JJMS **18**, No. 3, 341-349 (2025)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/18.3.2

When does $(P^x_{n+1} + P^x_{n-1})/2$ equal a Pell-Lucas number?

Djamel Bellaouar

Department of Mathematics, University 8 Mai 1945-Guelma, B.P. 401 Guelma 24000, Algeria

Received: March 14, 2024 Accepted : April 10, 2025

Abstract: In the present paper, we will find all solutions (m,n,x) for the equation $P^x_{n+1} + P^x_{n-1} = 2Q_m$, where P_i is the i-th term of Pell sequence and Q_i is the i-th term of Pell-Lucas sequence.

Keywords: Exponential diophantine equations; Pell numbers; Pell-Lucas numbers; Linear forms in logarithms.

2020 Mathematics Subject Classification. 11B39; 11J86; 11D72.

1 Introduction

Pell numbers P_n and *Pell-Lucas numbers* Q_n are defined recursively:

$$P_1 = 1, P_2 = 2 \qquad Q_1 = 1, Q_2 = 3$$
$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 3 \qquad Q_n = 2Q_{n-1} + Q_{n-2} \text{ for } n \geq 3.$$

The first few elements of the sequence (P_n) are

$$1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots.$$

and the *Binet formula* is given by

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ for } n \geq 1,$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ are the roots of $x^2 - 2x - 1$ which is called the characteristic equation of the sequence (P_n). For precise details, see [2]. We can prove by induction on n that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \text{ for } n \geq 1 \tag{1}$$

and

$$\frac{P_{n+1}}{P_n} \geq \frac{7}{3} \text{ for } n \geq 2. \tag{2}$$

The first few elements of the sequence (Q_n) are

$$1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, \ldots.$$

Similarly, the Binet formula of Pell-Lucas sequence is given by

$$Q_n = \frac{\alpha^n + \beta^n}{2} \text{ for } n \geq 1, \tag{3}$$

where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ are the roots of the same characteristic equation $x^2 - 2x - 1$. We also know that

$$\alpha^{n-1} \leq Q_n \leq 2\alpha^n \text{ for } n \geq 1. \tag{4}$$

* Corresponding author e-mail: bellaouar.djamel@univ-guelma.dz

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Various exponential Diophantine equations involving powers of some terms of recurrence sequences were investigated by many authors. For example, Trojovsk [8] studied the sum of two terms in the powers of some types of Fibonacci numbers which are Lucas numbers. In [7], Patel and Chaves solved the exponential Diophantine equation $F_{n+1}^x - F_{n-1}^x = F_m$, where (F_n) is the Fibonacci sequence. In the Pell sequence, one of the latest works along this line can be seen in the works [1],[4],[6],[9] and [10].

We start by recalling the following well-known identity (see [5, page 122]): For all $n \geq 2$, one has

$$P_{n-1} + P_{n+1} - 2Q_n = 0. \tag{5}$$

This identity suggests us to ask the natural question: Are there some triples of nonnegative integers (m,n,x) which satisfy the new identity:

$$P_{n-1}^x + P_{n+1}^x - 2Q_m = 0\ ? \tag{6}$$

2 Main Theorem and Tools

The purpose of this paper is to prove the following result:

Theorem 1(Main Theorem). *The Diophantine equation (6) has the only solutions $(m,n,x) = (1,n,0)$ or $(m,n,x) = (n,n,1)$ with $n \geq 1$.*

The main tools that we will use to prove the above theorem are the linear forms in logarithms defined by algebraic numbers. In addition, the reduction techniques involving continued fractions are also used. For details, see [1],[4],[6].

Definition 1. *Let $\gamma \in \mathbb{R}$ be algebraic whose degree is $d \geq 2$ and minimal polynomial over $\mathbb{Z}$ given by*

$$a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 = a_d \prod_{i=1}^{d} (x - \gamma^{(i)}),$$

where $a_d > 0$ and the numbers $\gamma^{(i)}$ are called the conjugates of γ. The logarithmic height of γ is denoted by $h(\gamma)$ and defined as

$$h(\gamma) = \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\gamma^{(i)}|, 1\})).$$

Theorem 2(Matveev's Theorem). *Let $\mathbb{L}$ be a number field of degree D and let $\gamma_1, \ldots, \gamma_t \in \mathbb{L}$ be positive real algebraic numbers. Let $b_1, \ldots, b_t$ be nonzero integers. If $\Lambda = \gamma_1^{b_1} \gamma_2^{b_2} \ldots \gamma_t^{b_t} - 1 \neq 0$, then*

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 \cdot (1 + \log D)(1 + \log B) A_1 \ldots A_t,$$

where $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for $i = 1, \ldots, t$.

In the proof of Theorem 1, the next result is used to decrease the upper bound on n. For its proof, one can see Dujella and Pethő [3].

Lemma 1(Legendre's Formula). *Let M be a positive integer. Let $p_0/q_0, p_1/q_1, p_2/q_2, \ldots$ be the convergents of an irrational number κ and let $[a_0; a_1, a_2, \ldots]$ be its continued fraction.*

1. *Let u, v be integers such that*

$$\left|\kappa - \frac{u}{v}\right| < \frac{1}{2v^2}.$$

Then u/v is a convergent of κ, i.e., $u/v = p_k/q_k$ for some $k \geq 1$.

2. *If $a(M) = \max(a_i)_{0 \leq i \leq N}$, where N is a positive integer such that $q_N > M$, then*

$$\left|\kappa - \frac{u}{v}\right| > \frac{1}{(a(M)+2)v^2} \tag{7}$$

is valid for all pairs of positive integers (u,v) with $0 < v < M$.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

3 Proof of Theorem 1

At first, if $x=0$, then clearly $Q_m = 1 = Q_1$. That is, the triple $(m,n,x) = (1,n,0)$ is a solution for all $n \geq 1$. Also, if $x = 1$, then by (5) the triple $(m,n,x) = (n,n,1)$ is also a solution for all $n \geq 1$. Next we need to show that there is no other solution.

In the case when $n,x \geq 2$, we can easily show that $m \geq 4$. In fact, we have

$$2Q_m = P_{n+1}^x + P_{n-1}^x \geq P_3^2 + P_1^2 = 26,$$

which implies that $m \geq 4$.

To complete the proof of Theorem 1, we need to have an inequality between x, m and n.

3.1 An Upper Bound for x in Terms of m and n

In this subsection, we prove that if (m,n,x) is a solution of the Diophantine equation (6) with $n,x \geq 2$ and $m \geq 4$, then

$$(n-1)x - 2 < m < (n+1)x + 1, \tag{8}$$

and

$$x < 1.56 \times 10^{10} n \log((n+2)x + 2). \tag{9}$$

In fact, applying (1) and (4), we get

$$\alpha^{m+2} \geq 4\alpha^m > 2Q_m = P_{n+1}^x + P_{n-1}^x \geq P_{n+1}^x > \alpha^{x(n-1)}.$$

and

$$2\alpha^{m-1} < 2Q_m = P_{n-1}^x + P_{n+1}^x \leq P_{n+2}^x < \alpha^{(n+1)x} \leq 2\alpha^{x(n+1)}.$$

Thus, by combining these inequalities we get (8). Next we prove (9). By (3), we rewrite the equation (6) as

$$\alpha^m - P_{n+1}^x = P_{n-1}^x - \beta^m. \tag{10}$$

Dividing this equation by P_{n+1}^x and using (2), we conclude that

$$\begin{aligned} \left|\alpha^m P_{n+1}^{-x} - 1\right| &= \left|\frac{P_{n-1}^x - \beta^m}{P_{n+1}^x}\right| \leq \left(\frac{P_n}{P_{n+1}}\right)^x + \frac{|\beta|^m}{P_{n+1}^x} < \left(\frac{P_n}{P_{n+1}}\right)^x + \frac{P_n^x}{P_{n+1}^x} \\ &= 2\left(\frac{P_n}{P_{n+1}}\right)^x \leq 2\left(\frac{3}{7}\right)^x < \frac{2}{(2.3)^x}, \end{aligned}$$

since $2.3 < 7/3$ and so $(3/7)^x < 1/(2.3)^x$ for all $x \geq 2$. Therefore,

$$\left|\alpha^m P_{n+1}^{-x} - 1\right| < \frac{2}{(2.3)^x}. \tag{11}$$

We put $\Lambda_1 = \alpha^m P_{n+1}^{-x} - 1$. If $\Lambda_1 = 0$, then $\alpha^m \in \mathbb{Z}$ for $m \geq 4$ which is not valid. Thus, we have $\Lambda_1 \neq 0$.

Now, we will apply Theorem 2 to find a lower bound of Λ_1. We put $t=2$, $\gamma_1 = \alpha$, $\gamma_2 = P_{n+1}$, $b_1 = m$ and $b_2 = -x$. Note that $\gamma_1, \gamma_2 \in \mathbb{Q}\left(\sqrt{2}\right)$. Thus, we take $D = 2$. Since $h(\gamma_1) = \log\alpha/2$ and by (1), $h(\gamma_2) = \log P_{n+1} < n\log\alpha$. Thus, we can choose $A_1 = \log\alpha$ and $A_2 = 2n\log\alpha$.

Therefore, the left inequality of (8) gives $m+2 > (n-1)x \geq x$, so it suffices to consider $B = m+2 \geq \max\{|b_1|, |b_2|\}$. Hence, by Theorem 2 we may immediately deduce

$$\begin{aligned} \log|\Lambda_1| &> -1.4 \times 30^5 \times 2^{4.5} \times 2^2 \times (1+\log 2)(1+\log(m+2))(\log\alpha)(2n\log\alpha) \\ &\geq -8.1 \times 10^9 n(1+\log(m+2)). \end{aligned} \tag{12}$$

Thus, the inequalities (11) and (12) give

$$\log 2 - x\log(2.3) > \log|\Lambda_1| \geq -8.1 \times 10^9 n(1+\log(m+2)).$$

and so

$$\begin{aligned} x &< \frac{8.1 \times 10^9 n(1+\log(m+2)) - \log 2}{\log(2.3)} \\ &\leq 9.73 \times 10^9 n(\log(m+2)+1) \\ &\leq 1.56 \times 10^{10} n\log(m+2), \end{aligned}$$

where the inequality $1+\log(m+2) < 1.6\log(m+2)$ holds for $m \geq 4$. It comes next that $x < 1.56 \times 10^{10} n\log(m+2)$. This proves (9) since $m < (n+2)x$.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

3.2 An Upper Bound for x in Terms of n

We find an upper bound for x which is only depending on n. We will use the following comprehended fact. Let $\ell \geq 3$ and $x \geq 2$. If $x < \ell \cdot \log x$, then

$$x < 2\ell \cdot \log \ell. \tag{13}$$

If (m,n,x) satisfies (6) with $x \geq 2$, $m \geq 4$ and $n \geq 59$, then

$$x < 5.69 \times 10^{11} n \log n. \tag{14}$$

For the proof of (14), we single out two cases:

Case 1. Suppose that $n+2 < x$. By (9) we see that

$$x < 1.56 \times 10^{10} n \log\left(x^2+2\right) \leq 1.56 \times 10^{10} n \log\left(1.51x^2\right) \leq 4.06 \times 10^{10} n \log x,$$

since $\log\left(x^2+2\right) \leq \log\left(1.51x^2\right) \leq 2.6 \log x$ for $x \geq 2$. It follows that

$$\frac{x}{\log x} < 4.06 \times 10^{10} n.$$

Applying (13) with $\ell = 4.06 \times 10^{10} n$ and since $\log \ell < 7 \log n$ holds for $n \geq 59$, we get

$$x < 2\left(4.06 \times 10^{10} n\right) \log\left(4.06 \times 10^{10} n\right) < 2\left(4.06 \times 10^{10} n\right)(7 \log n) \leq 5.69 \times 10^{11} n \log n.$$

This proves (14). We are done.

Case 2. When $x \leq n+2$. This immediately implies the inequality (14). We are done.

3.3 An Absolute Upper Bound on x

We will prove that if (m,n,x) satisfies (6) with $m \geq 4$ and $n \geq 59$, then

$$x < 10^{26}. \tag{15}$$

Consider the number $y = \dfrac{x}{\alpha^{2(n-1)}}$. By (14), for every $n \geq 39$ we obtain

$$y < \frac{5.69 \times 10^{11} n \log n}{\alpha^{2(n-1)}} < \frac{1}{\alpha^n},$$

and so $y < \alpha^{-39} < 10^{-14}$. On the other hand, since $\beta = -1/\alpha$, we can also write

$$P_{n-1}^x = \left(\frac{\alpha^{n-1} - \beta^{n-1}}{2\sqrt{2}}\right)^x = \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}\left(1 - \frac{(-1)^{n-1}}{\alpha^{2(n-1)}}\right)^x,$$

and

$$P_{n+1}^x = \left(\frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}}\right)^x = \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\left(1 - \frac{(-1)^{n+1}}{\alpha^{2(n+1)}}\right)^x.$$

There are two cases:

Case 1. Assume that n is even. Therefore,

$$1 < \left(1 - \frac{(-1)^{n-1}}{\alpha^{2(n-1)}}\right)^x = \left(1 + \frac{1}{\alpha^{2(n-1)}}\right)^x = \left(1 + \frac{x/\alpha^{2(n-1)}}{x}\right)^x$$
$$< \lim_{t\to\infty}\left(1 + \frac{x/\alpha^{2(n-1)}}{t}\right)^t = e^y < 1 + 2y,$$

since $y < 10^{-14}$.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Case 2. Assume that n is odd. Here, we can write

$$1 > \left(1 - \frac{(-1)^{n-1}}{\alpha^{2(n-1)}}\right)^x = \left(1 - \frac{x/\alpha^{2(n-1)}}{x}\right)^x$$
$$> \lim_{t\to\infty}\left(1 - \frac{x/\alpha^{2(n-1)}}{t}\right)^t = e^{-y} > 1-2y,$$

this is since $y < 10^{-14}$. It follows that

$$(1-2y)\frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x} < P_{n-1}^x < (1+2y)\frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x},$$

or, equivalently,

$$\left|P_{n-1}^x - \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}\right| < \frac{2y\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}.$$

The above inequality is also true when $n-1$ is replaced by $n+1$.

Now, return to the original equation (6), which can be rewritten as

$$\alpha^m + \beta^m = 2Q_m = \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x} + \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x} + \left(P_{n+1}^x - \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\right) + \left(P_{n-1}^x - \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}\right).$$

Equivalently,

$$\begin{aligned}\left|\alpha^m - \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\left(1+\alpha^{-2x}\right)\right| &= \left|-\beta^m + \left(P_{n+1}^x - \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\right) + \left(P_{n-1}^x - \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}\right)\right| \\ &\le \frac{1}{\alpha^m} + \left|P_{n+1}^x - \frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\right| + \left|P_{n-1}^x - \frac{\alpha^{x(n-1)}}{\left(2\sqrt{2}\right)^x}\right| \\ &< \frac{1}{\alpha^m} + 2y\left(\frac{\alpha^{x(n+1)}}{\left(2\sqrt{2}\right)^x}\left(1+\alpha^{-2x}\right)\right).\end{aligned}$$

Thereby dividing both sides of the above by $\alpha^{x(n+1)}/\left(2\sqrt{2}\right)^x$, we get

$$\begin{aligned}\left|\alpha^{m-x(n+1)}\left(2\sqrt{2}\right)^x - \left(1+\alpha^{-2x}\right)\right| &< \frac{\left(2\sqrt{2}\right)^x}{\alpha^{m+x(n+1)}} + 2y\left(1+\alpha^{-2x}\right) \\ &< \frac{\left(2\sqrt{2}\right)^x}{\alpha^{m+x(n-2)}} + 2y\left(1+\alpha^{-2x}\right) \\ &< \frac{1}{2\alpha^{n-2}} + \frac{35}{17}y \\ &\le \frac{1}{2\alpha^{n-2}} + \frac{35}{17\alpha^n} \\ &\le \left(\frac{1}{2}+\frac{35}{17}\right)\frac{1}{\alpha^{n-2}} < \frac{3}{\alpha^{n-2}},\end{aligned} \tag{16}$$

where $\left(2\sqrt{2}\right)^x/\alpha^{x(n+1)} < \left(2\sqrt{2}\right)^x/\alpha^{x(n-2)} \le \left(2\sqrt{2}/\alpha^{59}\right)^x < 1/2$, $n-2 \le x(n-2) \le m$ and $\alpha^{2x} \ge \alpha^4 > 34$. Thus, we see that

$$\alpha^{m-x(n+1)}\left(2\sqrt{2}\right)^x - 1 \le \frac{3}{\alpha^{n-2}} + \left(1+\alpha^{-2x}\right) - 1,$$

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

and so

$$\left|\alpha^{m-x(n+1)}\left(2\sqrt{2}\right)^x - 1\right| \le \frac{1}{\alpha^{2x}} + \frac{3}{\alpha^{n-2}} \le \frac{4}{\alpha^l}, \quad (17)$$

where $l = \min(2x, n-2)$.

Next, we can put

$$\Lambda_2 = \alpha^{m-x(n+1)}\left(2\sqrt{2}\right)^x - 1. \quad (18)$$

We can check easily that Λ_2 is different from zero. Otherwise, $8^{\frac{x}{2}} = \alpha^{x(n+1)-m}$. This means that $\alpha^{2x(n+1)-2m} \in \mathbb{N}$ and this is only true whenever $m = x(n+1)$, and so $x = 0$, but this is a contradiction with the hypothesis that $x \ge 2$. Moreover, since $n \ge 39$, we deduce from (17) and (18) that

$$|\Lambda_2| \le \frac{1}{\alpha^4} + \frac{3}{\alpha^{37}} < \frac{1}{2},$$

from which we conclude that $\frac{1}{2} \le \alpha^{m-(n+1)x}\left(2\sqrt{2}\right)^x \le \frac{3}{2}$. It comes next that

$$-\log 2 \le (m - x(n+1))\log\alpha + \frac{x}{2}\log 8 \le \log 3 - \log 2,$$

that is

$$-\log 3 + \log 2 \le (-m + x(n+1))\log\alpha - \frac{x}{2}\log 8 \le \log 2$$

so

$$-m + x(n+1) \le \frac{1}{\log\alpha}\left\{\frac{x}{2}\log 8 + \log 2\right\} \le 1.6x, \quad (19)$$

and also

$$-m + x(n+1) > \frac{1}{\log\alpha}\left\{-\log 3 + \log 2 + \frac{x}{2}\log 8\right\} \ge 1.16x - 0.47 > 0. \quad (20)$$

In view of (17), we apply once again Theorem 2 as follows; we set $t = 2$, $\gamma_1 = \alpha$, $\gamma_2 = 2\sqrt{2}$, $b_1 = m - (n+1)x$ and $b_2 = x$. Similarly, as above, we can choose $(A_1, A_2) = (\log\alpha, \log 8)$, $D = 2$ and $B = 1.6x$. Hence,

$$\begin{aligned}\log|\Lambda_2| &> -1.4\times 30^5 \times 2^{4.5} \times 2^2 (1+\log 2)(1+\log 1.6x)(\log\alpha)(\log 8) \\ &> -9.56\times 10^9 (1+\log 1.6x).\end{aligned} \quad (21)$$

From inequalities (17) and (21) one obtains

$$\log 4 - l\log\alpha > -9.56\times 10^9 (1+\log 1.6x),$$

that is

$$l < \frac{9.56\times 10^9 (1+\log 1.6x) + \log 4}{\log\alpha} < 2.1\times 10^{10}(\log 1.6x) < 3.6\times 10^{10}\log x,$$

where $2.1\log(1.6x) < 3.6\log x$ whenever $x \ge 2$. There are two possibilities:

- Assume that $l = 2x$. Then $2x < 3.6\times 10^{10}\log x$, which gives $x < 10^{11}$.
- Assume that $l = n-2$. Here, by (14) we get

$$n - 2 < 3.6\times 10^{10}\log\left(5.69\times 10^{11} n\log n\right).$$

Actually, from the above inequality we deduce that $n < 2.1\times 10^{12}$ and by (14) once again, we also get

$$x < 5.69\times 10^{11}\left(2.1\times 10^{12}\right)\log\left(2.1\times 10^{12}\right) \le 3.4\times 10^{25}.$$

In both cases, $x < 3.4\times 10^{25} < 10^{26}$. This completes the proof of (15).

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

3.4 Reducing the Bound on x

In this subsection, we improve the inequality (15). Indeed, we will prove that if (m,n,x) satisfies (6) with $m \geq 4$ and $n \geq 76$, then $x \leq 37$. Let us take $\Gamma_2 = \log\left(\gamma_1^{b_1}/\gamma_2^{b_2}\right)$, where $(b_1,b_2) = (x,(n+1)x-m)$ and $(\gamma_1,\gamma_2) = \left(2\sqrt{2},\alpha\right)$. Note that $\Lambda_2 = e^{\Gamma_2}-1$ is given in (18). Since $|\Lambda_2| < \frac{1}{2}$, we conclude that $e^{|\Gamma_2|} < 2$ and by (17) we have

$$|\Gamma_2| < e^{|\Gamma_2|}\left|e^{\Gamma_2}-1\right| < 2|\Lambda_2| < \frac{2}{\alpha^{2x}} + \frac{6}{\alpha^{n-2}}.$$

This gives

$$\left|\frac{\log\left(2\sqrt{2}\right)}{\log\alpha} - \frac{-m+x(n+1)}{x}\right| < \frac{1}{x\log\alpha}\left(\frac{2}{\alpha^{2x}} + \frac{6}{\alpha^{n-2}}\right). \tag{22}$$

Now, assume that $n \geq 76$. Therefore, $\alpha^{n-2} > \alpha^{74} > 10^2 x$ (this is by applying (15)). Suppose further that $x > 37$. Then $\alpha^{2x} > 10^2 x$. It follows that

$$\frac{1}{x\log\alpha}\left(\frac{2}{\alpha^{2x}} + \frac{6}{\alpha^{n-2}}\right) < \frac{8}{10^2 x^2\log\alpha} < \frac{1}{2x^2}. \tag{23}$$

The inequalities (22) and (23) imply that

$$\frac{-1}{2x^2} < \frac{\log\left(2\sqrt{2}\right)}{\log\alpha} - \frac{-m+x(n+1)}{x} < \frac{1}{2x^2}. \tag{24}$$

By Lemma 1 and the above inequalities we deduce that $(-m+x(n+1))/x$ is a convergent to the irrational number $\gamma = \log\left(2\sqrt{2}\right)/\log\alpha$. Let $[a_0;a_1,a_2,a_3,a_4,\ldots] = [1;5,1,1,3,\ldots]$ be the infinite continued fraction of γ and let p_k/q_k be the fraction formed from its k-th convergent. Suppose further that $\frac{-m+x(n+1)}{x} = p_k/q_k$ for some positive integer k. Since $\gcd(p_k,q_k) = 1$ (this holds from the fact that $p_k q_{k-1} = p_{k-1}q_k + (-1)^{k-1}$), we conclude that $x = d\cdot q_k$ for some positive integer d, that is, $q_k \leq x$.

We research for the smallest q_{k_0+1} such that $q_{k_0+1} > x$. After computation using Maple, we get

$$q_{k_0+1} = q_{49} = 297581592712700128741090663 > 10^{26} > x.$$

Thus, $k_0 \in \{0,1,\ldots,48\}$. Moreover, $a_i \leq 66$ for any $i = 0,1,\ldots,48$. From a well-known property, we conclude that

$$\left|\gamma - \frac{-m+x(n+1)}{x}\right| = \left|\gamma - \frac{p_k}{q_k}\right| > \frac{1}{(a_k+2)q_k^2} \geq \frac{1}{68x^2},$$

which contradicts (24). Thus, $x \leq 37$. As required.

3.5 The Final Step

From the above, we deduce that there are no solutions (m,n,x) of the diophantine equation (6) with $m \geq 4$, $n \geq 76$ and $x \leq 37$. Indeed, we rewrite the inequality (16) as follows:

$$\left|\alpha^{m-x(n+1)}\left(2\sqrt{2}\right)^x\left(1+\alpha^{-2x}\right)^{-1} - 1\right| < \frac{3}{\alpha^{n-2}(\alpha^{-2x}+1)} < \frac{3}{\alpha^{n-2}}.$$

Note that $2 \leq x \leq 37$, and so by (19) and (20) we get

$$1.16x - 0.47 < (n+1)x - m < 1.6x.$$

We let $s = (n+1)x - m$. After simple computation, for every $x \in [2,37]$ and $s \in [\lfloor 1.16x-0.47\rfloor, \lceil 1.6x\rceil] \subset [1,59]$ we get $\left|\alpha^{-s}\left(\sqrt{8}\right)^x\left(1+\alpha^{-2x}\right)^{-1} - 1\right| > 0.9$. Thus, $\frac{10}{9} < \frac{3}{\alpha^{n-2}}$, from which we have $\alpha^{n-2} < \frac{30}{9}$. Hence $n \leq 3$, contradicting the assumption. So, there are no solutions for $m \geq 4$, $n \geq 76$ and $x \leq 37$. As claimed.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Similarly, we can prove that there are no solutions to the Diophantine equation (6) for $m \geq 4$ and $n \leq 75$. In fact, by (9), we get

$$x < 1.56 \times 10^{10} \times 75 \log((n+2)x+2) \leq 1.18 \times 10^{12} \log(77x+2) \leq 1.27 \times 10^{12} \log(78x),$$

which gives $x < 4.54 \times 10^{13}$. Thus, $m < (n+2)x \leq 77x < 3.5 \times 10^{15}$. Let Λ_1 as above and put $\Gamma_1 = m \log\alpha - x \log P_{n+1}$. Clearly, by (10), $\alpha^m > \alpha^m - P_{n+1}^x = P_{n-1}^x - \beta^m > 0$ since $|\beta| < 1$. Hence, $\alpha^m > P_{n+1}^x$ and so $\Gamma_1 > 0$. It follows from (11) that

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{2}{(2.3)^x}.$$

That is,

$$0 < m \frac{\log\alpha}{\log P_{n+1}} - x < \frac{2}{(2.3)^x \log P_{n+1}}. \tag{25}$$

We divide both sides of the last inequality by m, one obtains the inequalities

$$0 < \frac{\log\alpha}{\log P_{n+1}} - \frac{x}{m} < \frac{2}{m(2.3)^x \log P_{n+1}} \leq \frac{1}{2(2.3)^x \log 5},$$

for $x \geq 2$. Since $x^2 < (2.3)^x \log 5$, we deduce that

$$0 < \frac{\log\alpha}{\log P_{n+1}} - \frac{x}{m} < \frac{1}{2x^2},$$

for $x \geq 2$ and $m \geq 4$. By Legendre's Formula (see Lemma 1), we infer that x/m is a convergent to the continued fraction of $\log\alpha/\log P_{n+1}$.

Let $[a_0; a_1, a_2, \ldots]$ be the continued fraction of the irrational number $\gamma = \log\alpha/\log P_{n+1}$ and let $p_k(n)/q_k(n)$ be its k-th convergent ($2 \leq n \leq 75$). Let $a_n(m) = \max\{a_i : i = 0, 1, \ldots, k\}$, where $q_{k-1}(n) < m < q_k(n)$. Recall that $m < 3.5 \times 10^{15}$. From the property of continued fraction, see (7) in Lemma 1, we have

$$m \frac{\log\alpha}{\log P_{n+1}} - x > \frac{1}{(a_n(m)+2)m}. \tag{26}$$

It follows from (25) and (26) that

$$\frac{1}{(a_n(m)+2)m} < \frac{2}{x(2.3)^x \log P_{n+1}}. \tag{27}$$

Let $a_{n_0}(m) = \max_{2 \leq n \leq 75}(a_n(m))$. A quick inspection using Maple reveals that $q_k(n_0) = 1311738121$ and $a_{n_0}(m) = 1565$. Since the inequalities

$$\begin{cases} \dfrac{x(2.3)^x \log P_3}{2} - (1565+2)m > 0, \text{ for } x \geq 48 \\[2ex] \dfrac{x(2.3)^x \log P_3}{2} - (1565+2)m < 0, \text{ for } x \leq 47 \end{cases}$$

hold, we conclude that for all $n = 2, 3, \ldots, 75$, if $x \geq 48$ then the inequality

$$\frac{x(2.3)^x \log P_{n+1}}{2} > (a_n(m)+2)m$$

holds. This contradicts (27). We deduce that if (m,n,x) is a solution of the diophantine equation (6) with $m \geq 4$ and $n \leq 75$, then $x \leq 47$.

Finally, a simple computation by Pari/Gp for $x \in [2,47]$ and $n \in [2,75]$ turned up that the only solutions (m,n,x) of (6) are the triples $(1,n,0)$ and $(n,n,1)$ with $n \geq 1$. Thus, we have finished the proof of Theorem 1.

4 Open Problem

As our final conclusion, we propose for further research the following interesting question: Let $N \geq 1$ and let a be an odd positive integer with $a \leq N$ (a is odd since p_{n+1} and p_{n-a} have the same parity for $n \geq a+1$). When does $\frac{P_{n+1}^x + P_{n-a}^x}{2}$ equal a Pell-Lucas number? That is, we research for all indices m, n, a with a odd and the exponent x such that $P_{n+1}^x + P_{n-a}^x = 2Q_m$.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Declarations

Competing interests: The author has not disclosed any competing interests.
Authors' contributions: D. Bellaouar wrote all the manuscript.
Funding: This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT)-Algeria.
Availability of data and materials: The research data and materials associated with this article are available in the references.
Acknowledgments: The author thank the referees and the editor for the useful comments and suggestions that have greatly improved on the quality of presentation of the current paper.

References

[1] G. C. Alexis, J. C. Gómez, F. Luca, On the Exponential Diophantine equation $F_{n+1}^x - P_{n-1}^x = F_m^y$. *Taiwanese J. Math.,* **26** (4) (2022), 685–712.
[2] A. Al-Kateeb, Generalization of Jacobsthal and Jacobsthal-Lucas numbers. *Jordan J. Math. Stat.*, **14** (3) (2021), 467–481.
[3] A. Dujella, A. Pethő, A generalization of a theorem of Baker and Davenport. *Quart. J. Math.,* Oxford Ser. **49** (2) (1998), 291–306.
[4] B. Faye, C. Gómez, F. Luca, S. E. Rihane, A. Togbé, Complete solution of the exponential Diophantine equation $P_n^x + P_{n+1}^x = P_m^y$. *Math. Commun.*, **27** (2) (2022), 163–185.
[5] T. Koshy, *Pell and Pell-Lucas numbers with applications*, Vol. 431, New York: Springer, (2014).
[6] F. Luca, E. Tchammou, A. Togbé, On the exponential Diophantine equation $P_n^x + P_{n+1}^x + \ldots + P_{n+k-1}^x = P_m$. *Math. Slovaca*, **70** (6) (2020): 1333–1348.
[7] B. K. Patel, A. P. Chaves, On the Exponential Diophantine Equation $F_{n+1}^x - F_{n-1}^x = F_m$. *Mediterr. J. Math.,* **18** (5) (2021): 187.
[8] T. Pavel, On the sum of powers of two k-Fibonacci numbers which belongs to the sequence of k-Lucas numbers. *Tatra Mt. Math. Publ.*, **67** (2016), 41–46.
[9] S. E. Rihane, B. Faye, F. Luca, A. Togbé, On the exponential Diophantine equation $P_n^x + P_{n+1}^x = P_m$, *Turk. J. Math.*, **43** (3) (2019), 1640–1649.
[10] S. E. Rihane, A. Togbé, An exponential Diophantine equation involving the sum or difference of powers of two Pell numbers. *Math. Slovaca*, **72** (3) (2022), 601–610.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.