] Başar F., *Summability Theory and its Applications*, 2^{nd} ed., CRC Press/Taylor&Francis Group, Boca Raton · London · New York, 2022.
[2] Bilgic, H., Furkan H., *On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_1 and bv,* Math. Comput. Model. **46** 7-8 (2007) 883-891.
[3] Fares, A., Ayad A., de Malafosse, B., *Calculations on matrix transformations involving an infinite tridiagonal matrix,* Axioms (2021), **10**, 218. https://doi.org/10.3390/axioms10030218.
[4] Furkan, H., Bilgiç H., Altay B., *On the fine spectrum of the operator $B(r,s,t)$ over c_0 and c,* Comput. Math. Appl. **53** 6 (2007) 989-998.
[5] Furkan, H., Bilgiç H., Başar F., *On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p,* Comput. Math. Appl. **60** 7 (2010) 2141-2152.
[6] Kreyszig, E., *Introductory functional analysis with applications,* John Wiley and Sons Inc. New-York-Chichester-Brisbane-Toronto, 1978.
[7] de Malafosse, B., *Tauberian Theorems for the Operator of Weighted Means*, Commun. Math. Anal. **5** 2 (2008), 1-12.
[8] de Malafosse, B., Fares, A., Ayad A., *On the solvability of certain (SSIE) and (SSE), with operators of the form $B(r,s,t)$*, Filomat, **35** (12), (2021), 3957-3970; https://doi.org/102298/FIL.2112957M.
[9] de Malafosse, B., Malkowsky, E., Rakocević, V., *Operators between sequence spaces and applications,* Springer: Singapore, 2021; doi:10.1007/978-981-15-97/42-8.
[10] Wilansky, A., *Summability through Functional Analysis,* North-Holland Mathematics Studies 85, 1984.
[11] Yeşiljayagil, M, Başar F., *A survey for the spectrum of triangles over sequence spaces*, Numer. Funct. Anal. Optim. **40** (2019), No.16, 1898-1917.