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Some Fixed Point Results in Complete b-Metric Spaces

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Abstract: The aims of this paper are threefold: the first is to prove the existence and uniqueness of common fixed points for two pairs of maps satisfying a new concept in a complete b-metric space, the second purpose is to produce two illustrative examples; on one hand, to show the applicability, validity and credibility of our results, and on the other hand, to show their superiority over several results, for instance the one's of Roshan et al. [17], and the third and last objective is to apply one of our results to solve an integral equation.

Keywords: Complete b-metric spaces; occasionally weakly biased maps of type (A); unique common fixed points; integral equation.

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1 Introduction

The subject of fixed points is nowadays a forceful branch. Furthermore, the contraction theorem is looked at as the starting point to prove the existence and uniqueness of fixed points. To improve, prolong and generalize this famous result, the authors were divided into three groups; in the first group, many authors raised the number of maps to search common fixed points, in the second group, many others focused on the underlying complete metric space, more precisely, they introduced a number of other spaces like, weak partial, rectangular, quasi-metric, s-metric, s_b-metric, b-metric, b-metric-like, multiplicative, ultra metric spaces, and so on. In particular, according to [1] and [2], the idea of b-metric spaces was initiated by [4], [7] and [8]. Note that, several authors worked in b-metric spaces (see for instance [1]-[4], [5]-[12], [14]-[18]). In the third group, several mathematicians emphasized the contractive condition, they improved, extended and generalized the contraction in Banach's theorem by giving different conditions. For instance, recently in 2022, a novel notion was introduced in order to find common fixed points under minimum conditions. In this paper, we will use our notion to refine and extend some results in complete metric and complete b-metric spaces especially the main results of [17].

2 Preliminary

Definition 1.([4], [7], [8]) *Let $\mathbb{X} \neq \emptyset$ be a set and let $s \geq 1$ be a real value. Consider the function $d : \mathbb{X} \times \mathbb{X} \to [0, +\infty)$. $(\mathbb{X}, d, s)$ is a **b-metric space** if for each λ, μ, $\nu \in \mathbb{X}$, the next axioms are satisfied:*

1. $d(\lambda, \mu) = 0 \Leftrightarrow \lambda = \mu$;
2. $d(\lambda, \mu) = d(\mu, \lambda)$;
3. $d(\lambda, \nu) \leq s(d(\lambda, \mu) + d(\mu, \nu))$.

Definition 2.([13]) *Let $(\mathbb{X}, d)$ be a metric space and let $\mathbb{F}$, $\mathbb{G} : \mathbb{X} \to \mathbb{X}$. We say that $\mathbb{F}$ and $\mathbb{G}$ are **compatible** if and only if*

$$\lim_{n \to \infty} d(\mathbb{F}\mathbb{G}t_n, \mathbb{G}\mathbb{F}t_n) = 0,$$

whenever $\{t_n\}$ is a sequence in $\mathbb{X}$ such that $\lim_{n \to \infty} \mathbb{F}t_n = \lim_{n \to \infty} \mathbb{G}t_n = l$ for some $l \in \mathbb{X}$.

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Definition 3.*([6]) $\mathbb{Y}$, $\mathbb{Z}:\mathbb{X}\to\mathbb{X}$ are* ***occasionally weakly*** *$\mathbb{Y}$**-biased (respectively,** $\mathbb{Z}$**-biased) of type** $(\mathscr{A})$ iff, there is ξ in $\mathbb{X}$ such that $\mathbb{Y}\xi=\mathbb{Z}\xi$ implies*

$$d(\mathbb{Y}\mathbb{Y}\xi,\mathbb{Z}\xi)\le d(\mathbb{Z}\mathbb{Y}\xi,\mathbb{Y}\xi),$$
$$d(\mathbb{Z}\mathbb{Z}\xi,\mathbb{Y}\xi)\le d(\mathbb{Y}\mathbb{Z}\xi,\mathbb{Z}\xi),$$

respectively.

3 Results of Roshan et al.

In their paper [17], the following theorems are proved:

Theorem 1.*Let $(\mathbb{X},d,s)$ be a complete b-metric space and let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}:\mathbb{X}\to\mathbb{X}$ satisfying:*

1.$\mathbb{C}$ and $\mathbb{D}$ are continuous,
2.$\mathbb{A}(\mathbb{X})\subseteq\mathbb{D}(\mathbb{X})$, $\mathbb{B}(\mathbb{X})\subseteq\mathbb{C}(\mathbb{X})$,
3.$\mathbb{A}$ and $\mathbb{C}$ (respectively $\mathbb{B}$ and $\mathbb{D}$) are compatible,
4.for every x, $y\in\mathbb{X}$

$$d(\mathbb{A}x,\mathbb{B}y)\le\frac{\tau}{\varsigma^4}\max\{d(\mathbb{C}x,\mathbb{D}y),d(\mathbb{A}x,\mathbb{C}x),d(\mathbb{B}y,\mathbb{D}y),\frac{1}{2}[dm(\mathbb{C}x,\mathbb{B}y)+d(\mathbb{A}x,\mathbb{D}y)]\},$$

where $\varsigma\ge1$ is a given real number and $0<\tau<1$.

Then, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ have a unique common fixed point in $\mathbb{X}$.

Theorem 2.*Let $(\mathbb{X},d,s)$ be a complete b-metric space and let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}:\mathbb{X}\to\mathbb{X}$ be maps satisfying:*

1.$\mathbb{C}$ and $\mathbb{D}$ are continuous,
2.$\mathbb{A}(\mathbb{X})\subseteq\mathbb{D}(\mathbb{X})$ and $\mathbb{B}(\mathbb{X})\subseteq\mathbb{C}(\mathbb{X})$,
3.$\mathbb{A}$ and $\mathbb{C}$ as well as $\mathbb{B}$ and $\mathbb{D}$ are compatible,
4.for all x, $y\in\mathbb{X}$,

$$d(\mathbb{A}x,\mathbb{B}y)\le\frac{1}{\delta^4}(\zeta_1d(\mathbb{C}x,\mathbb{D}y)+\zeta_2d(\mathbb{A}x,\mathbb{D}y)+\zeta_3d(\mathbb{C}x,\mathbb{B}y)+\zeta_4d(\mathbb{A}x,\mathbb{C}x)+\zeta_5d(\mathbb{B}y,\mathbb{D}y)),$$

where $\delta\ge1$ is a given real number and $\zeta_i\ge0$ $(i=1,2,3,4,5)$ are real constants with $\zeta_1+\rho\zeta_2+\rho\zeta_3+\zeta_4+\zeta_5<1$, where $\rho+\rho=2$, for ρ, $\rho\in\mathbb{N}\cup\{0\}$.

Then $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ have a unique common fixed point.

In this paper, we will improve the above results and some other ones by removing the continuity and the inclusions, using our recent notion which is more general than the concept of compatibility.

4 Main Results

Theorem 3.*Let $(\mathbb{X},d,s)$ be a complete b-metric space. Consider four maps $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}:\mathbb{X}\to\mathbb{X}$. Assume that for each x, $y\in\mathbb{X}$ these four maps satisfy the following inequality:*

$$d(\mathbb{A}x,\mathbb{B}y)\le\frac{q}{k^4}\max\{d(\mathbb{C}x,\mathbb{D}y),d(\mathbb{A}x,\mathbb{C}x),d(\mathbb{B}y,\mathbb{D}y),d(\mathbb{C}x,\mathbb{B}y),d(\mathbb{A}x,\mathbb{D}y)\}\tag{1}$$

where $k\ge1$ and $0<q<\frac{1}{2}$ are real values. If the pairs of maps $(\mathbb{A},\mathbb{C})$ (respectively, $(\mathbb{B},\mathbb{D})$) are occasionally weakly $\mathbb{C}$-biased (respectively $\mathbb{D}$-biased) of type $(\mathscr{A})$. Then, there is a unique term say δ in $\mathbb{X}$ which satisfies $\delta=\mathbb{A}\delta=\mathbb{B}\delta=\mathbb{C}\delta=\mathbb{D}\delta$.

Proof. According to the conditions, there exist elements s and t which verify

$\mathbb{A}s = \mathbb{C}s$ *implies* $d(\mathbb{C}\mathbb{C}s, \mathbb{A}s) \leq d(\mathbb{A}\mathbb{C}s, \mathbb{C}s)$,

$\mathbb{B}t = \mathbb{D}t$ *implies* $d(\mathbb{D}\mathbb{D}t, \mathbb{B}t) \leq d(\mathbb{B}\mathbb{D}t, \mathbb{D}t)$.

We will use four steps to prove the existence and uniqueness of the common fixed point.

Firstly: We show that $\mathbb{A}s = \mathbb{B}t$. If we assume that $d(\mathbb{A}s, \mathbb{B}t) > 0$, then, we obtain

$$\begin{aligned} d(\mathbb{A}s, \mathbb{B}t) &\leq \frac{q}{k^4} \max\{d(\mathbb{C}s, \mathbb{D}t), d(\mathbb{A}s, \mathbb{C}s), d(\mathbb{B}t, \mathbb{D}t), d(\mathbb{C}s, \mathbb{B}t), d(\mathbb{A}s, \mathbb{D}t)\} \\ &= \frac{q}{k^4} \max\{d(\mathbb{A}s, \mathbb{B}t), 0, 0, d(\mathbb{A}s, \mathbb{B}t), d(\mathbb{A}s, \mathbb{B}t)\} \\ &= \frac{q}{k^4} d(\mathbb{A}s, \mathbb{B}t) \\ &< d(\mathbb{A}s, \mathbb{B}t) \end{aligned}$$

thus, $\mathbb{A}s = \mathbb{B}t$.

Secondly: If $d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) > 0$, then, we acquire

$$\begin{aligned} d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) = d(\mathbb{A}\mathbb{A}s, \mathbb{B}t) &\leq \frac{q}{k^4} \max\{d(\mathbb{C}\mathbb{A}s, \mathbb{D}t), d(\mathbb{A}\mathbb{A}s, \mathbb{C}\mathbb{A}s), d(\mathbb{B}t, \mathbb{D}t), \\ &\quad d(\mathbb{C}\mathbb{A}s, \mathbb{B}t), d(\mathbb{A}\mathbb{A}s, \mathbb{D}t)\} \\ &= \frac{q}{k^4} \max\{d(\mathbb{C}\mathbb{C}s, \mathbb{A}s), d(\mathbb{A}\mathbb{A}s, \mathbb{C}\mathbb{C}s), 0, d(\mathbb{C}\mathbb{C}s, \mathbb{A}s), \\ &\quad d(\mathbb{A}\mathbb{A}s, \mathbb{A}s)\} \\ &\leq \frac{q}{k^4} \max\{d(\mathbb{C}\mathbb{C}s, \mathbb{A}s), k(d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) + d(\mathbb{A}s, \mathbb{C}\mathbb{C}s)), \\ &\quad 0, d(\mathbb{C}\mathbb{C}s, \mathbb{A}s), d(\mathbb{A}\mathbb{A}s, \mathbb{A}s)\}. \end{aligned}$$

Since $\mathbb{A}$ and $\mathbb{C}$ are occasionally weakly $\mathbb{C}$-biased of type $(\mathscr{A})$, we get

$$\begin{aligned} d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) &\leq \frac{q}{k^4} \max\{d(\mathbb{A}\mathbb{C}s, \mathbb{C}s), k(d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) + d(\mathbb{C}s, \mathbb{A}\mathbb{C}s)), 0, \\ &\quad d(\mathbb{A}\mathbb{C}s, \mathbb{C}s), d(\mathbb{A}\mathbb{A}s, \mathbb{A}s)\} \\ &= \frac{q}{k^4} \max\{d(\mathbb{A}\mathbb{A}s, \mathbb{A}s), 2kd(\mathbb{A}\mathbb{A}s, \mathbb{A}s), 0, d(\mathbb{A}\mathbb{A}s, \mathbb{A}s), \\ &\quad d(\mathbb{A}\mathbb{A}s, \mathbb{A}s)\} \\ &= \frac{2q}{k^3} d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) \\ &< d(\mathbb{A}\mathbb{A}s, \mathbb{A}s) \end{aligned}$$

a contradiction, thus, $\mathbb{A}\mathbb{A}s = \mathbb{A}s$, consequently, $\mathbb{C}\mathbb{A}s = \mathbb{A}s$.

Thirdly: We prove that $\mathbb{B}\mathbb{B}t = \mathbb{B}t$. If we suppose that we have the contrary, then, (1) gives

$$\begin{aligned} d(\mathbb{A}s, \mathbb{B}\mathbb{B}t) &\leq \frac{q}{k^4} \max\{d(\mathbb{C}s, \mathbb{D}\mathbb{B}t), d(\mathbb{A}s, \mathbb{C}s), d(\mathbb{B}\mathbb{B}t, \mathbb{D}\mathbb{B}t), d(\mathbb{C}s, \mathbb{B}\mathbb{B}t), \\ &\quad d(\mathbb{A}s, \mathbb{D}\mathbb{B}t)\}; \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathbb{B}t, \mathbb{B}\mathbb{B}t) &\leq \frac{q}{k^4} \max\{d(\mathbb{B}t, \mathbb{D}\mathbb{D}t), 0, d(\mathbb{B}\mathbb{B}t, \mathbb{D}\mathbb{D}t), d(\mathbb{B}t, \mathbb{B}\mathbb{B}t), d(\mathbb{B}t, \mathbb{D}\mathbb{D}t)\} \\ &\leq \frac{q}{k^4} \max\{d(\mathbb{B}t, \mathbb{D}\mathbb{D}t), 0, k(d(\mathbb{B}\mathbb{B}t, \mathbb{B}t) + d(\mathbb{B}t, \mathbb{D}\mathbb{D}t)), d(\mathbb{B}t, \mathbb{B}\mathbb{B}t), \\ &\quad d(\mathbb{B}t, \mathbb{D}\mathbb{D}t)\}. \end{aligned}$$

By the relationship between $\mathbb{B}$ and $\mathbb{D}$, we find

$$\begin{aligned} d(\mathbb{B}t, \mathbb{B}\mathbb{B}t) &\leq \frac{q}{k^4} \max\{d(\mathbb{B}\mathbb{D}t, \mathbb{D}t), 0, k(d(\mathbb{B}\mathbb{B}t, \mathbb{B}t) + d(\mathbb{B}\mathbb{D}t, \mathbb{D}t)), d(\mathbb{B}t, \mathbb{B}\mathbb{B}t), \\ &\quad d(\mathbb{B}\mathbb{D}t, \mathbb{D}t)\} \\ &= \frac{q}{k^4} \max\{d(\mathbb{B}\mathbb{B}t, \mathbb{B}t), 0, 2kd(\mathbb{B}\mathbb{B}t, \mathbb{B}t), d(\mathbb{B}t, \mathbb{B}\mathbb{B}t), d(\mathbb{B}\mathbb{B}t, \mathbb{B}t)\} \\ &= \frac{2q}{k^3} d(\mathbb{B}t, \mathbb{B}\mathbb{B}t) \\ &< d(\mathbb{B}t, \mathbb{B}\mathbb{B}t) \end{aligned}$$

this contradiction confirms that $\mathbb{B}\mathbb{B}t = \mathbb{B}t$ and consequently $\mathbb{D}\mathbb{B}t = \mathbb{B}t$. Therefore $\mathbb{A}s = \mathbb{C}s = \mathbb{B}t = \mathbb{D}t = \delta$.

Fourthly and lastly: Let σ be another fixed point, then, we get

$$\begin{aligned} d(\mathbb{A}\delta,\mathbb{B}\sigma) \leq \frac{q}{k^4}\max\{&d(\mathbb{C}\delta,\mathbb{D}\sigma),d(\mathbb{A}\delta,\mathbb{C}\delta),d(\mathbb{B}\sigma,\mathbb{D}\sigma),d(\mathbb{C}\delta,\mathbb{B}\sigma),\\ &d(\mathbb{A}\delta,\mathbb{D}\sigma)\}; \end{aligned}$$

i.e.,

$$\begin{aligned} d(\delta,\sigma) &\leq \frac{q}{k^4}\max\{d(\delta,\sigma),0,0,d(\delta,\sigma),d(\delta,\sigma)\}\\ &= \frac{q}{k^4}d(\delta,\sigma)\\ &< d(\delta,\sigma) \end{aligned}$$

which is a contradiction, hence, $\sigma = \delta$.

The next example clarifies our theorem.

Example 1. Endow $\mathbb{X} = [0,+\infty)$ with the b-metric $d(x,y) = (x-y)^2$, where $k = 2$. Take $q = \frac{1}{3}$ and define

$$\mathbb{A}x = \begin{cases} 1 \text{ if } x \in [0,1] \\ \frac{3}{4x} \text{ if } x \in (1,+\infty), \end{cases} \quad \mathbb{B}x = \begin{cases} 1 \text{ if } x \in [0,1] \\ \frac{1}{2y} \text{ if } x \in (1,+\infty), \end{cases}$$

$$\mathbb{C}x = \begin{cases} \frac{2}{x+1} \text{ if } x \in [0,1] \\ 500 \text{ if } x \in (1,+\infty), \end{cases} \quad \mathbb{D}x = \begin{cases} \frac{3}{y+2} \text{ if } x \in [0,1] \\ 1000 \text{ if } x \in (1,+\infty). \end{cases}$$

First of all, the bias condition is satisfied. We have

1. in $[0,1]$, $\mathbb{A}x = 1$, $\mathbb{B}y = 1$, $\mathbb{C}x = \frac{2}{x+1}$, $\mathbb{D}y = \frac{3}{y+2}$ and

$$\begin{aligned} 0 \leq \frac{1}{48}\max\Bigg\{&\left(\frac{2}{x+1}-\frac{3}{y+2}\right)^2,\left(1-\frac{2}{x+1}\right)^2,\\ &\left(1-\frac{3}{y+2}\right)^2\Bigg\}, \end{aligned}$$

2. in $(1,+\infty)$, $\mathbb{A}x = \frac{3}{4x}$, $\mathbb{B}y = \frac{1}{2y}$, $\mathbb{C}x = 500$, $\mathbb{D}y = 1000$ and

$$\begin{aligned} \left(\frac{3}{4x}-\frac{1}{2y}\right)^2 \leq \frac{1}{48}\max\Bigg\{&(500)^2,\left(\frac{3}{4x}-500\right)^2,\left(\frac{1}{2y}-1000\right)^2,\\ &\left(500-\frac{1}{2y}\right)^2,\left(\frac{3}{4x}-1000\right)^2\Bigg\}, \end{aligned}$$

3. for $x \in [0,1]$, $y \in (1,+\infty)$, $\mathbb{A}x = 1$, $\mathbb{B}y = \frac{1}{2y}$, $\mathbb{C}x = \frac{2}{x+1}$, $\mathbb{D}y = 1000$ and

$$\begin{aligned} \left(1-\frac{1}{2y}\right)^2 \leq \frac{1}{48}\max\Bigg\{&\left(\frac{2}{x+1}-1000\right)^2,\left(1-\frac{2}{x+1}\right)^2,\\ &\left(\frac{1}{2y}-1000\right)^2,\left(\frac{2}{x+1}-\frac{1}{2y}\right)^2,(999)^2\Bigg\}, \end{aligned}$$

4. for $x \in (1,+\infty)$, $y \in [0,1]$, $\mathbb{A}x = \frac{3}{4x}$, $\mathbb{B}y = 1$, $\mathbb{C}x = 500$, $\mathbb{D}y = \frac{3}{y+2}$ and

$$\begin{aligned} \left(\frac{3}{4x}-1\right)^2 \leq \frac{1}{48}\max\Bigg\{&\left(500-\frac{3}{y+2}\right)^2,\left(\frac{3}{4x}-500\right)^2,\\ &\left(1-\frac{3}{y+2}\right)^2,(499)^2,\left(\frac{3}{4x}-\frac{3}{y+2}\right)^2\Bigg\}, \end{aligned}$$

therefore, all the requirements are satisfied and 1 is the unique common fixed point.

*Remark.*It is noticed that Theorem 2.1 of [17] is not usable since the four maps are discontinuous, $\mathbb{A}\mathbb{X} = \left(0, \frac{1}{4}\right) \cup \{1\} \nsubseteq \mathbb{D}\mathbb{X} = \left[1, \frac{3}{2}\right] \cup \{1000\}$ and $\mathbb{B}\mathbb{X} = \left(0, \frac{1}{2}\right) \cup \{1\} \nsubseteq \mathbb{C}\mathbb{X} = [1,2] \cup \{500\}$, and neither $\mathbb{A}$ and $\mathbb{C}$ nor $\mathbb{B}$ and $\mathbb{D}$ are compatible. For this end, consider the sequence $x_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$. We have

$$\mathbb{A}x_n = 1 \to 1 \text{ when } n \to +\infty,$$
$$\mathbb{C}x_n = \frac{2}{x_n+1} \to 1 \text{ when } n \to +\infty,$$
$$\mathbb{B}x_n = 1 \to 1 \text{ when } n \to +\infty,$$
$$\mathbb{D}x_n = \frac{3}{x_n+2} \to 1 \text{ when } n \to +\infty,$$

however,

$$d(\mathbb{A}\mathbb{C}x_n, \mathbb{C}\mathbb{A}x_n) \to \left(\frac{3}{4} - 1\right)^2 = \frac{1}{16} \neq 0,$$
$$d(\mathbb{B}\mathbb{D}x_n, \mathbb{D}\mathbb{B}x_n) \to \left(\frac{1}{2} - 1\right)^2 = \frac{1}{4} \neq 0;$$

i.e., neither $\mathbb{A}$ and $\mathbb{C}$ nor $\mathbb{B}$ and $\mathbb{D}$ are compatible.

*Remark.*In Theorem 3, we can change condition (1) by:

$$d(\mathbb{A}x, \mathbb{B}y) \leq \frac{q}{k^4} \max\left\{d(\mathbb{C}x, \mathbb{D}y), \frac{1}{2}d(\mathbb{A}x, \mathbb{C}x), \frac{1}{2}d(\mathbb{B}y, \mathbb{D}y), d(\mathbb{C}x, \mathbb{B}y), d(\mathbb{A}x, \mathbb{D}y)\right\}$$

with $0 < q < 1$.

$$d(\mathbb{A}x, \mathbb{B}y) \leq \frac{q}{k^4} \max\left\{d(\mathbb{C}x, \mathbb{D}y), \frac{1}{4}(d(\mathbb{A}x, \mathbb{C}x) + d(\mathbb{B}y, \mathbb{D}y)), d(\mathbb{C}x, \mathbb{B}y), d(\mathbb{A}x, \mathbb{D}y)\right\}$$

with $0 < q < 1$.

$$d(\mathbb{A}x, \mathbb{B}y) \leq \frac{q}{k^4} \max\left\{d(\mathbb{C}x, \mathbb{D}y), \frac{1}{2}(d(\mathbb{A}x, \mathbb{C}x) + d(\mathbb{B}y, \mathbb{D}y)), d(\mathbb{C}x, \mathbb{B}y), d(\mathbb{A}x, \mathbb{D}y)\right\}$$

with $0 < q < \frac{1}{2}$.

Corollary 1.*Let $(\mathbb{X}, d, s)$ be a complete b-metric space. Let $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ be a map which verifies*

$$d(\mathbb{A}x, \mathbb{A}y) \leq \frac{q}{k^4} \max\{d(x,y), d(\mathbb{A}x, x), d(\mathbb{A}y, y), d(x, \mathbb{A}y), d(\mathbb{A}x, y)\} \tag{2}$$

for any x, $y \in \mathbb{X}$, where $0 < q < \frac{1}{2}$ and $k \geq 1$ is a given real number. Then, $\mathbb{A}$ has a unique fixed point.

*Proof.*Let $\eta_0 \in \mathbb{X}$, then, there is a point η_1 in $\mathbb{X}$ such that $\eta_1 = \mathbb{A}\eta_0$, for this point η_1 there exists another element η_2 such that $\eta_2 = \mathbb{A}\eta_1$, continuing in this manner we get the sequence $\eta_{n+1} = \mathbb{A}\eta_n$ for $n = 0, 1, 2, \ldots$. If there is a natural element m which verifies $\eta_{m+1} = \eta_m$ then $\eta_m = \mathbb{A}\eta_m$; i.e., $\mathbb{A}$ has a fixed point and the proof is ended. Suppose that $\eta_{n+1} \neq \eta_n$ for each $n \in \mathbb{N} \cup \{0\}$ then

$$\begin{aligned} d(\mathbb{A}\eta_n, \mathbb{A}\eta_{n+1}) \leq \frac{q}{k^4} \max\{& d(\eta_n, \eta_{n+1}), d(\mathbb{A}\eta_n, \eta_n), d(\mathbb{A}\eta_{n+1}, \eta_{n+1}), \\ & d(\eta_n, \mathbb{A}\eta_{n+1}), d(\mathbb{A}\eta_n, \eta_{n+1})\} \end{aligned}$$

i.e.,

$$\begin{aligned} d(\eta_{n+1}, \eta_{n+2}) &\leq \frac{q}{k^4} \max\{d(\eta_n, \eta_{n+1}), d(\eta_{n+1}, \eta_n), d(\eta_{n+2}, \eta_{n+1}), \\ &\quad d(\eta_n, \eta_{n+2}), d(\eta_{n+1}, \eta_{n+1})\} \\ &= \frac{q}{k^4} \max\{d(\eta_n, \eta_{n+1}), d(\eta_{n+2}, \eta_{n+1}), d(\eta_n, \eta_{n+2}), 0\} \\ &\leq \frac{q}{k^4} \max\{d(\eta_n, \eta_{n+1}), d(\eta_{n+2}, \eta_{n+1}), \\ &\quad k(d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+2})), 0\} \\ &= \frac{q}{k^3}(d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+2})), \end{aligned}$$

which implies that

$$d(\eta_{n+1},\eta_{n+2}) \le \frac{\frac{q}{k^3}}{1-\frac{q}{k^3}} d(\eta_n,\eta_{n+1}).$$

By the same manner, we get

$$d(\eta_n,\eta_{n+1}) \le \frac{\frac{q}{k^3}}{1-\frac{q}{k^3}} d(\eta_{n-1},\eta_n),$$

in consequence

$$\begin{aligned} d(\eta_n,\eta_{n+1}) &\le \frac{\frac{q}{k^3}}{1-\frac{q}{k^3}} d(\eta_{n-1},\eta_n) \le \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^2 d(\eta_{n-2},\eta_{n-1}) \\ &\le \ldots \le \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^n d(\eta_0,\eta_1), \end{aligned}$$

at infinity $d(\eta_n,\eta_{n+1}) \to 0$ because $\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}} < 1$.

Now, for each integer $m > 0$, we obtain

$$\begin{aligned} d(\eta_n,\eta_{n+m}) &\le k(d(\eta_n,\eta_{n+1}) + d(\eta_{n+1},\eta_{n+2}) + \ldots + d(\eta_{n+m-1},\eta_{n+m})) \\ &\le k\left(d(\eta_n,\eta_{n+1}) + \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right) d(\eta_n,\eta_{n+1}) \right. \\ &\qquad \left. + \ldots + \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^{m-1} d(\eta_n,\eta_{n+1})\right) \\ &= k\left(1 + \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right) + \ldots + \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^{m-1}\right) d(\eta_n,\eta_{n+1}) \\ &= k\left[\frac{1-\left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^m}{1-\left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)}\right] d(\eta_n,\eta_{n+1}) \\ &\le k\left[\frac{1-\left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^m}{1-\left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)}\right] \left(\frac{\frac{q}{k^3}}{1-\frac{q}{k^3}}\right)^n d(\eta_0,\eta_1), \end{aligned}$$

at infinity $d(\eta_n,\eta_{n+m}) \to 0$ which means that $\{\eta_n\}$ is a Cauchy sequence and since $\mathbb{X}$ is complete, $\{\eta_n\}$ converges to ς. Moreover, we have

$$\begin{aligned} d(\mathbb{A}\varsigma,\mathbb{A}\eta_n) \le \frac{q}{k^4} \max\{&d(\varsigma,\eta_n), d(\mathbb{A}\varsigma,\varsigma), d(\mathbb{A}\eta_n,\eta_n), d(\varsigma,\mathbb{A}\eta_n), \\ &d(\mathbb{A}\varsigma,\eta_n)\}; \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathbb{A}\varsigma,\eta_{n+1}) \le \frac{q}{k^4} \max\{&d(\varsigma,\eta_n), d(\mathbb{A}\varsigma,\varsigma), d(\eta_{n+1},\eta_n), d(\varsigma,\eta_{n+1}), \\ &d(\mathbb{A}\varsigma,\eta_n)\}, \end{aligned}$$

at infinity, we obtain

$$\begin{aligned} d(\mathbb{A}\varsigma,\varsigma) &\le \frac{q}{k^4} \max\{d(\varsigma,\varsigma), d(\mathbb{A}\varsigma,\varsigma), d(\varsigma,\varsigma), d(\varsigma,\varsigma), d(\mathbb{A}\varsigma,\varsigma)\} \\ &= \frac{q}{k^4} \max\{0, d(\mathbb{A}\varsigma,\varsigma)\} \\ &= \frac{q}{k^4} d(\mathbb{A}\varsigma,\varsigma) \\ &< d(\mathbb{A}\varsigma,\varsigma) \end{aligned}$$

a contradiction, hence, $\mathbb{A}\varsigma = \varsigma$.

If we assume that there exists another fixed point say θ, then

$$d(\mathbb{A}\varsigma, \mathbb{A}\theta) \le \frac{q}{k^4} \max\{d(\varsigma,\theta), d(\mathbb{A}\varsigma,\varsigma), d(\mathbb{A}\theta,\theta), d(\varsigma,\mathbb{A}\theta), d(\mathbb{A}\varsigma,\theta)\};$$

i.e.,

$$\begin{aligned} d(\varsigma,\theta) &\le \frac{q}{k^4}\max\{d(\varsigma,\theta), d(\varsigma,\varsigma), d(\theta,\theta), d(\varsigma,\theta), d(\varsigma,\theta)\} \\ &= \frac{q}{k^4}\max\{d(\varsigma,\theta), 0\} \\ &= \frac{q}{k^4} d(\varsigma,\theta) \\ &< d(\varsigma,\theta) \end{aligned}$$

a contradiction, thus $\theta = \varsigma$.

Theorem 4. *Consider a complete b-metric space $(\mathbb{X}, d, s)$. Define four maps $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D} : \mathbb{X} \to \mathbb{X}$. Suppose that the following conditions hold*

1. *$\mathbb{A}$ and $\mathbb{C}$ are occasionally weakly $\mathbb{C}$-biased of type $(\mathscr{A})$,*
2. *$\mathbb{B}$ and $\mathbb{D}$ are occasionally weakly $\mathbb{D}$-biased of type $(\mathscr{A})$,*
3. *for every x, $y \in \mathbb{X}$*

$$\begin{aligned} d(\mathbb{A}x, \mathbb{B}y) \le \frac{1}{k^4}[&\alpha d(\mathbb{C}x,\mathbb{D}y) + \beta d(\mathbb{A}x,\mathbb{C}x) + \gamma d(\mathbb{B}y,\mathbb{D}y) \\ &+ \delta d(\mathbb{C}x,\mathbb{B}y) + \eta d(\mathbb{A}x,\mathbb{D}y)], \end{aligned} \tag{3}$$

where $k \ge 1$ is a real value and α, β, γ, δ, $\eta \in (0,+\infty)$ such that $\alpha + 2k\beta + \delta + \eta < 1$ or $\alpha + 2k\gamma + \delta + \eta < 1$. Then, there is a unique value ξ which verifies $\mathbb{A}\xi = \mathbb{B}\xi = \mathbb{C}\xi = \mathbb{D}\xi = \xi$.

Proof. According to hypotheses, there exist m and n such that

$\mathbb{A}m = \mathbb{C}m$ *implies* $d(\mathbb{C}\mathbb{C}m, \mathbb{A}m) \le d(\mathbb{A}\mathbb{C}m, \mathbb{C}m)$ *and*

$\mathbb{B}n = \mathbb{D}n$ *implies* $d(\mathbb{D}\mathbb{D}n, \mathbb{B}n) \le d(\mathbb{B}\mathbb{D}n, \mathbb{D}n)$.

If we suppose that $\mathbb{A}m \ne \mathbb{B}n$, then, using inequality (3) we get

$$\begin{aligned} d(\mathbb{A}m,\mathbb{B}n) &\le \frac{1}{k^4}[\alpha d(\mathbb{C}m,\mathbb{D}n) + \beta d(\mathbb{A}m,\mathbb{C}m) + \gamma d(\mathbb{B}n,\mathbb{D}n) + \delta d(\mathbb{C}m,\mathbb{B}n) \\ &\quad + \eta d(\mathbb{A}m,\mathbb{D}n)] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{A}m,\mathbb{B}n) + \beta d(\mathbb{A}m,\mathbb{A}m) + \gamma d(\mathbb{B}n,\mathbb{B}n) + \delta d(\mathbb{A}m,\mathbb{B}n) \\ &\quad + \eta d(\mathbb{A}m,\mathbb{B}n)] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{A}m,\mathbb{B}n) + \delta d(\mathbb{A}m,\mathbb{B}n) + \eta d(\mathbb{A}m,\mathbb{B}n)] \\ &= \frac{\alpha+\delta+\eta}{k^4} d(\mathbb{A}m,\mathbb{B}n) \\ &< d(\mathbb{A}m,\mathbb{B}n) \end{aligned}$$

a contradiction, so, $\mathbb{A}m = \mathbb{B}n$.

Now, assume that $AAm \ne Am$, the use of condition (3) gives

$$\begin{aligned} d(\mathbb{A}\mathbb{A}m,\mathbb{B}n) \le \frac{1}{k^4}[&\alpha d(\mathbb{C}\mathbb{A}m,\mathbb{D}n) + \beta d(\mathbb{A}\mathbb{A}m,\mathbb{C}\mathbb{A}m) + \gamma d(\mathbb{B}t,\mathbb{D}n) \\ &+ \delta d(\mathbb{C}\mathbb{A}m,\mathbb{B}n) + \eta d(\mathbb{A}\mathbb{A}m,\mathbb{D}n)]; \end{aligned}$$

i.e.,

$$\begin{aligned} d(\mathbb{A}\mathbb{A}m,\mathbb{A}m) &\le \frac{1}{k^4}[\alpha d(\mathbb{C}\mathbb{C}m,\mathbb{A}m) + \beta d(\mathbb{A}\mathbb{A}m,\mathbb{C}\mathbb{C}m) + \delta d(\mathbb{C}\mathbb{C}m,\mathbb{A}m) \\ &\quad + \eta d(\mathbb{A}\mathbb{A}m,\mathbb{A}m)] \\ &\le \frac{1}{k^4}[\alpha d(\mathbb{C}\mathbb{C}m,\mathbb{A}m) + k\beta(d(\mathbb{A}\mathbb{A}m,\mathbb{A}m) + d(\mathbb{A}m,\mathbb{C}\mathbb{C}m)) \\ &\quad + \delta d(\mathbb{C}\mathbb{C}m,\mathbb{A}m) + \eta d(\mathbb{A}\mathbb{A}m,\mathbb{A}m)]. \end{aligned}$$

Since $\mathbb{A}$ and $\mathbb{C}$ are occasionally weakly $\mathbb{C}$-biased of type $(\mathscr{A})$, we obtain

$$\begin{aligned}
d(\mathbb{A}\mathbb{A}m,\mathbb{A}m) &\leq \frac{1}{k^4}[\alpha d(\mathbb{A}\mathbb{C}m,\mathbb{C}m)+k\beta(d(\mathbb{A}\mathbb{A}m,\mathbb{A}m)+d(\mathbb{C}m,\mathbb{A}\mathbb{C}m)) \\
&\quad +\delta d(\mathbb{A}\mathbb{C}m,\mathbb{C}m)+\eta d(\mathbb{A}\mathbb{A}m,\mathbb{A}m)] \\
&= \frac{\alpha+2k\beta+\delta+\eta}{k^4}d(\mathbb{A}\mathbb{A}m,\mathbb{A}m) \\
&< d(\mathbb{A}\mathbb{A}m,\mathbb{A}m)
\end{aligned}$$

thus $\mathbb{A}\mathbb{A}m=\mathbb{A}m$, which implies that $\mathbb{C}\mathbb{A}m=\mathbb{A}m$.

Now, suppose that $\mathbb{B}\mathbb{B}n\neq\mathbb{B}n$, then

$$\begin{aligned}
d(\mathbb{A}m,\mathbb{B}\mathbb{B}n) &\leq \frac{1}{k^4}[\alpha d(\mathbb{C}m,\mathbb{D}\mathbb{B}n)+\beta d(\mathbb{A}m,\mathbb{C}m)+\gamma d(\mathbb{B}\mathbb{B}n,\mathbb{D}\mathbb{B}n) \\
&\quad +\delta d(\mathbb{C}m,\mathbb{B}\mathbb{B}n)+\eta d(\mathbb{A}m,\mathbb{D}\mathbb{B}n)];
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathbb{B}n,\mathbb{B}\mathbb{B}n) &\leq \frac{1}{k^4}[\alpha d(\mathbb{B}n,\mathbb{D}\mathbb{D}n)+\gamma d(\mathbb{B}\mathbb{B}n,\mathbb{D}\mathbb{D}n)+\delta d(\mathbb{B}n,\mathbb{B}\mathbb{B}n) \\
&\quad +\eta d(\mathbb{B}n,\mathbb{D}\mathbb{D}n)] \\
&\leq \frac{1}{k^4}[\alpha d(\mathbb{B}n,\mathbb{D}\mathbb{D}n)+k\gamma(d(\mathbb{B}\mathbb{B}n,\mathbb{B}n)+d(\mathbb{B}n,\mathbb{D}\mathbb{D}n)) \\
&\quad +\delta d(\mathbb{B}n,\mathbb{B}\mathbb{B}n)+\eta d(\mathbb{B}n,\mathbb{D}\mathbb{D}n)].
\end{aligned}$$

Using the relationship between $\mathbb{B}$ and $\mathbb{D}$, we get

$$\begin{aligned}
d(\mathbb{B}n,\mathbb{B}\mathbb{B}n) &\leq \frac{1}{k^4}[\alpha d(\mathbb{D}n,\mathbb{B}\mathbb{D}n)+k\gamma(d(\mathbb{B}\mathbb{B}n,\mathbb{B}n)+d(\mathbb{D}n,\mathbb{B}\mathbb{D}n)) \\
&\quad +\delta d(\mathbb{B}n,\mathbb{B}\mathbb{B}n)+\eta d(\mathbb{D}n,\mathbb{B}\mathbb{D}n)] \\
&= \frac{\alpha+2k\gamma+\delta+\eta}{k^4}d(\mathbb{B}n,\mathbb{B}\mathbb{B}n) \\
&< d(\mathbb{B}n,\mathbb{B}\mathbb{B}n)
\end{aligned}$$

this contradiction implies that $\mathbb{B}n=\mathbb{B}\mathbb{B}n$ and hence $\mathbb{B}n=\mathbb{D}\mathbb{B}n$. Therefore, $\mathbb{A}m=\mathbb{C}m=\mathbb{B}n=\mathbb{D}n=\xi$ is a common fixed point of maps $\mathbb{A}$, $\mathbb{C}$, $\mathbb{B}$ and $\mathbb{D}$.

Now, suppose there is another common fixed point called ζ, of $\mathbb{A}$, $\mathbb{C}$, $\mathbb{B}$ and $\mathbb{D}$, then

$$\begin{aligned}
d(\xi,\zeta)=d(\mathbb{A}\xi,\mathbb{B}\zeta) &\leq \frac{1}{k^4}[\alpha d(\mathbb{C}\xi,\mathbb{D}\zeta)+\beta d(\mathbb{A}\xi,\mathbb{C}\xi)+\gamma d(\mathbb{B}\zeta,\mathbb{D}\zeta)+\delta d(\mathbb{C}\xi,\mathbb{B}\zeta) \\
&\quad +\eta d(\mathbb{A}\xi,\mathbb{D}\zeta)] \\
&= \frac{1}{k^4}[\alpha d(\xi,\zeta)+\beta d(\xi,\xi)+\gamma d(\zeta,\zeta)+\delta d(\xi,\zeta)+\eta d(\xi,\zeta)] \\
&\leq \frac{\alpha+\delta+\eta}{k^4}d(\xi,\zeta) \\
&< d(\xi,\zeta)
\end{aligned}$$

a contradiction, so, $\zeta=\xi$.

Now, we give an example to support our result.

*Example 2.*Let $\mathbb{X}=[0,+\infty)$ be endowed with the b-metric $d(x,y)=(x-y)^2$, where $k=2$. Define

$$\mathbb{A}x=\begin{cases} 2 \text{ if } x\in[0,2] \\ \frac{3}{2x} \text{ if } x\in(2,+\infty), \end{cases} \quad \mathbb{B}x=\begin{cases} 2 \text{ if } x\in[0,2] \\ \frac{5}{2x} \text{ if } x\in(2,+\infty), \end{cases}$$

$$\mathbb{C}x=\begin{cases} \frac{6}{x+1} \text{ if } x\in[0,2] \\ 1000 \text{ if } x\in(2,+\infty), \end{cases} \quad \mathbb{D}x=\begin{cases} \frac{12}{x+4} \text{ if } x\in[0,2] \\ 2000 \text{ if } x\in(2,+\infty). \end{cases}$$

First of all, the bias hypothesis is satisfied. Take $\alpha=\frac{3}{4}$, $\beta=\gamma=\delta=\eta=\frac{1}{41}$, we get

1.for $x, y \in [0,2]$, $\mathbb{A}x = 2$, $\mathbb{B}y = 2$, $\mathbb{C}x = \frac{6}{x+1}$, $\mathbb{D}y = \frac{12}{y+4}$ and

$$\begin{aligned} d(\mathbb{A}x,\mathbb{B}y) &= 0 \\ &\leq \frac{1}{16}\left[\frac{3}{4}\left(\frac{6}{x+1}-\frac{12}{y+4}\right)^2+\frac{1}{41}\left(2-\frac{6}{x+1}\right)^2\right. \\ &\quad +\frac{1}{41}\left(2-\frac{12}{y+4}\right)^2+\frac{1}{41}\left(\frac{6}{x+1}-2\right)^2 \\ &\quad \left.+\frac{1}{41}\left(2-\frac{12}{y+4}\right)^2\right] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{C}x,\mathbb{D}y)+\beta d(\mathbb{A}x,\mathbb{C}x)+\gamma d(\mathbb{B}y,\mathbb{D}y)+\delta d(\mathbb{C}x,\mathbb{B}y) \\ &\quad +\eta d(\mathbb{A}x,\mathbb{D}y)], \end{aligned}$$

2.for $x, y \in (2,+\infty)$, $\mathbb{A}x = \frac{3}{2x}$, $\mathbb{B}y = \frac{5}{2y}$, $\mathbb{C}x = 1000$, $\mathbb{D}y = 2000$ and

$$\begin{aligned} d(\mathbb{A}x,\mathbb{B}y) &= \left(\frac{3}{2x}-\frac{5}{2y}\right)^2 \\ &\leq \frac{1}{16}\left[\frac{3}{4}(1000)^2+\frac{1}{41}\left(\frac{3}{2x}-1000\right)^2+\frac{1}{41}\left(\frac{5}{2y}-2000\right)^2\right. \\ &\quad \left.+\frac{1}{41}\left(1000-\frac{5}{2y}\right)^2+\frac{1}{41}\left(\frac{3}{2x}-2000\right)^2\right] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{C}x,\mathbb{D}y)+\beta d(\mathbb{A}x,\mathbb{C}x)+\gamma d(\mathbb{B}y,\mathbb{D}y)+\delta d(\mathbb{C}x,\mathbb{B}y) \\ &\quad +\eta d(\mathbb{A}x,\mathbb{D}y)], \end{aligned}$$

3.for $x \in [0,2]$, $y \in (2,+\infty)$, $\mathbb{A}x = 2$, $\mathbb{B}y = \frac{5}{2y}$, $\mathbb{C}x = \frac{6}{x+1}$, $\mathbb{D}y = 2000$ and

$$\begin{aligned} d(\mathbb{A}x,\mathbb{B}y) &= \left(2-\frac{5}{2y}\right)^2 \\ &\leq \frac{1}{16}\left[\frac{3}{4}\left(\frac{6}{x+1}-2000\right)^2+\frac{1}{41}\left(2-\frac{6}{x+1}\right)^2\right. \\ &\quad \left.+\frac{1}{41}\left(\frac{5}{2y}-2000\right)^2+\frac{1}{41}\left(\frac{6}{x+1}-\frac{5}{2y}\right)^2+\frac{1}{41}(1998)^2\right] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{C}x,\mathbb{D}y)+\beta d(\mathbb{A}x,\mathbb{C}x)+\gamma d(\mathbb{B}y,\mathbb{D}y)+\delta d(\mathbb{C}x,\mathbb{B}y) \\ &\quad +\eta d(\mathbb{A}x,\mathbb{D}y)], \end{aligned}$$

4.for $x \in (2,+\infty)$, $y \in [0,2]$, $\mathbb{A}x = \frac{3}{2x}$, $\mathbb{B}y = 2$, $\mathbb{C}x = 1000$, $\mathbb{D}y = \frac{12}{y+4}$ and

$$\begin{aligned} d(\mathbb{A}x,\mathbb{B}y) &= \left(\frac{3}{2x}-2\right)^2 \\ &\leq \frac{1}{16}\left[\frac{3}{4}\left(1000-\frac{12}{y+4}\right)^2+\frac{1}{41}\left(\frac{3}{2x}-1000\right)^2\right. \\ &\quad \left.+\frac{1}{41}\left(2-\frac{12}{y+4}\right)^2+\frac{1}{41}(998)^2+\frac{1}{41}\left(\frac{3}{2x}-\frac{12}{y+4}\right)^2\right] \\ &= \frac{1}{k^4}[\alpha d(\mathbb{C}x,\mathbb{D}y)+\beta d(\mathbb{A}x,\mathbb{C}x)+\gamma d(\mathbb{B}y,\mathbb{D}y)+\delta d(\mathbb{C}x,\mathbb{B}y) \\ &\quad +\eta d(\mathbb{A}x,\mathbb{D}y)], \end{aligned}$$

so, all hypotheses of Theorem 4 are satisfied and 2 is the unique common fixed point of the four maps.

*Remark.*Note that Theorem 2.7 of [17] is not applicable because the four maps are discontinuous, $\mathbb{A}\mathbb{X} = \left(0, \frac{3}{4}\right) \cup \{2\} \nsubseteq \mathbb{D}\mathbb{X} = [2,3] \cup \{2000\}$ and $\mathbb{B}\mathbb{X} = \left(0, \frac{5}{4}\right) \cup \{2\} \nsubseteq \mathbb{C}\mathbb{X} = [2,6] \cup \{1000\}$, and neither $\mathbb{A}$ and $\mathbb{C}$ nor $\mathbb{B}$ and $\mathbb{D}$ are compatible. For this end, consider the sequence $x_n = 2 - \frac{1}{n}$ for $n \in \mathbb{N}$. We have

$$\begin{aligned}
\mathbb{A}x_n &= 2 \to 2 \textit{ when } n \to +\infty, \\
\mathbb{C}x_n &= \frac{6}{x_n + 1} \to 2 \textit{ when } n \to +\infty, \\
\mathbb{B}x_n &= 2 \to 2 \textit{ when } n \to +\infty, \\
\mathbb{D}x_n &= \frac{12}{x_n + 4} \to 2 \textit{ when } n \to +\infty,
\end{aligned}$$

however,

$$d(\mathbb{A}\mathbb{C}x_n, \mathbb{C}\mathbb{A}x_n) \to \left(\frac{3}{4} - 2\right)^2 = \frac{25}{16} \neq 0,$$

$$d(\mathbb{B}\mathbb{D}x_n, \mathbb{D}\mathbb{B}x_n) \to \left(\frac{5}{4} - 2\right)^2 = \frac{9}{16} \neq 0;$$

i.e., neither $\mathbb{A}$ and $\mathbb{C}$ nor $\mathbb{B}$ and $\mathbb{D}$ are compatible.

Corollary 2.*Let $(\mathbb{X}, d, s)$ be a complete b-metric space. Let $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ be a map such that*

$$\begin{aligned}
d(\mathbb{A}x, \mathbb{A}y) \leq \frac{1}{k^4}[&\alpha d(x,y) + \beta d(\mathbb{A}x, x) + \gamma d(\mathbb{A}y, y) \\
&+ \delta d(x, \mathbb{A}y) + \eta d(\mathbb{A}x, y)]
\end{aligned}$$

holds for all x, $y \in \mathbb{X}$, with a given real number $k \geq 1$, where α, β, γ, δ, $\eta \in (0, +\infty)$ such that $\alpha + 2k\beta + \delta + \eta < 1$ or $\alpha + 2k\gamma + \delta + \eta < 1$. Then, $\mathbb{A}$ has a unique common fixed point.

*Proof.*Let $\psi_0 \in \mathbb{X}$, then, there is a point ψ_1 in $\mathbb{X}$ such that $\psi_1 = \mathbb{A}\psi_0$, for this point ψ_1 there exists another element ψ_2 such that $\psi_2 = \mathbb{A}\psi_1$, continuing in this manner we get the sequence $\psi_{n+1} = \mathbb{A}\psi_n$ for $n = 0, 1, 2, \ldots$. If there is a natural element m which verifies $\psi_{m+1} = \psi_m$ then $\psi_m = \mathbb{A}\psi_m$; i.e., $\mathbb{A}$ has a fixed point and the proof is ended. Suppose that $\psi_{n+1} \neq \psi_n$ for each $n \in \mathbb{N} \cup \{0\}$ then

$$\begin{aligned}
d(\mathbb{A}\psi_n, \mathbb{A}\psi_{n+1}) \leq \frac{1}{k^4}[&\alpha d(\psi_n, \psi_{n+1}) + \beta d(\mathbb{A}\psi_n, \psi_n) + \gamma d(\mathbb{A}\psi_{n+1}, \psi_{n+1}) \\
&+ \delta d(\psi_n, \mathbb{A}\psi_{n+1}) + \eta d(\mathbb{A}\psi_n, \psi_{n+1})];
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\psi_{n+1}, \psi_{n+2}) &\leq \frac{1}{k^4}[\alpha d(\psi_n, \psi_{n+1}) + \beta d(\psi_{n+1}, \psi_n) + \gamma d(\psi_{n+2}, \psi_{n+1}) \\
&\quad + \delta d(\psi_n, \psi_{n+2}) + \eta d(\psi_{n+1}, \psi_{n+1})] \\
&= \frac{1}{k^4}[(\alpha + \beta) d(\psi_n, \psi_{n+1}) + \gamma d(\psi_{n+2}, \psi_{n+1}) + \delta d(\psi_n, \psi_{n+2})] \\
&\leq \frac{1}{k^4}[(\alpha + \beta) d(\psi_n, \psi_{n+1}) + \gamma d(\psi_{n+2}, \psi_{n+1}) \\
&\quad + k\delta(d(\psi_n, \psi_{n+1}) + d(\psi_{n+1}, \psi_{n+2}))] \\
&= \frac{1}{k^4}[(\alpha + \beta + k\delta) d(\psi_n, \psi_{n+1}) + (\gamma + k\delta) d(\psi_{n+2}, \psi_{n+1})]
\end{aligned}$$

which implies that

$$d(\psi_{n+1}, \psi_{n+2}) \leq \frac{\alpha + \beta + k\delta}{k^4(1 - \gamma + k\delta)} d(\psi_n, \psi_{n+1}).$$

By the same manner, we get

$$d(\psi_n, \psi_{n+1}) \leq \frac{\alpha + \beta + k\delta}{k^4(1 - \gamma + k\delta)} d(\psi_{n-1}, \psi_n),$$

consequently

$$d(\psi_n,\psi_{n+1}) \leq \frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)} d(\psi_{n-1},\psi_n) \leq \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^2 d(\psi_{n-2},\psi_{n-1})$$
$$\leq \ldots \leq \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^n d(\psi_0,\psi_1)$$

at infinity $d(\psi_n,\psi_{n+1}) \to 0$ because $\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)} < 1$.

Now, for each integer $m > 0$, we obtain

$$\begin{aligned}
d(\psi_n,\psi_{n+m}) &\leq k(d(\psi_n,\psi_{n+1}) + d(\psi_{n+1},\psi_{n+2}) + \ldots + d(\psi_{n+m-1},\psi_{n+m}))\\
&\leq k\Bigg(d(\psi_n,\psi_{n+1}) + \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right) d(\psi_n,\psi_{n+1})\\
&\quad + \ldots + \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^{m-1} d(\psi_n,\psi_{n+1})\Bigg)\\
&= k\left(1 + \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right) + \ldots + \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^{m-1}\right) d(\psi_n,\psi_{n+1})\\
&= k\left[\frac{1-\left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^m}{1-\left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)}\right] d(\psi_n,\psi_{n+1})\\
&\leq k\left[\frac{1-\left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^m}{1-\left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)}\right] \left(\frac{\alpha+\beta+k\delta}{k^4(1-\gamma+k\delta)}\right)^n d(\psi_0,\psi_1),
\end{aligned}$$

at infinity $d(\psi_n,\psi_{n+m}) \to 0$ which means that $\{\psi_n\}$ is a Cauchy sequence and since $\mathbb{X}$ is complete, $\{\psi_n\}$ converges to σ. Moreover, we have

$$\begin{aligned}
d(\mathbb{A}\sigma,\mathbb{A}\psi_n) \leq \frac{1}{k^4}[&\alpha d(\sigma,\psi_n) + \beta d(\mathbb{A}\sigma,\sigma) + \gamma d(\mathbb{A}\psi_n,\psi_n)\\
&+\delta d(\sigma,\mathbb{A}\psi_n) + \eta d(\mathbb{A}\sigma,\psi_n)];
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathbb{A}\sigma,\psi_{n+1}) \leq \frac{1}{k^4}[&\alpha d(\sigma,\psi_n) + \beta d(\mathbb{A}\sigma,\sigma) + \gamma d(\psi_{n+1},\psi_n)\\
&+\delta d(\sigma,\psi_{n+1}) + \eta d(\mathbb{A}\sigma,\psi_n)],
\end{aligned}$$

at infinity, we obtain

$$\begin{aligned}
d(\mathbb{A}\sigma,\sigma) &\leq \frac{1}{k^4}[\alpha d(\sigma,\sigma) + \beta d(\mathbb{A}\sigma,\sigma) + \gamma d(\sigma,\sigma)\\
&\quad +\delta d(\sigma,\sigma) + \eta d(\mathbb{A}\sigma,\sigma)]\\
&= \frac{1}{k^4}[\beta+\eta] d(\mathbb{A}\sigma,\sigma)\\
&< d(\mathbb{A}\sigma,\sigma)
\end{aligned}$$

a contradiction, hence, $\mathbb{A}\sigma = \sigma$.

If we assume that there exists another fixed point say ϖ, then

$$\begin{aligned}
d(\mathbb{A}\sigma,\mathbb{A}\varpi) \leq \frac{1}{k^4}[&\alpha d(\sigma,\varpi) + \beta d(\mathbb{A}\sigma,\sigma) + \gamma d(\mathbb{A}\varpi,\varpi)\\
&+\delta d(\sigma,\mathbb{A}\varpi) + \eta d(\mathbb{A}\sigma,\varpi)];
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\sigma,\varpi) &\leq \frac{1}{k^4}[\alpha d(\sigma,\varpi) + \beta d(\sigma,\sigma) + \gamma d(\varpi,\varpi) + \delta d(\sigma,\varpi) + \eta d(\sigma,\varpi)]\\
&= \frac{1}{k^4}[\alpha+\delta+\eta] d(\sigma,\varpi)\\
&< d(\sigma,\varpi)
\end{aligned}$$

a contradiction, thus $\varpi = \sigma$.

5 Application

Consider the following integral equation:

$$m(\mu) = \chi(\mu) + \alpha \int_0^1 \Phi(\rho, \mu, m(\mu)) d\mu \tag{4}$$

where $0 < \alpha < \frac{1}{\sqrt{2}}$, χ, $\Phi \in C([0,1], \mathbb{R})$. Take the following b-metric:

$$d(m,n) = \sup_{\rho \in [0,1]} |m(\rho) - n(\rho)|^2.$$

Theorem 5. *Suppose that for all m, $n \in C([0,1], \mathbb{R})$ and ρ, $\mu \in [0,1]$*

$$|\Phi(\rho, \mu, m(\mu)) - \Phi(\rho, \mu, n(\mu))| \leq |m(\mu) - n(\mu)|.$$

Then, integral equation (4) *has a unique solution.*

Proof. Define $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ by $\mathbb{A}m(\mu) = \chi(\mu) + \alpha \int_0^1 \Phi(\rho, \mu, m(\mu)) d\mu$ for all ρ, $\mu \in [0,1]$, where $\mathbb{X} = C([0,1])$. For all m, $n \in \mathbb{X}$, we have

$$\begin{aligned}
|\mathbb{A}m(\mu) - \mathbb{A}n(\mu)|^2 &= \left| \alpha \int_0^1 [\Phi(\rho, \mu, m(\mu)) - \Phi(\rho, \mu, n(\mu))] d\mu \right|^2 \\
&\leq \left(\alpha \int_0^1 |\Phi(\rho, \mu, m(\mu)) - \Phi(\rho, \mu, n(\mu))| d\mu \right)^2 \\
&\leq \alpha^2 \left(\int_0^1 |m(\mu) - n(\mu)| d\mu \right)^2 \\
&\leq \alpha^2 \sup_{\mu \in [0,1]} |m(\mu) - n(\mu)|^2,
\end{aligned}$$

hence,

$$\begin{aligned}
d(\mathbb{A}m, \mathbb{A}n) &= \sup_{\mu \in [0,1]} |\mathbb{A}m(\mu) - \mathbb{A}n(\mu)|^2 \\
&\leq \alpha^2 \sup_{\mu \in [0,1]} |m(\mu) - n(\mu)|^2 \\
&= \alpha^2 d(m,n) \\
&\leq \alpha^2 \max\{d(m,n), d(\mathbb{A}m, m), d(\mathbb{A}n, n), d(m, \mathbb{A}n), d(\mathbb{A}m, n)\};
\end{aligned}$$

that is, condition (2) of Corollary 1 is satisfied, therefore, there is only one element (say p) in $\mathbb{X}$ which verifies $\mathbb{A}p = p$, in consequence, p is the only solution of integral equation (4).

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