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The Global Existence And Attractor For $m(x)$-Laplacian Equation With Nonlinear Boundary Conditions

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Abstract: In this paper, we consider a doubly $m(x)$-Laplacian equation

$$\frac{\partial \alpha(v)}{\partial t} - \mathrm{div}(|\nabla v|^{m(x)-2}\nabla v) + F(v) = G, \qquad in\ \Omega \times (0,+\infty),$$

with nonlinear boundary conditions and initial data given. Firstly, we use the regularization method to determine the existence and uniqueness of weak solutions in the Sobolev space with variable exponents. Secondly, in the frame of the dynamical systems approach, a standard limiting process and a method to generate a series of approximation solutions are used to study the long behavior of solutions for the above problem (1.1). We formulate our problem as a dynamical system, and then, by using Hölder continuity solutions and assuming appropriate hypotheses, we prove also the existence of a global attractor in $L^2(\Omega)$.

Keywords: $m(x)$-Laplacian; nonlinear boundary conditions; existence; uniqueness; variable exponents; global attractors.

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1 Introduction

This manuscript is concerned with a class of pseudo-parabolic equations involving $\mathscr{A}_{m(x)}$ operators with nonlinearities of variable exponents and with nonlinear boundary conditions:

$$\begin{aligned} &\frac{\partial \alpha(v(x,t))}{\partial t} + \mathscr{A}_{m(x)} v(x,t) + F(v(x,t)) = G(x), && (x,t) \in Q_T, \\ &|\nabla v(x,t)|^{m(x)-2}\frac{\partial v(x,t)}{\partial \eta} + H(x,v(x,t)) = 0, && (x,t) \in S_T, \\ &\alpha(v(x,0)) = \alpha(v_0(x)), && x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d, (d \geq 2)$, is a regular bounded domain with a Lipschitz continuous boundary $\partial\Omega; Q_T := \Omega \times]0,T[;\ \ S_T := \partial\Omega \times]0,T[,\ \ m(.)$ is logarithm Hölder continuous, and the nonlinear term $\mathscr{A}_{m(x)} u = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(|\nabla u|^{m(x)-2}\frac{\partial u}{\partial x_i}\right)$ is known as the $m(x)$-Laplacian, and when $m(x) = p$, it will be reduced to the $p - Laplacian$. Precise conditions concerning α, F, G, H and v_0 will be given hereafter.
Electrorheological fluids have been modeled using the $m(x)$-Laplacian operator, see [21,30,31]. In the field of elastic mechanics, see [37], within the image restoration in [10,32,35], and in issues involving magnetostatics, see [9].

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Our demonstration proves the presence of global attractors in appropriate spaces under appropriate conditions on the exponent $m(.)$ and on α, F, G, and H.

The findings are demonstrated by applying a standard limiting process and a technique to create a series of approximation solutions. In the last year, it has been possible to determine the presence of solutions and their asymptotic behavior to these operator-related equations for Steklov boundary conditions. We cite the articles and references of [7, 12, 27, 34, 38] as an example. Regarding the question for the asymptotic behavior to the $\mathscr{A}_p$ parabolic problem with nonlinear boundary condition, we refer to [7, 27], respectively, and the references therein, when $m(.) = p$.
Numerous findings about the existence and regularity of the attractors can be found in the cases $\alpha(.) = \alpha$ and $m(.) = p$. We just discuss the work of [7, 11, 12] to be exhaustive. Regarding the case $m(.) = p$ and α is increasing locally Lipschitz function with $\alpha(0) = 0$, the author of [12, 13] investigated the existence of a global attractor in $L^r(\Omega)$, with ($r = 2$ and ∞). The current study also aims to create, in the spirit of articles [5, 11, 12, 25, 33], a variational technique in the parabolic situation.

There are two steps in the proof. We first demonstrate that there is a global solution to the approximate problem, after which we perform some uniform estimates for these solutions. Our primary tools are the approximation solution method and the ability to estimate inequality. We determine that a problem of type (1.1) exists using a conventional limiting procedure.

The sections of this article are organized as follows: In Section 2, it is shown that the bounded weak solutions to problem (1.1) exist and are unique. Additionally, some fundamental Lebesgue and Sobolev spaces are introduced. In Section 3, we prove the global attractor for the semigroup associated with the problem (1.1).

2 Preliminary results

We define for $s \in \mathbb{R}$ a continuous function α with $\alpha(0) = 0$.

$$\phi(s) = \int_0^s \alpha(t)dt.$$

The Legendre transform of ϕ, ϕ^* is then defined by

$$\phi^*(t) = \sup_{z\in\mathbb{R}} \{tz - \phi(z)\}.$$

Specifically, we have

$$\phi^*(\alpha(t)) = t\alpha(t) - \phi(t). \tag{2.0}$$

One needs to understand the fundamentals of spaces $L^{m(.)}(\Omega)$ and $W^{1,m(.)}(\Omega)$ in order to examine problems involving variable exponents. Here is a quick review of them for the benefit of the readers. [16, 17] contains the basic result's properties of variable exponent and Lebesgue Sobolev spaces. The exponent $m(x)$ should be in $C(\overline{\Omega})$. Additionally, $m(x)$ is logarithmic Hölder continuous, that is, a constant D exists such that :

$$|m(x_1) - m(x_2)| \leq \frac{-D}{\log|x_1 - x_2|}, \forall x_1, x_2 \in \Omega, |x_1 - x_2| < \frac{1}{2}. \tag{2.1}$$

For $m^- > 1$, we designates the variable exponent Lebesgue space by

$$L^{m(.)}(\Omega) := \left\{w : \Omega \to \mathbb{R}; |\rho_{m(x)}(\lambda w) < \infty, for\ some\ \lambda > 0\right\},$$

where,

$$\rho_{m(x)}(z) = \int_\Omega |z(x)|^{m(x)}\, dx < \infty.$$

We regard this space as possessing the Luxemburg norm.

$$\|z\|_{m(.)} = \inf\left\{\lambda > 0 | \rho_{m(x)}(\frac{z}{\lambda}) \leq 1\right\}.$$

The space $L^{m'(.)}(\Omega)$, stands for the dual space $L^{m(.)}(\Omega)$, such that $\frac{1}{m(x)} + \frac{1}{m'(x)} = 1\ \forall x \in \overline{\Omega}$. The variable exponent Sobolev space $W^{1,m(.)}(\Omega)$ is defined as

$$W^{1,m(.)}(\Omega) = \left\{z \in L^{m(.)}(\Omega) :\ |\nabla z| \in L^{m(.)}(\Omega)\right\},$$

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equipped with the norm

$$\|z\|_{W^{1,m(.)}}(\Omega) = \|z\|_{m(.)} + \|\nabla z\|_{m(.)}.$$

We denote the closure of $C_0^\infty(\Omega)$ in $W^{1,m(.)}(\Omega)$ by $W_0^{1,m(.)}(\Omega)$. With these norms, the space $L^{m(.)}(\Omega)$, $W_0^{1,m(.)}(\Omega)$ and $W^{1,m(.)}(\Omega)$ are separable reflexive Banach spaces, see [17,18]. Proposition 1 contains some of the results we found about the Luxembourg norm.

Proposition 1. *([17])*
There is a positive constant C depending only on m and Ω such that for every $w \in W_0^{1,m(.)}(\Omega)$,

$$\|w\|_{m(.)} \leq C\|\nabla w\|_{m(.)},$$

this suggests that the equivalent norms of $W_0^{1,m(.)}(\Omega)$ are $\|\nabla w\|_{m(.)}$ and $\|w\|_{1,m(.)}$.

The lemma that follows is practical and widely applied.

Lemma 1. *(Ghidaghia Lemma, cf.[33])*
Let Y be a positive absolutely continuous function on $(0,\infty)$ such that it satisfies

$$Y' + \mu_0 Y^\beta \leq \mu_1,$$

with $\beta > 1, \mu_0 > 0, \mu_1 \geq 0$. Then for $s > 0$

$$Y(s) \leq \left[\frac{\mu_1}{\mu_0}\right]^{1/\beta} + \frac{1}{(\mu_0(\beta-1)s)^{1/(\beta-1)}}.$$

Lemma 2. *(Uniform Gronwall Lemma)*
Assume that z and h be non-negative locally integrable functions on $(t_0,+\infty)$ such that

$$z' \leq h, \forall s \geq t_0,$$

$$\int_s^{s+\xi} z(y)dy \leq b_1, \quad \int_s^{s+\xi} |h(y)|\,dy \leq b_2, \quad \forall s \geq t_0,$$

where ξ, b_1 and b_2 are positive constants. Then

$$z(s+\xi) \leq \frac{b_1}{\xi} + b_2, \quad \forall s \geq t_0.$$

For a proof of the above lemma, see [33].

In the sense that follows, we thus present weak solutions of equation (1.1).

Definition 1. *If $v(x,t)$ satisfies the following, it is referred to as a weak solution of (1.1): $v(x,t) \in L^\infty(Q_T) \cap L^{m^-}(0,T,W^{1,m(x)}(\Omega))$ such that*

$$\int_0^T \langle(\alpha(v))_t, \psi\rangle + \iint_{Q_T} |\nabla v|^{m(x)-2}\nabla v \nabla\psi + \iint_{S_T} H(x,v)\psi + F(v)\psi = \iint_{Q_T} G(x)\psi;$$

for every function $\psi \in L^\infty(Q_T) \cap L^{m^-}(0,T,W^{1,m(x)}(\Omega))$.

If $\psi \in L^{m(x)}(0,T;W^{1,m(x)}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$, with $\psi(.,T) = 0$, then

$$\int_0^T \langle(\alpha(v))_t, \psi\rangle = -\iint_{Q_T} (\alpha(v) - \alpha(v_0))(\psi)_t,$$

where $\langle .,. \rangle$ denote the duality product between $W^{1,m(.)}(\Omega)$ and $W^{-1,m'(.)}(\Omega)$.

Throughout this work, the functions $m(x), \alpha(x), v_0(x), F(s)$ and $H(x,s)$ satisfy the following conditions:

(H1) $v_0 \in L^\infty(\Omega) \cap W^{1,m(x)}(\Omega)$, G is a bounded function.

(H2) α is a function with $\alpha(0) = 0$ that maps from $\mathbb{R}$ to $\mathbb{R}$, satisfying the requirement $0 < \alpha_1 \leq \alpha(x) \leq \alpha_2$.

(H3) The nonlinearity function satisfies a very general condition, $F : \mathbb{R} \to \mathbb{R}$ is Carathéodory mapping such that

$$L_1 |s|^{p(x)} - L_0 \leq sF(s) \leq L_2 |s|^{p(x)} + L_0,$$

$$s \to F(s) + L_3\alpha(s) \text{ is increasing}$$

for some $p(.) \in C(\overline{\Omega})$, with $2 < p^- \leq p^+ < \infty$, and for some $C_0 \geq 0, L_1 \geq 0, L_2 \geq 0, L_3 \geq 0$.

(H4) $s \to H(x,s)$ is an increasing Lipschitz continuous function such that, $\forall s \in \mathbb{R}, \forall x \in \Omega$, $H(x,s)s \geq 0$ and $H(x,s)\alpha(s) \geq 0$.

2.1 Existence of Weak Solutions

The primary reason for our existence is as follows:

Theorem 1. *Assuming that $2 \leq m^- \leq m^+ < \infty$, let $m(.) \in C(\overline{\Omega})$. This will satisfy (2.1) and, under the given assumptions, (H1) through (H4). There is just one bounded solution for equation* (1.1) *such that $\alpha(v) \in C([0,T);L^1(\Omega))$.*

Proof. A priori estimates are used in the theorem 1.

• Consider a sequence α_ε in $C^1(\mathbb{R})$ from α, in a way that $\alpha_\varepsilon(0) = 0$, $\alpha_\varepsilon \to \alpha$ in $C_{loc}(\mathbb{R})$ $\alpha_1 \leq \alpha_\varepsilon \leq \alpha_2$, and $|\alpha_\varepsilon| \leq |\alpha|$.
• F_ε in $C^\infty(\mathbb{R})$, $F_\varepsilon(s) \to F(s)$ in $L^1(\Omega)$ $\forall s$ in $C_{loc}(\mathbb{R})$, F_ε satisfies uniformly (H3).
• H_ε in $C^\infty(\Omega \times \mathbb{R})$, $H_\varepsilon(x,s) \to H(x,s)$ in $L^1(\Omega)$ $\forall s$ in $C_{loc}(\mathbb{R})$, x in Ω, H_ε satisfies uniformly (H4).
• Assume that the sequence $(v_{0\varepsilon})_{\varepsilon>0}$ in $C^3(\overline{\Omega})$, $v_{0\varepsilon} \to v_0$ almost everywhere in $L^1(\Omega)$, $\|v_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} + 1$, and fulfills the requirement for compatibility :

$$(|\nabla v_{0\varepsilon}|^2 + \varepsilon)^{\frac{(m(x)-2)}{2}} \frac{\partial v_{0\varepsilon}}{\partial \eta} + H_\varepsilon(x, v_{0\varepsilon}) = 0.$$

Let $\varepsilon > 0$, consider approximation solutions (v_ε) of problem (P_ε)

$$\frac{\partial \alpha_\varepsilon(v_\varepsilon)}{\partial t} - div\left((|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{m(x)-2}{2}} \nabla v_\varepsilon\right) + F_\varepsilon(v_\varepsilon) = G(x), \quad \text{in } Q_T, \tag{2.2}$$

with

$$(|\nabla v_\varepsilon|^2 + \varepsilon)^{\frac{m(x)-2}{2}} \frac{\partial v_\varepsilon}{\partial \eta} + H_\varepsilon(x, u_\varepsilon) = 0, \quad on\ S_T, \tag{2.3}$$

and

$$\alpha_\varepsilon(v_\varepsilon(x,0)) = \alpha_\varepsilon(v_{0\varepsilon}(x)), \quad \text{in } \Omega. \tag{2.4}$$

Note that problems (2.2) - (2.4) have a unique classical solution, v_ε by the classical results of Ladyzenskaya et al. $[23, \text{Chapter } V]$.

Remark. We'll use the same symbol, C, in the sequel to denote a few positive constants that may differ from one another, showing up in different calculations and hypotheses and relying solely on the data rather than ε. In situations where we must determine the exact value of a single constant, we will utilize a notation such as $M_i, i = 1, 2, ...,$.
First, we offer a technical lemma that will be utilized numerous times in the following..

Lemma 3. *There exists a positive constant C depending on the problem data, but independent of ε*

$$||v_\varepsilon(s)||_{L^\infty(\Omega)} \leq C, \forall s > 0. \tag{2.5}$$

Proof. Multiplying $|\alpha_\varepsilon(v_\varepsilon)|^k \alpha_\varepsilon(v_\varepsilon)$ by the first equation in (2.2). We arrive at the conclusion that by applying the growth condition on $F_\varepsilon, H_\varepsilon$,

$$F_\varepsilon(v_\varepsilon) |\alpha_\varepsilon(v_\varepsilon)|^k \alpha_\varepsilon(v_\varepsilon) \geq -C_1 |\alpha_\varepsilon(v_\varepsilon)|^{p(x)+k},\ H_\varepsilon(x, v_\varepsilon) |\alpha_\varepsilon(v_\varepsilon)|^k \alpha_\varepsilon(v_\varepsilon) \geq 0,$$

and

$$\frac{1}{k+2} \frac{d}{dt} \int_\Omega |\alpha_\varepsilon(v_\varepsilon)|^{k+2} dx + C_2 \int_\Omega |\alpha_\varepsilon(v_\varepsilon)|^{p(x)+k} dx \leq C_3 \int_\Omega |\alpha_\varepsilon(v_\varepsilon)|^{k+1} dx. \tag{2.6}$$

Setting $y_{\varepsilon,k}(t) = \|\alpha_\varepsilon(v_\varepsilon)\|_{L^{k+2}(\Omega)}$ and on both sides of (2.6), applying Hölder's inequality, there exist two constants such that $\lambda_0 > 0$ and $\lambda_1 > 0$.

$$\frac{dy_{\varepsilon,k}(t)}{dt} + \lambda_1 y_{\varepsilon,k}^{(p^- - 1)}(t) \leq \lambda_0,$$

which suggests, according to Ghidaglia's Lemma 2, that

$$y_{\varepsilon,k}(t) \leq (\frac{\lambda_0}{\lambda_1})^{1/(p^- - 1)} + \frac{1}{[\lambda_0(p^- - 2)t]^{1/(p^- - 2)}} = C_4 \forall t > 0. \tag{2.7}$$

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For all $t \geq t_1 > 0$, and as $k \to +\infty$, we have

$$||\alpha_\varepsilon(v_\varepsilon(t))||_{L^\infty(\Omega)} \leq C(t_1). \tag{2.8}$$

With assumption (H2), we obtain

$$||v_\varepsilon(t)||_{L^\infty(\Omega)} \leq C(t_1). \tag{2.9}$$

Remark. The estimates (2.5) are important to our work because they aid in various stages of the a priori estimate proof.

Lemma 4. *The constants C_i exist under the hypotheses (H1)–(H4) such that the following estimates hold for any $\varepsilon > 0$ and any $t_1 > 0$.*

$$\|v_\varepsilon\|_{L^{m(x)}(0,T;W^{1,m(x)}(\Omega))} \leq C_5(T), \tag{2.10}$$

$$\|v_\varepsilon\|_{W^{1,m(x)}(\Omega)} \leq C_6(t_1), \textit{ for any } t \geq t_1 > 0, \tag{2.11}$$

$$\int_{t_0}^{T}\int_\Omega \alpha'_\varepsilon(v_\varepsilon)(\frac{\partial v_\varepsilon}{\partial t})^2 dxds \leq C_7(t_1), \textit{ for any } t \geq t_1 > 0. \tag{2.12}$$

Proof. Multiplying the first equation in (2.2) by v_ε, we obtain, after integrating over Ω

$$\frac{d}{dt}\Big(\int_\Omega \phi^*_\varepsilon(\alpha_\varepsilon(v_\varepsilon))dx\Big) + \int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)-2}{2}}|\nabla v_\varepsilon|^2 dx + \int_\Omega F_\varepsilon(v_\varepsilon)v_\varepsilon dx$$
$$+\int_{\partial\Omega} H_\varepsilon(x,v_\varepsilon)v_\varepsilon dx + \int_\Omega G(x)v_\varepsilon dx. \tag{2.13}$$

If we take assumptions (H3) and (H4), we have

$$\int_\Omega F(v_\varepsilon)v_\varepsilon dx \geq L_1\int_\Omega |v_\varepsilon|^{p(x)}dx - L_0, \int_{\partial\Omega} H_\varepsilon(x,v_\varepsilon)v_\varepsilon dx \geq 0. \tag{2.14}$$

Young's inequality allows us to obtain

$$\int_\Omega |v_\varepsilon|^2 dx \leq \int_\Omega \frac{2}{p(x)}|v_\varepsilon|^{p(x)}dx + \int_\Omega \frac{p(x)-2}{p(x)}|1|^{\frac{p(x)}{p(x)-2}}dx \leq \int_\Omega |v_\varepsilon|^{p(x)}dx + M_1,$$
$$\int_\Omega G(x)v_\varepsilon dx \leq C_8\int_\Omega |G(x)|^2 dx + C_9\int_\Omega |v_\varepsilon|^2 dx + M_2. \tag{2.15}$$

Putting the previously stated inequality into (2.13), we obtain:

$$\frac{d}{dt}\Big(\int_\Omega \phi^*_\varepsilon(\alpha_\varepsilon(v_\varepsilon))dx\Big) + \int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)-2}{2}}|\nabla v_\varepsilon|^2 dx + \int_\Omega |v_\varepsilon|^{p(x)}dx \leq M_3. \tag{2.16}$$

Lemma 3 and the equality (2.0) lead us to the conclusion that $\phi^*(\alpha(v))$ is bounded as well. By hypotheses (H1), we can assume that:

$\int_\Omega \phi^*_\varepsilon(\alpha_\varepsilon(v_{0\varepsilon}))dx$ converges to $\int_\Omega \phi^*(\alpha(v_0))dx \leq C$.

By hypotheses (H3) and using the boundedness of v_ε, we obtain after integrating (2.16) from 0 to T

$$\int_0^T\int_\Omega |\nabla v_\varepsilon|^{m(x)}dxdt \leq \int_0^T\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)-2}{2}}|\nabla v_\varepsilon|^2 dxdt \leq C. \tag{2.17}$$

The result (2.10) is obtained instantaneously. Now multiplying the first equation of (2.2) by $(v_\varepsilon)_t$ and integrating over Ω

$$\int_\Omega \alpha'_\varepsilon(v_\varepsilon)(\frac{\partial v_\varepsilon}{\partial t})^2 + \frac{d}{dt}\big(\frac{1}{m(x)}\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)}{2}})\big) + \int_\Omega F(v_\varepsilon)\frac{\partial v_\varepsilon}{\partial t}$$
$$+\int_{\partial\Omega} H(x,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial t} = \int_\Omega G(x)\frac{\partial v_\varepsilon}{\partial t}. \tag{2.18}$$

We set
$f_\varepsilon(s) = \int_0^s F_\varepsilon(y)dy$ and $h_\varepsilon(x,s) = \int_0^s H_\varepsilon(x,y)dy$, From assumption (H3), we have

$$C_1|s|^{p(x)} - C \leq f(s) \leq C_2|s|^{p(x)} + C. \tag{2.19}$$

By the hypothesis of H_ε and the boundedness of (v_ε), we can determine that there is a constant M_2 such that, based on assumption (H3), we deduce

$$\int_{\partial\Omega} h_\varepsilon(x,s)ds + M_2 \geq 0. \tag{2.20}$$

Putting the previously stated inequality into (2.18), we obtain

$$\frac{d}{dt}\Big(\frac{1}{m(x)}\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)}{2}})dx + \int_\Omega |v_\varepsilon|^{p(x)}\,dx + \int_{\partial\Omega} h_\varepsilon(x,s)ds + M_2\Big) \leq C. \tag{2.21}$$

Let us fix $T_0 > 0$, now integrating (2.21) on $[s, s+T_0]$, we can derive

$$\int_s^{s+T_0} \Big[\frac{1}{m(x)}\int_\Omega (|\nabla v_\varepsilon|^2+\varepsilon)^{\frac{m(x)}{2}})\Big]dxdy \leq C_{10}(T_0),$$
$$\int_s^{s+T_0} \Big(\int_\Omega |v_\varepsilon|^{p(x)}\,dx + \int_{\partial\Omega} h_\varepsilon(x,s)dx\Big)dsdy \leq C_{11}(T_0).$$

Therefore, (2.11) is true according to the uniform Gronwall Lemma 3.
After integrating (2.18) across (T_0, T), we arrive at

$$\int_{T_0}^T\int_\Omega \alpha_\varepsilon'(v_\varepsilon)(\frac{\partial v_\varepsilon}{\partial t})^2 dxds \leq \frac{1}{m(x)}\int_\Omega \Big((|\nabla v_\varepsilon(T_0|^2+\varepsilon)^{\frac{m(x)}{2}} - (|\nabla v_\varepsilon(T|^2+\varepsilon)^{\frac{m(x)}{2}}\Big)\,dx$$
$$+\int_\Omega (f_\varepsilon(T_0) - f_\varepsilon(T))\,dx + \int_{\partial\Omega} (h_\varepsilon(x,T_0) - h_\varepsilon(x,T))\,dx + C_{12}(T-T_0). \tag{2.22}$$

Thus, we can infer using the uniform Gronwall Lemma 3.

$$\int_{T_0}^T\int_\Omega \alpha_\varepsilon(v_\varepsilon)(\frac{\partial v_\varepsilon}{\partial t})^2 dxds \leq C(T_0,T),\ \text{for any } T \geq T_0 > 0.$$

By hypothesis of (H2), we obtain

$$\int_{T_0}^T\int_\Omega (\frac{\partial \alpha_\varepsilon(v_\varepsilon)}{\partial t})^2 dxds \leq C_{11}(T_0,T),\ \text{for any } T \geq T_0 > 0.$$

Lemma 4 is fully proved, and this (2.12) is demonstrated.

There exists a subsequence of v_ε (again indicated by itself) and v by (2.5), (2.10), (2.11), and (2.12), such that as $\varepsilon \to +\infty$:

$$v_\varepsilon \to v \ \ weakly\ \ star\ \ in\ \ L^\infty(Q_T),$$
$$v_\varepsilon \to v \ \ weakly\ \ in\ \ L^{m(x)}(0,T;W^{1,m(x)}(\Omega)),$$
$$v_\varepsilon \to v \ \ weakly\ \ star\ \ in\ \ L^\infty(T_0,T;W^{1,m(x)}(\Omega)), \forall T_0 > 0,$$
$$\frac{\partial v_\varepsilon}{\partial t} \to \frac{\partial v}{\partial t}\ \ \ in\ \ L^2(Q_T),$$
$$|\nabla v_\varepsilon|^{m(x)-2}\nabla v_\varepsilon \to \chi\ \ weakly\ \ in\, L^2(0,T;L^{m'(x)}(\Omega)).$$

We claim that $\chi = |\nabla v|^{m(x)-2}\nabla v$ by making use of the same reasoning as in [3]. The observation that $F(v_\varepsilon) \to F(v)$ and $H(x,v_\varepsilon) \to H(x,v)$ strongly in $L^1(Q_T)$ and in $L^r(0,T;L^r(\Omega))$ $(\forall r \geq 1)$ suffices to conclude the existence of weak bounded solution.

2.2 Uniqueness of solution

Lemma 5. *Let (H1) to (H4) be satisfied. There is only one solution to* (1.1)*. Furthermore, if* (v_1, v_2) *corresponds to the initial data* (u_0, v_0) *in such a way that* $u_0 \leq v_0$*, then* $v_1 \leq v_2$.

Proof. Assume that there are two solutions, (v_1, v_2) and that correspond to the initial data, respectively, such that $u_0 \leq v_0$ then $v_1 \leq v_2$. Following [1, Theorem 2.2], we consider the following test function:

$$U_\theta(s) := \min\Big(1, \max(0, \frac{s}{\theta})\Big), \text{ for all } s \in \mathbb{R}, \text{ and for } \theta > 0 \text{ small.}$$

Notice that $U_\theta(v_1 - v_2) \in L^{m^-}(0,T; W^{s,m(x)}(\Omega)) \cap L^\infty(Q_T)$, $\forall T > 0$, and that

$$\nabla U_\theta = \begin{cases} \nabla \frac{1}{\theta}(v_1 - v_2) & \text{if } 0 < v_1 - v_2 < \theta \\ 0 & \text{otherwise} \end{cases}.$$

Taking $U_\theta(v_1 - v_2)$ as the test function and considering v_1,v_2, two solutions for the problem (1.1), we arrive at

$$\iint_{Q_t} \Big(\frac{\partial(\alpha(v_1) - \alpha(v_2))}{\partial t}\Big) U_\theta(v_1 - v_2) + \iint_{Q_t} |\nabla v_1|^{m(x)-2}\nabla v_1 - |\nabla v_2|^{m(x)-2}\nabla v_2)\nabla(v_1 - v_2) U'_\theta$$
$$+ \iint_{Q_t} (F(v_1) - F(v_2)) U_\theta + \int_0^t \int_{\partial\Omega} (H(x, v_1) - H(x, v_2)) U_\theta = 0. \tag{2.23}$$

Furthermore, the following inequality results from $r(x) \geq 2 \; \forall \; x \in \overline{\Omega}$ (see [22])

$$(|\nabla v_1|^{r(x)-2}\nabla v_1 - |\nabla v_2|^{r(x)-2}\nabla v_2)\nabla(v_1 - v_2) \geq \left(\frac{1}{2}\right)^{r(x)} |\nabla v_1 - \nabla v_2|^{r(x)} \geq 0. \tag{2.24}$$

Using the fact $\frac{\partial\alpha(v_1)}{\partial t}$, $\frac{\partial\alpha(v_2)}{\partial t} \in L^1(Q_T)$, we get

$$\iint_{Q_t} \Big(\frac{\partial(\alpha(v_1) - \alpha(v_2))}{\partial t}\Big) U_\theta \to \iint_{Q_t} (\alpha(v_1(t)) - \alpha(v_2(t)))^+.$$

By (H3) and (H4), we get

$$\iint_{Q_t} (F(v_1) - F(v_2)) U_\theta \to \iint_{Q_t} (F(v_1) - F(v_2)) \chi_{\{(v_1 - v_2) > 0\}}) \geq -L_3 \int_\Omega ((\alpha(v_1) - \alpha(v_2))^+,$$

and

$$\int_0^t \int_{\partial\Omega} (H(x, v_1) - H(x, v_2)) U_\theta \to \int_0^t \int_{\partial\Omega} ((H(x, v_1) - H(x, v_2)) \chi_{\{(v_1 - v_2) > 0\}}) \geq 0,$$

where the positive part of s and the characteristic function are denoted by χ and $s^+ := \max(s, 0)$, respectively. Now, let tends $\theta \to 0$, Thus, we obtain

$$\int_\Omega (\alpha(v_1(t)) - \alpha(v_2(t)))^+ \leq L_3 \int_0^t \int_\Omega (\alpha(v_1) - \alpha(v_2))^+. \tag{2.25}$$

By standard Gronwall Lemma 3, we can therefore infer that $\alpha(v_1) - \alpha(v_2) \leq 0$. In the set $\{(v_1 - v_2) > 0\}$, we have $\alpha(v_1) = \alpha(v_2)$ according to hypothesis (H1). By (2.24), we obtain

$$\nabla(v_1 - v_2) = 0 \text{ in the set } \{0 < (v_1 - v_2) < \theta\}.$$

Therefore, since $\max(0, \min(v_1 - v_2), \theta) = const$ holds true on S_T, we can conclude that the solution is unique and that $v_1 \leq v_2$.

Remark. Under the suppositions of Theorem 1, one obtains that the solution operator $S(t)v_0 = v(t)$ $(t \geq 0)$ of Problem (1.1) generates a semigroup that verifies the following properties:

1. $S(t) : L^2(\Omega) \to L^2(\Omega)$ for $t \geq 0$ and $S(0)v_0 = v_0$, for $v_0 \in L^2(\Omega)$;
2. $S(t+s) = S(t)S(s)$ for $t, s \geq 0$;
3. $S(t)\phi \to S(s)\phi$ in $L^2(\Omega)$ as $t \to s$ for every $\phi \in L^2(\Omega)$;
4. $\frac{\partial\alpha(v)}{\partial t} \in L^2(s, +\infty; L^2(\Omega))$, for every $s > 0$.

3 Global attractor

The reader is directed to [33] for the definitions of global attractors and absorbing sets used here.

Theorem 2. *Assuming that (H1) - (H4) are satisfied, the corresponding semigroup* $(S(s))_{s\geq 0}$ *possesses an absorbing set in* $W^{1,m(.)}(\Omega)\cap L^{p(.)}(\Omega)$*, and there is a bounded set* $\mathscr{B}_0 \subset W^{1,m(.)}(\Omega)\cap L^{p(.)}(\Omega)$ *such that, for any bounded set* $\mathscr{B}$ *in* $L^2(\Omega)$, $\exists T_0 > 0$ *such that* $S(s)\mathscr{B} \subset \mathscr{B}_0\ \forall s \geq T_0$*. In this case,* T_0 *depends only on* $\mathscr{B}$*.*

Proof. Let v represent the (1.1) solution, and v_ε represent the (P_ε) solution that approximates v. Lemma 3 can be utilized to obtain

$$\|v_\varepsilon(s)\|_{L^\infty(\Omega)} \leq c(\tau), \text{ for any } s \geq \tau. \tag{3.0}$$

By letting ε tend to 0 in (3.0), we get

$$\|v(s)\|_{L^\infty(\Omega)} \leq c(\tau). \tag{3.1}$$

Therefore, the open ball $B(0,c_\tau)$, with radius $c(\tau)$ and center at 0, is an absorbing set in $L^\infty(\Omega)$ by (3.0) and (3.1).
By (2.11), we have

$$\|v_\varepsilon\|_{W^{1,m(.)}(\Omega)} \leq C_8(t_0), \text{ for any } t \geq t_0 > 0.$$

By letting ε tend to 0 in this equality, we obtain

$$\|v(t)\|_{W^{1,m(.)}(\Omega)} \leq c_{t_0}, \text{ for any } t \geq t_0 > 0.$$

Thus, in $W^{1,m(.)}(\Omega)$, the ball $B(0,c_{t_0})$, centered at 0 and with radius c_{t_0}, is an absorbing set.

Applying the theorem 1 in [26] and the compact imbedding results in [17], we get:

Corollary 1. *Under the conditions of theorem 2 and for* $2 \leq m^- \leq m^+ < \infty$*, the corresponding semigroup generated by* (1.1) *with initial data* v_0 *in* $L^\infty(\Omega)\cap W^{1,m(x)}(\Omega)$ *possesses a global attractor* $\mathscr{A}$ *in* $L^2(\Omega)$*, that is,* $\mathscr{A}$ *is compact, invariant in* $L^2(\Omega)$ *and attracts every bounded subset of* $L^2(\Omega)$ *in the topology of* $L^2(\Omega)\cap L^\infty(\Omega)$*.*

4 Conclusion

In this paper, we have studied the $m(x)$-Laplacian problem with nonlinear boundary conditions. Due to the presence of doubly nonlinear linearity, we obtain the existence of classical solutions for regularized problems associated with problem (1.1), which can be solved in a classical sense by well-known results of [23]. We use some a priori estimates in suitable functional spaces to study the convergence of these solutions. Global attractors' existence in appropriate spaces is demonstrated by making general assumptions under some sufficient conditions.

Declarations

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