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On The Maximum Likelihood Estimation For The Transmuted Extreme Value Distribution Parameters

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Abstract: Recently, researchers have been interested in the transmuted family of distributions via the quadratic rank transformation map, which was studied by Shaw and Buckley [21] (2009). Among the important distributions is the generalized extreme value distribution, due to its wide use in various fields. For this reason, we focus on estimating the parameters of the transmuted generalized extreme value (TGEV) distribution. Therefore, we develop transformed parameters of a generalized Pareto distribution (GPD) with a scale parameter and a shape parameter and approximate it to the transformed conditional distribution function to estimate the parameters of the TGEV via the maximum likelihood estimation (MLE). In addition, we present a numerical method for estimating the unknown parameters of the transmuted GPD starting from the MLE based on simple random sampling (SRS) and ranked set sampling (RSS). Finally, we present a simulation study using this practical method to better illustrate the findings of this investigation.

Keywords: Generalized Pareto distributions; Excesses over high thresholds; Maximum Likelihood Estimation; Ranked set sampling; Modified Bisection Algorithm.

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1 Introduction

The quality of the operations used in statistical analysis depends entirely on the distributions. Considerable effort has been made to develop large classes of standard probability distributions along with related statistical methodologies. However, there are still many important problems, as the real data do not follow any of the classical or standard distributions. However, a model with a large amount of information provides more flexibility and covers more variation in the data. Several techniques exist to obtain generalized probability models. One of the techniques for adding additional parameters to existing models is the QRTM technique. Shaw and Buckley [21] (2009) introduced the QRTM technique for the generalization of classical probability models. And Aryall and Tsokos [1] (2009) focused on the transformed generalized maximum value distribution and studied its applications and properties. In addition, Faton and Llukan [4] (2009) studied the transmuted Pareto distribution. In application to real data, they observed that the transmuted Pareto distribution leads to a better fit than the Pareto distribution. And can be used to model flood data. There, more Habib et al. [17] (2016) had given the transmuted GPD with four parameters. They show that the subject distribution can be used to model reliability data.

As is known under the theorem presented by Pickands [20] (1975), we can approximate the conditional distribution function (df) to the GPD parameters. We also know that the GPD is strongly related to the generalized extreme value (GEV) distribution. Therefore, we find that the MLE method is based on this beautiful interconnection. So, the aim of this study is to use the transmuted GPD (TGPD) parameters, where the GPD has two parameters, the shape parameter and the scale parameter, to estimate the TGEV parameters using the MLE method. Therefore, we approximate the TGPD parameters to the transmuted condition distribution based on the theorem, which we will present and expand in sections (1). In section (2) we give a procedure that is performed by the MLE method via SRS and RSS. And to estimate the

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TGPD parameters numerically by the MLE method, we use the modified bisection algorithm (MBA) for multi-roots, which was presented by [11] (2019). For section (3), some simulations are given with three numerical examples. In the first, we applied the MLE using SRS with a sample of 15 data that follows the TGPD parameters. The second is given with real data. And for the last one, we apply the MLE method with 1000 replicates using the RSS. They are used to test the efficiency of the estimators of the TGPD parameters using the algorithms presented in section (2).

The cumulative distribution function (cdf) of the GPD parameters (shape and scale) is

$$P(X \le x) = G_{\gamma,\sigma}(x) := \begin{cases} 1 - \left(1 + \frac{\gamma}{\sigma}x\right)^{-1/\gamma} & \text{for } \gamma \ne 0 \\ 1 - \exp\left(-\frac{x}{\sigma}\right) & \text{for } \gamma = 0 \end{cases}, \tag{1}$$

And a random variable (rv) X is said to follow the GPD given in (1) if the probability density function (pdf) of X as:

$$g_{\gamma,\sigma}(x) := \begin{cases} \frac{1}{\sigma}\left(1 + \frac{\gamma}{\sigma}x\right)^{-1/\gamma - 1}, & \text{for } \gamma \ne 0 \\ \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right), & \text{for } \gamma = 0 \end{cases}, \tag{2}$$

where $\gamma \in \mathbb{R}$ is the shape parameter and $\sigma > 0$ is the scale parameter for $x > 0$ if $\gamma \ge 0$ and for $0 < x < \sigma/|\gamma|$ if $\gamma < 0$.

Let $X_1, \ldots, X_n$ be a sequence of random variables (rv's) of independent and identically distributed (iid) from some unknown df H. For $\tau_H = \sup\{x : H(x) < 1\} \le \infty$ and $1 - H(t) > 0$ with $t < \tau_H$ and $x > 0$, be the conditional df of $X - t$ given $X > t$,

$$H_t(x) = P(X < t + x | X > t) = \frac{H(x+t) - H(t)}{1 - H(t)} \tag{3}$$

where τ_H is the upper endpoint of H.

We denote the order statistics by $X_{1,n} < \cdots < X_{n,n}$. The weak convergence of the centered and standardized of $X_{n,n} = \max(X_i)$ for $i = 1, \ldots, n$ implies that there exist two strictly positive sequences of constants a_n and b_n with a continuous df $\Phi(x)$ such that for $x > 0$:

$$\lim_{n \to +\infty} P\left(\frac{X_{n,n} - a_n}{b_n} \le x\right) = \Phi(x) \tag{4}$$

However, the results of Fisher and Tippett [5] (1928), Gnedenko [6] (1943) and de Haan [7] (1970) characterized the classes of df having a certain limit in (4). This possible limiting of $\Phi(x)$ in (4) are given by the so-called extreme value distributions Ψ_γ with γ called the extreme value index, defined by

$$\Psi_\gamma(x) := \begin{cases} \exp\left(-(1 + \gamma x)^{-1/\gamma}\right) & \text{if } \gamma \ne 0 \\ \exp(-\exp(-x)) & \text{of } \gamma = 0 \end{cases}, \tag{5}$$

Then, Balkema and de Haan [2] (1974) and Pickands [20] (1975)] has been proved that there exists a normalizing function $\sigma(t) > 0$ such that for all $x > 0$

$$\lim_{t \to \tau_H} \sup_{0 < x < t - \tau_H} \left| H_t(x) - G_{\gamma,\sigma(t)}(x) \right| = 0 \tag{6}$$

if and only if $H \in D(\Psi_\gamma)$ that's mean if H satisfies the condition (4) where $\gamma \in \mathbb{R}$. Also from (1) and (5) we obtain that for the shape parameter $\gamma \in \mathbb{R}$ and the scale parameter $\sigma > 0$ we find

$$G_{\gamma,\sigma}(x) = 1 + \log\left(\Psi_\gamma\left(\frac{x}{\sigma}\right)\right) \tag{7}$$

Furthermore, if the rv X has GPD parameters (the shape and scale), then the conditional df of $X - t$ given $X > t$ is also the GPD parameters given in (1). In view of (6) with τ_H denoting the upper endpoint of H we can expect that observations of $C_i = X_{n-i+1,n} - X_{n-k,n}$ for $1 \le i \le k$, or, equivalently, on

$$C_0 = X_{n-k,n}, C_1 = X_{n-k+1,n} - X_{n-k,n}, \ldots, C_k = X_{n,n} - X_{n-k,n} \tag{8}$$

where in the asymptotic setting $k_n = k$ an intermediate sequence, that is, $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Therefore, in view of convergence (6), the conditional df of the sample $(C_0, C_1, \ldots, C_k)$ given $C_0 = c_0$ can be approximated by the df of an ordered sample of k random variables iid of the GPD with df $G_{\gamma,\sigma}$ defined in (1).

In this article, we use the QRTM approach, which was suggested by Shaw and Buckley [21] (2009) to define TGPD for simplicity if we take the GPD which is defined in (1) as the base df. Then, according to the QRTM approach, the cdf of TGPD satisfies the relationship.

$$F_{\gamma,\sigma,\lambda}(x) := (1 + \lambda) G_{\gamma,\sigma}(x) - \lambda G^2_{\gamma,\sigma}(x), \tag{9}$$

which in differentiation yields,

$$f_{\gamma,\sigma,\lambda}(x) := g_{\gamma,\sigma}(x)\left(1+\lambda-2\lambda G_{\gamma,\sigma}(x)\right) \tag{10}$$

where $G_{\gamma,\sigma}(x)$ is the cdf of the GPD given in (1) and $|\lambda| \le 1$. Observe that with $\lambda = 0$ we get the df of the base rv, i.e., $F_{\gamma,\sigma,\lambda}(x) = G_{\gamma,\sigma}(x)$. Then, a rv X is said to have TGPD with parameters (γ,σ,λ) if his pdf is defined as:

$$f_{\gamma,\sigma,\lambda}(x) := \begin{cases} \frac{1}{\sigma}\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma-1}\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right), & \text{for} \quad \gamma \neq 0 \\ \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)\left(1-\lambda+2\lambda\exp\left(-\frac{x}{\sigma}\right)\right), & \text{for} \quad \gamma = 0 \end{cases} \tag{11}$$

and its cdf as

$$F_{\gamma,\sigma,\lambda}(x) := \begin{cases} (1+\lambda)\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2, & \text{for} \quad \gamma \neq 0 \\ (1+\lambda)\left(1-\exp\left(-\frac{x}{\sigma}\right)\right)-\lambda\left(1-\exp\left(-\frac{x}{\sigma}\right)\right)^2, & \text{for} \quad \gamma = 0 \end{cases} \tag{12}$$

where $\gamma \in \mathbb{R}$ is the shape parameter and $\sigma > 0$ is the scale parameter for $x > 0$ if $\gamma \ge 0$ and for $0 < x < \sigma/|\gamma|$ if $\gamma < 0$ with $|\lambda| \le 1$ is the transmuted parameter. Likewise, the q^{th} quantile of x_q TGPD is derived in the following corollary.

Corollary 1. *The q^{th} quantile x_q of the random variable X having the cdf of the TGPD from (12) with $\gamma \neq 0$ is given by the nonlinear equation*

$$x_q = \frac{\sigma}{\gamma}\left(\left(\frac{\sqrt{4\lambda q+(1-\lambda)^2}-(1-\lambda)}{2\lambda}\right)^{-\gamma}-1\right) \tag{13}$$

Proof. The q^{th} quantile x_q of rv X, where X follows $F_{\gamma,\sigma,\lambda}(x)$ the cdf of TGPD for $\gamma \neq 0$ is obtained by inverting $\bar{F}^{-1}_{\gamma,\sigma,\lambda}(x)$ which obtained from (12) to obtain $x_q = \bar{F}^{-1}_{\gamma,\sigma,\lambda}(x)$ where $p = F_{\gamma,\sigma,\lambda}(x)$ by

$$p := (1+\lambda)\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2$$

Then it's easy to find that

$$1-p = (1-\lambda)\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}+\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-2/\gamma}$$

After calculating math with $q = 1-p$ we found

$$x_q = \frac{\sigma}{\gamma}\left(\left(\frac{\sqrt{4\lambda q+(1-\lambda)^2}-(1-\lambda)}{2\lambda}\right)^{-\gamma}-1\right)$$

Hence, the distribution median is

$$x_{0.5} = \frac{\sigma}{\gamma}\left(\left(\frac{\sqrt{1+\lambda^2}-(1-\lambda)}{2\lambda}\right)^{-\gamma}-1\right) \tag{14}$$

Also, if we consider the base df to be $H_t(x)$ the conditional df of $X-t$ given in (3) for $X > t$, we will have defined the transmuted conditional df for $X-t$ as follows:

$$F_{t,\lambda}(x) := (1+\lambda)H_t(x)-\lambda H_t^2(x), \quad \text{for } |\lambda| \le 1 \tag{15}$$

Theorem 1. *Let $X_1,\ldots,X_n$ be iid rv's with unknown common df H and let τ_H where $\tau_H = \sup\{x : H(x) < 1\} \le \infty$ denote the right endpoint of support of H. Consider a threshold t as $t \to \tau_H$ for $0 < x < t-\tau_H$, and if the condition (6) is met then*

$$\lim_{t\to\tau_H}\sup_{0<x<t-\tau_H}\left|F_{t,\lambda}(x)-F_{\gamma,\sigma(t),\lambda}(x)\right| = 0 \tag{16}$$

where $F_{t,\lambda}$ is the transmuted conditional df given in (15) and $F_{\gamma,\sigma(t),\lambda}$ is the TGPD presented in (12).

Proof. Under (12) and (15) we have

$$\sup\left|F_{t,\lambda}(x)-F_{\gamma,\sigma(t),\lambda}(x)\right| = \sup\left|(1+\lambda)\left(H_t(x)-G_{\gamma,\sigma(t)}(x)\right)-\lambda\left(H_t^2(x)-G_{\gamma,\sigma(t)}^2(x)\right)\right|$$

This is equivalent to that

$$\sup\left|F_{t,\lambda}(x)-F_{\gamma,\sigma(t),\lambda}(x)\right| \leq (1+\lambda)\sup\left|\left(H_t(x)-G_{\gamma,\sigma(t)}(x)\right)\right|-\lambda\sup\left|\left(H_t^2(x)-G_{\gamma,\sigma(t)}^2(x)\right)\right|$$

For $H_t^2(x)-G_{\gamma,\sigma(t)}^2(x)=\left(H_t(x)-G_{\gamma,\sigma(t)}(x)\right)\left(H_t(x)+G_{\gamma,\sigma(t)}(x)\right)$ and under (6) we check that $F_{t,\lambda}(x)\simeq F_{\gamma,\sigma(t)}(x)$. Then, it is easy to prove $F_{t,\lambda}(x)\simeq F_{\gamma,\sigma(t),\lambda}(x)$.

2 The MLE of the TGEV parameters

Since the GPD is considered an equivalent set of results in extreme value distributions Ψ_γ which is presented in (5) and under the relation (7) we can state the distribution of the TGEV, which is defined as follows:

$$\Psi_{\gamma,\lambda}\left(\frac{x}{\sigma}\right)=\exp\left(-\overline{F}_{\gamma,\sigma,\lambda}(x)\right) \tag{17}$$

for $\overline{F}_{\gamma,\sigma,\lambda}(x)=1-F_{\gamma,\sigma,\lambda}(x)$ with $F_{\gamma,\sigma,\lambda}(x)$ is the TGPD given in (12). Aryal and Tsokos [1] (2009) were the first to introduce the TGEV. Then Cira et al. [3] (2019) studied in detail the characteristics of this distribution and estimated its parameters using the MLE method using a series of Monte Carlo simulation experiments. Hence, in this paper we focus on estimating the parameters of TGEV through estimates of the parameters of TGPD. In addition, in view of (14) the transmuted conditional distribution can be approximated by the distribution of an ordered sample of k iid rv following df $F_{\gamma,\sigma,\lambda}(x)$ which is the cdf of TGPD. This allows us to estimate the parameters of TGEV with the MLE method to estimate the parameters of TGPD. Therefore, we will present the MLE method for TGPD using the SRS and RSS.

2.1 The MLE for the TGPD parameters using SRS

In this subsection, we discuss the estimation (γ,σ,λ) based on the MLE of TGPD using a simple random sample (SRS) with one set. To specify it, let $X_1,X_2,...,X_n$ be iid rv's with common cdf $F_{\gamma,\sigma,\lambda}(x)$ of the TGPD, which given in (12). We assume that both of $F_{\gamma,\sigma,\lambda}$ is absolutely continuous. Then for $\gamma\neq 0$ the likelihood function can be written as

$$\ell(x;\gamma,\sigma,\lambda)=\prod_{i=1}^{n}f_{\gamma,\sigma,\lambda}(x_i)=\prod_{i=1}^{n}\frac{1}{\sigma}\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma-1}\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right) \tag{18}$$

And with $\gamma=0$ the likelihood function is

$$\ell(x;\gamma,\sigma,\lambda)\left|_{\gamma=0}\right.=\prod_{i=1}^{n}\frac{1}{\sigma}\exp\left(-\frac{x_i}{\sigma}\right)\left(1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)\right) \tag{19}$$

By accumulation taking logarithm of equation (18),

$$\log\ell(x_i;\gamma,\sigma,\lambda)=\sum_{i=1}^{n}\left[\log\left(\frac{1}{\sigma}\right)-\left(\frac{1}{\gamma}+1\right)\log\left(1+\frac{\gamma}{\sigma}x_i\right)+\log\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)\right] \tag{20}$$

where $\sigma>0$ for $\gamma>0$ and $\sigma>-\gamma x_{n,n}$ for $\gamma<0$ with $0\leq\lambda\leq 1$. Hence, for any $\gamma<-1$ and $-1\leq\lambda\leq 0$ there is no MLE. And in the case $\gamma=0$ the log-likelihood function of (19) can be written as

$$\log\left(\ell(x;\gamma,\sigma,\lambda)\left|_{\gamma=0}\right.\right)=\sum_{i=1}^{n}\left[\log\left(\frac{1}{\sigma}\right)-\frac{x_i}{\sigma}+\log\left(1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)\right)\right] \tag{21}$$

The likelihood equations of (20) with $\gamma\neq 0$ are given in terms of the partial derivatives as

$$\begin{cases} \frac{\log(\ell(x;\gamma,\sigma,\lambda))}{\partial\gamma}=\sum_{i=1}^{n}\left[\log\left(1+\frac{\gamma}{\sigma}x_i\right)-(1+\gamma)\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}+(1-\zeta_i)\left(\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}\right)\right]=0. \\ \frac{\log(\ell(x;\gamma,\sigma,\lambda))}{\partial\sigma}=\sum_{i=1}^{n}\left[-1+\left(1+\frac{1}{\gamma}\right)\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}+\frac{(1-\zeta)}{\gamma}\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}\right]=0. \\ \frac{\log(\ell(x;\gamma,\sigma,\lambda))}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{-1+2\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}}{\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)}\right]=0. \end{cases} \tag{22}$$

With

$$\zeta_i = \frac{1-\lambda}{1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}}$$

Also, under (21) for $\gamma = 0$ we have

$$\begin{array}{c} \frac{\log\left(\ell(x;\gamma,\sigma,\lambda)\big|_{\gamma=0}\right)}{\partial\sigma} = \sum_{i=1}^{n}\left[-1+\frac{1}{\sigma}x_i+\frac{x_i}{\sigma}\left(1-\varepsilon_i\right)\right] = 0. \\ \frac{\log\left(\ell(x;\gamma,\sigma,\lambda)\big|_{\gamma=0}\right)}{\partial\lambda} = \sum_{i=1}^{n}\left[\frac{-1+2\exp\left(-\frac{x_i}{\sigma}\right)}{1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)}\right] = 0. \end{array} \tag{23}$$

where

$$\varepsilon_i = \frac{1-\lambda}{1-\lambda+2\lambda\exp\left(-\frac{x_i}{\sigma}\right)}$$

There more for $0 \le \lambda \le 1$ it is easy to hold that $0 \le \zeta_i \le 1$ and $0 \le \varepsilon_i \le 1$. The MLE of TGPD (γ,σ,λ)parameters via the SRS which denoting by $\left(\hat{\gamma}_{SRS},\hat{\sigma}_{SRS},\hat{\lambda}_{SRS}\right)$ are obtained by solving the equations $\Psi(\hat{\gamma}) = 0$, $\Psi(\hat{\sigma}) = 0$ and $\Psi\left(\hat{\lambda}\right) = 0$ for $-1 \le \gamma \le 0$, $\sigma > -\gamma x_{n,n}$ where $x_{n,n} = \max(x_i)$ for $i = 1,2,...,n$ and $\gamma > 0$ for $\sigma > 0$ and $0 \le \lambda \le 1$. These functions are related to the system equations (22). They are as follows:

$$\Psi(\gamma) = \sum_{i=1}^{n}[\log\left(1+\frac{\gamma x_i}{\sigma}\right) - (1+\gamma)\left(1-\left(1+\frac{\gamma x_i}{\sigma}\right)^{-1}\right) + (1-\zeta_i)\left(\log\left(1+\frac{\gamma x_i}{\sigma}\right)+\left(1+\frac{\gamma x_i}{\sigma}\right)^{-1}-1\right)] \tag{24}$$

$$\Psi(\sigma) = \sum_{i=1}^{n}\left[-1+\left(1+\frac{1}{\gamma}\right)\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1}\right)+\frac{1-\zeta_i}{\gamma}\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1}\right)\right] \tag{25}$$

And

$$\Psi(\lambda) = \sum_{i=1}^{n}\left(\left(-1+2\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^{-1}\right) \tag{26}$$

Where

$$\zeta_i = (1-\lambda)\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^{-1}$$

It is usually more convenient to use nonlinear optimization algorithms such as MBA for multi-roots, which is proposed by [11] (2019) to search a root of a function defined on the interval $[\alpha,\beta]$. As it is known that this new numerical method is based on the mean values theorem on the interval $[\alpha,\beta]$ with $f(\alpha)f(\beta) \le 0$.

First, to determine $\hat{\gamma}_{SRS}$ the estimator of the shape parameter γ one should find the roots of $\Psi(\gamma)$ which is defined in (24) for $\gamma \ge -1$. Since, the MBA for multi-roots was applied in a closed interval. For this, we will focus on checking the roots of the function $\Psi(\gamma)$ for $\gamma \in [-1,0]$. However, for $\gamma > 0$ we can express the function (24) in terms of $\theta \in]-1,0[$ where $\theta = -(1/\gamma)$. Then the problem is over.

Second, to determine $\hat{\sigma}_{SRS}$ the estimator of the scale parameter σ we must search the roots of the function $\Psi(\sigma)$ via the MBA. Such as $\sigma \ge -\gamma x_{n,n}$ where $x_{n,n} = \max(x_i)$ with $i = 1,2,...,n$ for $-1 \le \gamma \le 0$ and $\sigma > 0$ for $\gamma > 0$. But the MBA works with closed interval for the scale parameter. Therefore, we present the following theorem, which gives us a simple technique to implement the MBA to find the root of $\Psi(\sigma)$ which is defined in (25).

Theorem 2. *Let define the function $\Psi(\sigma)$ in (25). Then,*

1. $\lim_{\sigma\to+\infty} \zeta_i = \frac{1-\lambda}{1+\lambda}$ *where* $\zeta_i = (1-\lambda)\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^{-1}$.
2. $\lim_{\sigma\to+\infty} \Psi(\sigma) = -1$.
3.For $\Psi(\sigma)$ *with* $\gamma \ge 0$ *we find* $\sigma \ge \sigma_u$ *where* $\sigma_u = \left(1+\gamma+n-n\times\overline{\zeta}\right)\overline{x}$ *with* $\overline{x} = (1/n)\sum_{i=1}^{n} x_i$ *and* $\bar{\zeta} = (1/n)\sum_{i=1}^{n}\zeta_i$.

with $\sigma \ge -\gamma x_{n,n}$ *where* $x_{n,n} = \max(x_i)$ *with* $i = 1,2,...,n$ *for* $-1 \le \gamma \le 0$ *and* $\sigma > 0$ *for* $\gamma > 0$.

Clearly from the theorem (2) it is easy to apply the MBA method. Then, with $\sigma > 0$ for $\gamma > 0$, we can search the zero of the function $\Psi(\sigma)$ on (25) in the interval $[\varepsilon, \sigma_u]$ where $\varepsilon = 10^{-8}\bar{x}$ with $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$. Hence, for $\sigma \geq -\gamma x_{n,n}$ where $x_{n,n} = \max(x_i)$ with $i = 1, 2, ..., n$ for $-1 \leq \gamma \leq 0$ we will find that there is no problem. Because we can return to the previous state by taking $\beta = -\gamma$. Then we search the roots of $\Psi(\sigma)$ in the interval $[\varepsilon, \sigma_u]$ where $\sigma_u = \left(1 - \beta + n - n \times \overline{\zeta}\right)\bar{x}$.

Finally, for $\hat{\lambda}_{SRS}$ the estimator of the transmuted parameter λ. Is checked by determining the roots of $\Psi(\lambda)$ in (26) with $\lambda \in [0,1]$ because the MLE exists for all $\lambda \in [0,1]$. Hence, for $\lambda \neq 0$ we can rewrite the third equation from (22) as

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{1 - \lambda + 2\lambda\left(1 + \frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}}\right] - 1 = 0 \tag{27}$$

We solve the $\Psi(\lambda)$ through MBA for multi-roots in $[0,1]$. We find the root of $\Psi(\lambda)$ gives an estimator of λ with good estimator for σ and γ as we discus in section (3). All the methods that we explained for estimating TGPD parameters with MLE via SRS will also be discussed upon in Example 1 that we present in section (3).

2.2 The MLE for the TGPD parameters using RSS

In this subsection, we estimate the unknown parameters $(\gamma, \sigma, \lambda)$ of TGPD based on MLE using ranked set sampling (RSS). RSS is one of the traditional methods that are generally used to achieve the surveillance economy. The concept of RSS was initially shown by [18] (1951) in his work. Without the need for many substantial observations, RSS can increase the effectiveness of methods such as estimation. It is used to reduce the number of measured observations necessary to achieve the specified inference precision. The following is a description of the RSS scheme [23]: First, pick a simple random sample (SRS) of r sets of size n from the group you want to study. Second, pick the element with rank i from the j set in a cycle where $i \in \{1, 2, ..., n\}$ And $j \in \{1, 2, ..., r\}$. Then we get an RSS sample of size $m = r \times n$.

Let $X^{(1)}, X^{(2)}, ..., X^{(r)}$ be an SRS from the TGPD with pdf $f_{\gamma,\sigma,\lambda}$ given in (11). The rv/s $X_{1,n}^{(1)}, X_{2,n}^{(1)}, ..., X_{n,n}^{(1)}$ and $X_{1,n}^{(2)}, X_{2,n}^{(2)}, ..., X_{n,n}^{(2)}$ with $X_{1,n}^{(r)}, X_{2,n}^{(r)}, ..., X_{n,n}^{(r)}$ be independent where $X_{i,n}^{j}$ denotes the j^{th} and i^{th} sample of size n for $j \in \{1, 2, ..., r\}$ and $i \in \{1, 2, ..., n\}$ where $m = r \times n$ the size of RSS . For $j = 1$ we have $r = 1$ then $m = n$. Now, let $X_1, X_2, ..., X_n$ be a random sample with the common distribution cdf $F_{\gamma,\sigma,\lambda}$ given in (12) and pdf $f_{\gamma,\sigma,\lambda}$ given in (11). The pdf $f_{i,\gamma,\sigma,\lambda}(x)$ of RSS for the i^{th} order statistics as shown as

$$f_{i,\gamma,\sigma,\lambda}(x) = \frac{n!}{(i-1)!\,(n-i)!} f_{\gamma,\sigma,\lambda}(x)\left[F_{\gamma,\sigma,\lambda}(x)\right]^{i-1}\left[1 - F_{\gamma,\sigma,\lambda}(x)\right]^{n-i} \tag{28}$$

Using equations (11) and (12) in equation (28), we have for $\gamma \neq 0$.

$$\begin{aligned} f_{i,\gamma,\sigma,\lambda}(x) &= \tfrac{n!}{(i-1)!(n-i)!}\left(\tfrac{1}{\sigma}\left(1 + \tfrac{\gamma}{\sigma}x\right)^{\frac{-1}{\gamma}-1}\left(1 - \lambda + 2\lambda\left(1 + \tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)\right) \\ &\times \left[(1+\lambda)\left(1 - \left(1 + \tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right) - \lambda\left(1 - \left(1 + \tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right]^{i-1} \\ &\times \left[1 - \left[(1+\lambda)\left(1 - \left(1 + \tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right) - \lambda\left(1 - \left(1 + \tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right]\right]^{n-i} \end{aligned} \tag{29}$$

And with $\gamma = 0$ the pdf of RSS is

$$\begin{aligned} f_{i,\gamma,\sigma,\lambda}(x) &= \tfrac{n!}{(i-1)!(n-i)!}\left(\tfrac{1}{\sigma}\exp\left(-\tfrac{x}{\sigma}\right)\right)\left(1 - \lambda + 2\lambda\exp\left(-\tfrac{x}{\sigma}\right)\right) \\ &\times \left[(1+\lambda)\left(1 - \exp\left(-\tfrac{x}{\sigma}\right)\right) - \lambda\left(1 - \exp\left(-\tfrac{x}{\sigma}\right)\right)^2\right]^{i-1} \\ &\times \left[1 - \left[(1+\lambda)\left(1 - \exp\left(-\tfrac{x}{\sigma}\right)\right) - \lambda\left(1 - \exp\left(-\tfrac{x}{\sigma}\right)\right)^2\right]\right]^{n-i} \end{aligned} \tag{30}$$

Let $X_1, X_2, ..., X_n$ be an iid random sample with df of TGPD. Then for $\gamma \neq 0$ the likelihood function can be written as

$$\begin{aligned}\ell(x;\gamma,\sigma,\lambda) = \prod_{i=1}^{n}[\tfrac{n!}{(i-1)!(n-i)!}\left(\tfrac{1}{\sigma}\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{\frac{-1}{\gamma}-1}\left(1-\lambda+2\lambda\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)\right) \\ \times\left[(1+\lambda)\left(1-\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2\right]^{i-1} \\ \times\left[1-\left[(1+\lambda)\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right]\right]^{n-i}]\end{aligned} \tag{31}$$

And for $\gamma = 0$ the likelihood function can be written as

$$\begin{aligned}\ell(x;\gamma,\sigma,\lambda) = \prod_{i=1}^{n}[\tfrac{n!}{(i-1)!(n-i)!}\left(\tfrac{1}{\sigma}\exp\left(-\tfrac{x}{\sigma}\right)\right)\left(1-\lambda+2\lambda\exp\left(-\tfrac{x}{\sigma}\right)\right) \\ \times\left[(1+\lambda)\left(1-\exp\left(-\tfrac{x}{\sigma}\right)\right)-\lambda\left(1-\exp\left(-\tfrac{x}{\sigma}\right)\right)^2\right]^{i-1} \\ \times\left[1-\left[(1+\lambda)\left(1-\exp\left(-\tfrac{x}{\sigma}\right)\right)-\lambda\exp\left(-\tfrac{x}{\sigma}\right)^2\right]\right]^{n-i}]\end{aligned} \tag{32}$$

Using the logarithmic accumulation of equation (31),

$$\begin{aligned}\log\ell(x;\gamma,\sigma,\lambda) = \log[\prod_{i=1}^{k}\tfrac{n!}{(i-1)!(n-i)!}] + \sum_{i=1}^{n}[-\log\sigma-\left(1+\tfrac{1}{\gamma}\right)\log\left(1+\tfrac{\gamma}{\sigma}x_i\right) \\ +\log\left(1-\lambda+2\lambda\left(1+\tfrac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right) \\ +(i-1)\log\left((1+\lambda)\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right) \\ +(n-i)\log\left[1-\left[(1+\lambda)\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)-\lambda\left(1-\left(1+\tfrac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)^2\right]\right]]\end{aligned} \tag{33}$$

Using the logarithmic accumulation of equation (32),

$$\begin{aligned}\log\ell(x;\gamma,\sigma,\lambda) = \log[\prod_{i=1}^{n}\tfrac{n!}{(i-1)!(n-i)!}] + \sum_{i=1}^{n}[-\log\sigma-\left(\tfrac{x_i}{\sigma}\right)+\log\left(1-\lambda+2\lambda+\exp\left(\tfrac{x_i}{\sigma}\right)\right) \\ +(i-1)\log\left[(1+\lambda)\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)-\lambda\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)^2\right] \\ +(n-i)\log\left[1-\left[(1+\lambda)\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)-\lambda\left(1-\exp\left(\tfrac{x_i}{\sigma}\right)\right)^2\right]\right]]\end{aligned} \tag{34}$$

The likelihood equations of (31) with $\gamma \neq 0$ are then given in terms of the partial derivatives,

$$\begin{aligned}\frac{\partial\log\ell(x;\gamma,\sigma,\lambda)}{\partial\gamma} = \sum_{i=1}^{n}[\tfrac{1}{\gamma^2}\log\left(1+\tfrac{\gamma}{\sigma}x_i\right)-\left(1+\tfrac{1}{\gamma}\right)\frac{\frac{x_i}{\sigma}}{\left(1+\frac{\gamma}{\sigma}x\right)} \\ +(i-1)\frac{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(-\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2} \\ -(i-1)\frac{2\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(\log\left(1+\frac{\gamma}{\sigma}x\right)+\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2} \\ -(n-i)\frac{(1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\left(-\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{1-\left((1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2\right)} \\ +(n-i)\frac{2\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\left(\log\left(1+\frac{\gamma}{\sigma}x\right)+\frac{\frac{x_i}{\gamma\sigma}}{1+\frac{\gamma}{\sigma}x_i}\right)}{1-\left((1+\lambda)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}-\lambda\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^2\right)} = 0\end{aligned} \tag{35}$$

and

$$\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \sigma} = \sum_{i=1}^{k} [\frac{-1}{\sigma} - \left(1+\frac{1}{\gamma}\right) \frac{\frac{-\gamma}{\sigma^2}}{(1+\frac{\gamma}{\sigma}x_i)} + 2(1-\zeta_i)\left(1+\frac{\gamma}{\sigma}x_i\right)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}$$
$$+(i-1)\frac{\left((1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}\right)+2\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}\right)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}}{(1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}-\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}\right)^2} \tag{36}$$
$$-(n-i)\frac{\left((1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}\right)+2\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}\right)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}-1}\frac{x_i}{\sigma^2}}{1-\left((1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}-\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-\frac{1}{\gamma}}\right)^2\right)} = 0$$

$$\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} [\frac{-1+2(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}}{1-\lambda+2\lambda(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}} + (i-1)\frac{(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}-\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}\right)}{(1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}-\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}\right)^2}$$
$$+(n-i)\frac{-(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}+\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}\right)^2}{1-\left((1+\lambda)(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}-\lambda\left(1-(1+\frac{\gamma}{\sigma}x_i)^{-1/\gamma}\right)^2\right)} = 0 \tag{37}$$

Also, under (34) for $\gamma = 0$ we have.

$$\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \sigma}\Big|_{\gamma=0} = \sum_{i=1}^{n} [\frac{-1}{\sigma} + \frac{x_i}{\sigma^2} + 2(1-\varepsilon_i)\exp\left(-\frac{x_i}{\sigma}\right)\frac{x_i}{\sigma^2}$$
$$-(i-1)\frac{(1+\lambda)\exp(-\frac{x_i}{\sigma})\frac{x_i}{\sigma^2}+2\lambda(1-\exp(-\frac{x_i}{\sigma}))\exp(-\frac{x_i}{\sigma})\frac{x_i}{\sigma^2}}{(1+\lambda)(1-\exp(-\frac{x_i}{\sigma}))-\lambda(1-\exp(-\frac{x_i}{\sigma}))^2} \tag{38}$$
$$+(n-i)\frac{(1+\lambda)\exp(-\frac{x_i}{\sigma})\frac{x_i}{\sigma^2}+2\lambda(1-\exp(-\frac{x_i}{\sigma}))\exp(-\frac{x_i}{\sigma})\frac{x_i}{\sigma^2}}{1-\left((1+\lambda)(1-\exp(-\frac{x_i}{\sigma}))-\lambda(1-\exp(-\frac{x_i}{\sigma}))^2\right)}] = 0$$

and

$$\frac{\partial \log \ell(x;\gamma,\sigma,\lambda)}{\partial \lambda}\Big|_{\gamma=0} = \sum_{i=1}^{n} [\frac{-1+2\exp(-\frac{x_i}{\sigma})}{1-\lambda+2\lambda\exp(-\frac{x_i}{\sigma})} + (i-1)\frac{(1-\exp(-\frac{x_i}{\sigma}))-(1-\exp(-\frac{x_i}{\sigma}))^2}{(1+\lambda)(1-\exp(-\frac{x_i}{\sigma}))-\lambda(1-\exp(-\frac{x_i}{\sigma}))^2}$$
$$+(n-i)\frac{-(1-\exp(-\frac{x_i}{\sigma}))}{1-\left((1+\lambda)(1-\exp(-\frac{x_i}{\sigma}))-\lambda(1-\exp(-\frac{x_i}{\sigma}))^2\right)}] = 0 \tag{39}$$

To obtain the estimators of the TGPD parameters (γ,σ,λ) via MLE with RSS. We set the three equations (35)-(37) and the two equations (38)-(39) equal to zero. To solve these nonlinear equations with iterative approaches, we use numerical methods, such as the Newton-Raphson method or the MBA for multi-roots, which is presented by [10]. The MLE of the TGPD parameters (γ,σ,λ) through RSS, which is denoted by $\left(\hat{\gamma}_{RSS},\hat{\sigma}_{RSS},\hat{\lambda}_{RSS}.\right)$.

3 Simulation and results

3.1 Example1

In order to investigate the performance of the MLE via SRS for (γ,σ,λ). Then we applied the practical method which discuss in section(2) as we are given with the following cases :

1.For estimating the shape parameter. We use the MBA for multi-roots where $\gamma \in [0;-1]$ to search the root of the function $\Psi(\gamma)$ that is defined on (24). Under the Know parameter $\gamma \in [0;-1]$. Else if $\gamma > 0$ we estimated it via using the MBA for multi-roots to determine the root of the function $\Psi(\theta)$ for $\theta = -(1/\gamma)$ where

$$\Psi(\gamma) = \sum_{i=1}^{n} [\log\left(1-\frac{x_i}{\theta\sigma}\right) - \left(1-\frac{1}{\theta}\right)\left(1-\left(1-\frac{x_i}{\theta\sigma}\right)^{-1}\right) + (1-\zeta_i)\left(\log\left(1-\frac{x_i}{\theta\sigma}\right)+\left(1-\frac{x_i}{\theta\sigma}\right)^{-1}-1\right)] \tag{40}$$

Consequently, $\widehat{\gamma}_{SRS} = -\left(1/\widehat{\theta}\right)$.

2.For estimate the scale parameter. If $\sigma > 0$ for $\gamma > 0$ and if $\sigma \geq -\gamma x_{n,n}$ where $x_{n,n} = \max(x_i)$ with for all $i = 1,\dots,n$ for $\gamma \in]0;-1]$. We use the MBA for multi-roots on $[\varepsilon;\sigma_u]$ for search the root of the function $\Psi(\sigma)$ which define by

$$\Psi(\sigma) = \sum_{i=1}^{n}\left[-1+\left(1+\frac{1}{|\gamma|}\right)\left(1-\left(1+\frac{|\gamma|}{\sigma}x_i\right)^{-1}\right)+\frac{1-\zeta_i}{|\gamma|}\left(1-\left(1+\frac{|\gamma|}{\sigma}x_i\right)^{-1}\right)\right] \tag{41}$$

where $\sigma_u = \left(1+|\gamma|+n-n\times\overline{\zeta}\right)\overline{x}$.. And $\varepsilon = 10^{-8}/\overline{x}$.

3.For estimating the transmuted parameter. We use the MBA for multi-roots on $[0;1]$ For search the root of $\Psi(\lambda)$ which is defined on (26).

We will use this practical method to estimate the parameters (γ,σ,λ) of TGPD in this example. It is used when two parameters are known from (γ,σ,λ) to each function. Hence, when we applied this numerical method, we may find several roots for a function. However, it is possible to separate them by taking the parameters that have the maximum log-likelihood function, which is given in (20) for $\gamma \neq 0$ and (21) for $\gamma = 0$. The functions are (24) or (40) for γ, (41) for σ and (26) for λ.

Random samples of size $n = 15$ of the random variable follow the new generalized Pareto distribution (TGPD) with parameters (γ,σ,λ). These parameters are $\gamma = -0.7$, $\sigma = 2$, and $\lambda = 0.9$. Before we generate a random sample following the uniform distribution for size $n = 15$ with the R Core Team software as

1.$p_i = runif(15)$

2.Define the queue as $q_i = 1 - p_i$

3.Generating random samples of size $n = 15$ from the TGPD by the method of inversion using the quantiles as we given in (13) by

$$x_{q_i} = \frac{\sigma}{\gamma}\left(\left(\frac{\sqrt{4\lambda q_i+(1-\lambda)^2}-(1-\lambda)}{2\lambda}\right)^{-\gamma}-1\right)$$

A simulation study is performed using the TGPD. We generate a sample X_i of size 15 following the TGPD with parameters $\gamma = -0.7$, $\sigma = 2$ and $\lambda = 0.9$. The 15 values are listed in increasing order in the following table(1):

0.447	0.507	0.507	0.517	0.651
0.833	0.983	1.100	1.166	1.179
1.292	1.408	1.695	2.054	2.385

Table 1: X-sample of 15 size generated with the TGPD parameters $\gamma = -0.7$, $\sigma = 2$ and $\lambda = 0.9$

We use our practical method given before for the MLE of TGPD. We find that the shape parameter is $\gamma = -0.7$, the scale parameter is $\sigma = 2$ and the transmuted parameter is $\lambda = 0.9$. We calculate the bounds $\sigma_u = 16.5$ and $\varepsilon = 1.114933 \times 10^{-8}$, to apply MBA for multi-roots to determine the root of $\Psi(\gamma)$ which is defined in (24) for $\gamma \in [0;-1]$ and the root of the function $\Psi(\sigma)$ given in (41) on $[\varepsilon;\sigma_u]$ and the root of $\Psi(\lambda)$ which is defined in (26) for $\lambda \in [0;1]$. We find $\hat{\gamma}_{SRS} = -0.7238293$, $\hat{\sigma}_{SRS} = 1.915073$ and $\hat{\lambda}_{SRS} = 0.88865$.

3.2 Example2

Here, we apply the transmuted extreme value model with the transmuted Peak-Over-Threshold (TPOT) model which is supported by the theorem (1) that explained the relationship (16). Also, based on the GPD in (1) and the TGPD in (12) we find that they have the same parameters γ and σ. Then we can estimate γ and σ parameters of the TGPD through the MLE for the GPD parameters given in (1) with the algorithm given by Kouider et al [13] (2023) under full data. And to estimate the transmuted parameter, we reach the zero of (26) by the MBA on $[0;1]$. Obviously, we can formulate the function (26) as follows:

$$\Psi\left(\widehat{\lambda}^{ML}\right) = \sum_{i=1}^{n}\left(\left(1-2G_{\widehat{\gamma}^{ML},\widehat{\sigma}^{ML}}(x_i)\right)\left(1+\lambda-2\lambda G_{\widehat{\gamma}^{ML},\widehat{\sigma}^{ML}}(x_i)\right)^{-1}\right)$$

0.011	0.030	0.051	0.056	0.092
0.100	0.140	0.184	0.200	0.2860
0.338	0.365	0.518	0.561	0.876

Table 2: Set of 15 Nylon Carpet Fibers (in kg/mm2)

where γ^{ML} and σ^{ML} and λ^{ML} are the maximum likelihood (ML) estimators of γ and σ and λ, respectively. We fit a data set presented by [10] using our proposed TGPD model. Data were collected from ensile strength testing for a random value that exceeds the threshold of 15 nylon carpet fibers. The data set is listed in increasing order in Table (2):.

In his study, he found that the estimators of the GPD parameters by the MLE method were $\widehat{\gamma}^{ML} = -0.1176979$ and $\widehat{\sigma}^{ML} = 0.283040$. Noted that the SRS method with one set is the same as the MLE method and $(\widehat{\gamma}^{ML}, \widehat{\sigma}^{ML}, \widehat{\lambda}^{ML})$ becomes $(\widehat{\gamma}^{SRS}, \widehat{\sigma}^{SRS}, \widehat{\lambda}^{SRS})$. Then, we use a practical method given before for the MLE of TGPD to estimate the transmuted parameter. We will find this by determining the root of $\Psi(\lambda)$ defined in (26) via MBA for multi-roots in $[0;1]$. Then we get $\widehat{\lambda}^{ML} = 0.05885239$. And the TGPD maximum likelihood estimates for the data of 15 nylon carpet fibers are $\widehat{\gamma}^{ML} = -0.1176979$ and $\widehat{\sigma}^{ML} = 0.283040$ and $\widehat{\lambda}^{ML} = 0.05885239$. The P-P interaction graph of C-sample with TGPD$\left(\widehat{\gamma}^{ML}, \widehat{\sigma}^{ML}, \widehat{\lambda}^{ML}\right)$ and their histogram are shown in the following figures, respectively.

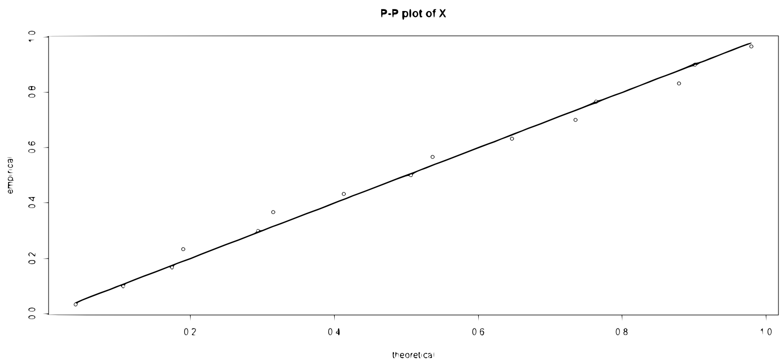

Fig. 1: P-P plot of 15 Nylon Carpet Fibers (in kg/mm2) with TGPD$\left(\widehat{\gamma}^{ML}, \widehat{\sigma}^{ML}, \widehat{\lambda}^{ML}\right)$

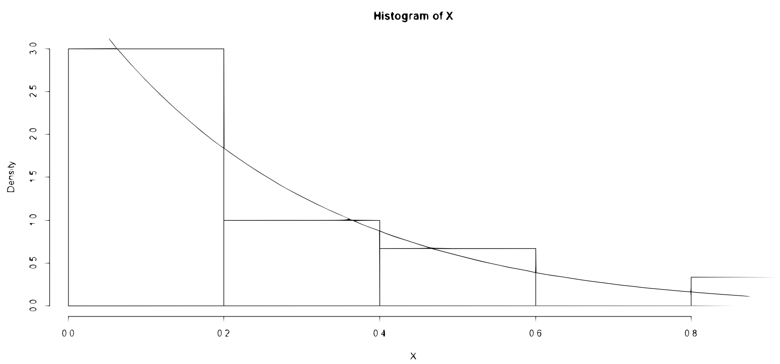

Fig. 2: Histogram of 15 Nylon Carpet Fibers (in kg/mm2) with TGPD$\left(\widehat{\gamma}^{ML}, \widehat{\sigma}^{ML}, \widehat{\lambda}^{ML}\right)$

Therefore, we can find some samples that follow the GPD parameters given in (1) which, in fact, follow the TGPD parameters defined in (12). And to determine the value of λ, we search the root of $\Psi(\lambda)$ given in (26) which has a maximum $\log \ell\left(x_i; \widehat{\gamma}^{ML}, \widehat{\sigma}^{ML}, \widehat{\lambda}^{ML}\right)$. Since if $\lambda = 0$ the TGPD becomes GPD.

3.3 Example3

We apply MLE using RSS to estimate the TGPD parameters given in (12). The tables from (3) to (5) show the estimator, the mean square error (MSE), and the bias for unknown parameters of TGPD using values $\gamma = -0.7$, $\sigma = 2$ and $\lambda = 0.9$. based on SRS and RSS techniques using the ML method with 1000 replications.

	SRS			RSS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	−0.6562	1.7378	0.6870	−0.7163	1.8757	0.6921
100	−0.6720	1.7530	0.6858	−0.6713	1.9155	0.6590
150	−0.6980	1.7551	0.7089	−0.6680	1.9138	0.6561
200	−0.6925	1.7346	0.7003	−0.7480	1.9726	0.7121

Table 3: The estimators of TGPD parameters via MLE and MLE with RSS techniques.

	SRS			RSS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	0.0178	0.0236	0.0863	0.0058	0.0035	0.0531
100	0.0053	0.0124	0.0661	0.0022	0.0019	0.0577
150	0.0026	0.0074	0.0588	0.0021	0.0010	0.0583
200	0.0020	0.0037	0.0227	0.0008	0.0026	0.0210

Table 4: MSEs of the SRS and RSS techniques generated with the TGPD parameters.

	SRS			RSS		
n	$\widehat{\gamma}_{SRS}$	$\widehat{\sigma}_{SRS}$	$\widehat{\lambda}_{SRS}$	$\widehat{\gamma}_{RSS}$	$\widehat{\sigma}_{RSS}$	$\widehat{\lambda}_{RSS}$
50	−0.0437	0.2621	0.2129	0.0163	0.1242	0.2078
100	−0.0279	0.2469	0.2141	−0.0286	0.0844	0.2409
150	−0.0019	0.2448	0.1910	−0.0319	0.0861	0.2438
200	−0.0074	0.2654	0.1996	0.0480	0.1878	0.0273

Table 5: Biases for SRS and RSS techniques for the TGPD parameters.

From tables (3) to (5) it can be seen that

1.The biases are very small in all cases.
2.The values of all estimators are very close to the true values of the parameters.
3.Estimators using RSS have lower MSEs than those using SRS.
4.The RSS produces results that are very close to the true values of the parameters with very small biases, and it outperforms the SRS.

4 Conclusion

In this paper, we introduce the GPD with three parameters called the TGPD parameters, which are given in (12) by taking the GPD given in (1) as the base distribution in the QRTM approach. The MLE are getting the most attention in this article. Therefore, we give the likelihood approach of the transmuted condition distribution function to the MLE of the TGPD parameters to estimate the TGEV parameters, which are presented in (17). Moreover, we define the MLE of the TGPD parameters on the basis of SRS and RSS techniques. Then, we presented a practical method to estimate the unknown parameters of the TGPD. We will explain as follows: For the estimate γ, we solve equations (24) with $\gamma \in [0;-1]$ or (40) with $\gamma > 0$ such that σ and λ are known. And to estimate σ we solve the equation (41) with γ and λ to be known, and finally, to estimate λ we solve the equation (26) and σ and γ must be known, and this is what we worked on in example1. Otherwise, if γ and σ and λ are not previously known, we work with the methods that were used in Example2 or Example3. The goal of presenting these methods is to provide the maximum-likelihood estimator of the transmuted parameter. Moreover, we can use the MLE method that we presented with the SRS and RSS techniques to prove an estimator of GPD parameters given in (1) which we can find if $\lambda = 0$. Finally, we expect this work to serve as a reference and help advance future research in this field.

5 Appendix

First, we have the first derivative of the log-likelihood from (20) for $\gamma \neq 0$ is given by:

$$\frac{\partial \log \ell (x_i;\gamma,\sigma,\lambda)}{\partial \gamma} = \sum_{i=1}^{n}\left[-\frac{\partial \left(\frac{1}{\gamma}+1\right)\log\left(1+\frac{\gamma x_i}{\sigma}\right)}{\partial \gamma} + \frac{\partial \log\left(1-\lambda+2\lambda\left(1+\frac{\gamma x_i}{\sigma}\right)^{\frac{-1}{\gamma}}\right)}{\partial \gamma}\right] \tag{42}$$

Then, we get

$$-\frac{\partial \left(\frac{1}{\gamma}+1\right)\log\left(1+\frac{\gamma}{\sigma}x\right)}{\partial \gamma} = \frac{1}{\gamma^2}\left(\log\left(1+\frac{\gamma}{\sigma}x\right) - (1+\gamma)\frac{\frac{\gamma}{\sigma}x}{1+\frac{\gamma}{\sigma}x}\right) \tag{43}$$

Since we get that

$$\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma} = \exp\left(\frac{-1}{\gamma}\log\left(1+\frac{\gamma}{\sigma}x\right)\right)$$

and we derive the previous formula with respect to γ, we have

$$\frac{\partial \left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}}{\partial \gamma} = \frac{1}{\gamma^2}\left(\log\left(1+\frac{\gamma}{\sigma}x\right) - \frac{\frac{\gamma}{\sigma}x}{1+\frac{\gamma}{\sigma}x}\right)\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}$$

Then, we find

$$\frac{\partial \log\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)}{\partial \gamma} = \frac{\frac{1}{\gamma^2}2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\left(\log\left(1+\frac{\gamma}{\sigma}x\right) - \frac{\frac{\gamma}{\sigma}x}{1+\frac{\gamma}{\sigma}x}\right)}{1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}} \tag{44}$$

We consider

$$\zeta(x) = \frac{1-\lambda}{1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}}$$

Equation (44) can be formulated in the form

$$\frac{\partial \log\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)}{\partial \gamma} = \frac{1}{\gamma^2}\left(1-\zeta(x)\right)\left(\log\left(1+\frac{\gamma}{\sigma}x\right) - \frac{\frac{\gamma}{\sigma}x}{1+\frac{\gamma}{\sigma}x}\right) \tag{45}$$

By adding the two equations (43) and (45) to Eq (42) we obtain the following.

$$\begin{aligned}\frac{\partial \log \ell (x_i;\gamma,\sigma,\lambda)}{\partial \gamma} = \frac{1}{\gamma^2}\sum_{i=1}^{n}\Bigg[&\log\left(1+\frac{\gamma x_i}{\sigma}\right) - (1+\gamma)\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i} \\ &+\frac{1}{\gamma^2}(1-\zeta_i)\left(\log\left(1+\frac{\gamma}{\sigma}x_i\right) - \frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}\right)\Bigg]\end{aligned}$$

with $\zeta_i = \zeta(x_i)$. Finally for take $\hat{\gamma}_{SRS}$ the estimator of γ via search the solution of the following equation

$$\sum_{i=1}^{n}\left[\log\left(1+\frac{\gamma}{\sigma}x_i\right)-(1+\gamma)\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}+\frac{1}{\gamma^2}(1-\zeta_i)\left(\log\left(1+\frac{\gamma}{\sigma}x_i\right)-\frac{\frac{\gamma}{\sigma}x_i}{1+\frac{\gamma}{\sigma}x_i}\right)\right]=0.$$

We use the same mathematical operations to arrive at the remaining equations in (22), which represent the basic equations to obtain the estimators of the parameters $\sigma$and λ which are denoted by $\hat{\sigma}_{SRS}$ and $\hat{\lambda}_{SRS}$ respectively.

5.1 ***Proof of Theorem(2)***

The proofs of (1) and (2) are unpretentious. The proof of (3) follows by noticing that the Cauchy-Schwarz inequality is written:

$$\sum_{i=1}^{n}(1-\zeta_i)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\leq\left(\sum_{i=1}^{n}(1-\zeta_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)^2\right)^{1/2}$$

Since $\left(1-\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1}\right)>0$ where σ is a positive value for $\gamma\geq-1$ and $0\leq\zeta_i\leq1$. Then, we have

$$\left(\sum_{i=1}^{n}(1-\zeta_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)^2\right)^{1/2}\leq\sum_{i=1}^{n}(1-\zeta_i)\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)$$

Noticing that, for $\gamma>0$, we get

$$\sum_{i=1}^{n}\frac{(1-\zeta_i)}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\leq\frac{1}{\gamma}\sum_{i=1}^{n}(1-\zeta_i)\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)$$

Then we can find that

$$\sum_{i=1}^{n}\left[-1+\left(1+\frac{1}{\gamma}\right)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)+\frac{(1-\zeta_i)}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\right]$$
$$\leq\sum_{i=1}^{n}\left[-1+\left(1+\frac{1+\gamma+k-k\overline{\zeta}}{\gamma}\right)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\right]$$

Under Titu's inequality, we find

$$\frac{n^2}{\sum_{i=1}^{n}\left(1+\frac{\gamma}{\sigma}x_i\right)}\leq\sum_{i=1}^{n}\left(\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)$$

It follows that

$$\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\leq1-\frac{n}{\sum_{i=1}^{n}\left(1+\frac{\gamma}{\sigma}x_i\right)}=1-\frac{1}{1+\frac{\gamma}{\sigma}\overline{x}}$$

Then we find the following.

$$\frac{1}{n}\sum_{i=1}^{n}\left[-1+\left(1+\frac{1}{\gamma}\right)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)+\frac{(1-\zeta_i)}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{\sigma}x_i}\right)\right]$$
$$\leq-1+\left(1+\frac{1+\gamma+n-n\times\overline{\zeta}}{\gamma}\right)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}\overline{x}}\right)$$

Next, we put

$$\widetilde{\Psi}(\sigma)=-1+\left(1+\frac{1+\gamma+n-n\times\overline{\zeta}}{\gamma}\right)\left(1-\frac{1}{1+\frac{\gamma}{\sigma}\overline{x}}\right)$$

where $\overline{\zeta} = \frac{1}{n}\sum_{i=1}^{n}\zeta_i$ for $\zeta_i = (1-\lambda)\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x_i\right)^{-1/\gamma}\right)^{-1}$ with $i = 1,\ldots,n$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n}x_i$. Therefore we have $\Psi(\sigma) \leq k\widetilde{\Psi}(\sigma)$. It follows that if $\widetilde{\Psi}(\sigma) \leq 0$ implies $\Psi(\sigma) \leq 0$. For $\widetilde{\Psi}(\sigma) \leq 0$ for all $\sigma \geq \sigma_u$. Also for all $\sigma \geq \sigma_u$, $\Psi(\sigma) \leq 0$. Then we get $\gamma > 0$ and $\sigma > 0$

$$\frac{1}{1+\frac{\gamma}{\sigma}\overline{x}} \geq 1 - \frac{\gamma}{1+2\gamma+n-n\times\overline{\zeta}}$$

Since $\gamma < 2\gamma$, it's easy to verify that $\left(1-\frac{\gamma}{1+2\gamma+n-n\times\overline{\zeta}}\right) \in [0;1]$. Hence,

$$\frac{\gamma}{\sigma}\overline{x} \leq \frac{1+2\gamma+n-n\times\overline{\zeta}}{1+\gamma+n-n\times\overline{\zeta}} - 1$$

Then we find

$$\frac{\gamma}{\sigma}\overline{x} \leq \frac{\gamma}{1+\gamma+n-n\times\overline{\zeta}}$$

Since for $\sigma > 0$ and $\gamma > 0$ we will find

$$\sigma \geq \left(1+\gamma+n-n\times\overline{\zeta}\right)\overline{x}.$$

then we have $\sigma_u = \left(1+\gamma+n-n\times\overline{\zeta}\right)\overline{x}$. On the other side, if -1$\leq \gamma < 0$ is negative, we can take γ as absolute value $|\gamma|$ and σ_u will still be correct as $\sigma_u = \left(1+|\gamma|+n-n\times\overline{\zeta}\right)\overline{x}$.

Next, we were provided mathematical formulations of the TGPD and also some of its properties, such as the reliability function, the hazard rate function, the statistical properties, specifically moments and moment-generating.

5.2 Reliability analysis

The TGPD can be a useful characterization of lifetime data analysis. The reliability function of the TGPD is denoted by R_{TGPD} also known as the survivor function and is defined for $\gamma \in \mathbb{R}$ as:

$$R_{TGPD}(x;(\gamma,\sigma,\lambda)) = 1 - F_{\gamma,\sigma,\lambda}(x). \tag{46}$$

It is important to note that $R_{TGPD}(x;(\gamma,\sigma,\lambda)) + F_{\gamma,\sigma,\lambda}(x) = 1$. One of the characteristic in reliability analysis is the Hazard Rate Function (HRF) defined with $\gamma \in \mathbb{R}$ by

$$h_{TGPD}(x;(\gamma,\sigma,\lambda)) = \frac{f_{\gamma,\sigma,\lambda}(x)}{R_{TGPD}(x;(\gamma,\sigma,\lambda))} \tag{47}$$

The cumulative hazard rate function based on

$$H_{TGPD}(x;(\gamma,\sigma,\lambda)) = \int_0^x h_{TGPD}(t;(\gamma,\sigma,\lambda))\,dt$$

and substituting equation (47) into the previous equation yields

$$H_{TGPD}(x;(\gamma,\sigma,\lambda)) = -\ln\left(1-F_{\gamma,\sigma,\lambda}(x)\right) \tag{48}$$

5.3 Statistical properties

We discuss the statistical properties of the TGPD, specifically the quantile function, median, moments, and moment generating function.

5.3.1 Moments

The r^{th} moment of a random variable X of the TGPD for $\gamma \neq 0$ can be obtained from the following theorem:

Theorem 3. *The r^{th} moment of a random variable X follows the TGPD is given by*

$$E\left(X^r\right) = \left(\frac{\sigma}{\gamma}\right)^r \sum_{j=0}^{r} (-1)^j C_r^j \left[\left(\frac{1}{(j-r)\times\gamma+1}\right)\left(\frac{-(\lambda+1)(j-r)\times\gamma-2}{(j-r)\times\gamma+2}\right)\right] \tag{49}$$

where $\gamma \neq 0$.

Proof. The r^{th} moment of a random variable X of the TGPD for $\gamma \neq 0$ can be obtained from

$$E\left(X^r\right) := \int_0^1 x^r f_{\gamma,\sigma,\lambda}(x)\,dx \tag{50}$$

Then, substituting pdf of TGPD for $\gamma \neq 0$ from (11) into (50) yields

$$E\left(X^r\right) := \int_0^1 x^r \frac{1}{\sigma}\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma-1}\left(1-\lambda+2\lambda\left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}\right)dx \tag{51}$$

Setting $u = \left(1+\frac{\gamma}{\sigma}x\right)^{-1/\gamma}$. Then $x = \frac{\sigma}{\gamma}\left(u^{-\gamma}-1\right)$ and d$x = -\sigma u^{-\gamma-1}du$. Substituting this to (51) yields then

$$E\left(X^r\right) := \left(\frac{\sigma}{\gamma}\right)^r \int_0^1 \left(u^{-\gamma}-1\right)^r (2\lambda(1-u)-1-\lambda)\,du$$

Using the binomial theorem yield

$$\left(u^{-\gamma}-1\right)^r = \sum_{j=0}^{r} C_r^j (-1)^j u^{(j-r)\times\gamma}$$

Then

$$E\left(X^r\right) = \left(\frac{\sigma}{\gamma}\right)^r \sum_{j=0}^{r} (-1)^j C_r^j \left[2\lambda\int_0^1 u^{(j-r)\times\gamma}(1-u)\,du - (1+\lambda)\int_0^1 u^{(j-r)\times\gamma}du\right]$$

Thus it's easy to checked that

$$E\left(X^r\right) = \left(\frac{\sigma}{\gamma}\right)^r \sum_{j=0}^{r} (-1)^j C_r^j \left[\left(\frac{1}{(j-r)\times\gamma+1}\right)\left(\frac{-(\lambda+1)(j-r)\times\gamma-2}{(j-r)\times\gamma+2}\right)\right]$$

The end of proof

For $\gamma \neq 0$ with $r = 1$,.(49) gives

$$E(X) = \frac{\sigma}{1-\gamma}\left(\frac{\lambda\gamma}{(2-\gamma)}+1\right) \tag{52}$$

Setting $r = 2$ with $\gamma \neq 0$, then (49) gives

$$E\left(X^2\right) = \left(\frac{\sigma}{\gamma}\right)^2 \left(\frac{\gamma\lambda+2\gamma^2}{(2\gamma-1)(1-\gamma)}\right) \tag{53}$$

Variance Let X be a random variable that follows the TGPD of (12) and (11) with $\gamma \neq 0$. Using (52) and (53), the variance is given as follows:

$$Var(X) = E\left(X^2\right) - (E(X))^2 = \left(\frac{\sigma}{1-\gamma}\right)^2 \left[\frac{(\lambda+2\gamma)(1-\gamma)}{\gamma(2\gamma-1)} - \left(\frac{\lambda\gamma}{2-\gamma} - 1\right)^2\right] \quad (54)$$

*Remark.*If $\lambda = 0$ with $\gamma \neq 0$, we have

1. $E(X) = \frac{\sigma}{1-\gamma}$.
2. $E\left(X^2\right) = \left(\frac{\sigma}{1-\gamma}\right)^2 \frac{2(1-\gamma)}{2\gamma-1}$.
3. $Var(X) = \left(\frac{\sigma}{1-\gamma}\right)^2 \frac{1}{2\gamma-1}$

5.3.2 The Moment Generating Function

The moment generating function of the TGPD where $\gamma \neq 0$ is obtained in the following theorem:

Theorem 4.*The moment generating function of the random variable X which has the pdf of the TGPD for from (12) is given by*

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \quad (55)$$

*Proof.*Clearly, from the following fact

$$M_x(t) = E(\exp(Xt))$$

Using the expansion of $\exp(Xt)$ yields

$$M_x(t) = \sum_{r=0}^{\infty} \frac{(Xt)^r}{r!}$$

Then

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

Declarations

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