JJMS **18**, No. 3, 369-379 (2025)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/18.3.4

Estimation of Inverse Pareto Distribution through Different Loss Functions on Mortality Rate of COVID Data

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Received: March 18, 2024 Accepted: Jun. 11, 2025

Abstract: In this paper, the main objective is to estimate the parameter of inverse Pareto distribution based on lower record values. Hence, we consider the classical and Bayesian approaches to estimate the parameters of Inverse Pareto distribution (IPD), such as Maximum likelihood estimator(MLE), the general entropy loss function (GELF), the square error loss function (SELF), and the linear exponential loss function (LINEX). Further, simulation studies are performed to compare the different loss functions and confidence interval is also obtained for the given parameter. At last, covid data are used to compare the different results of estimations.

Keywords: Inverse Pareto distribution, Record values, Maximum Likelihood Estimates (MLE), SELF, LINEX, GELF, Bayesian estimation, Covid data.

2010 Mathematics Subject Classification. 62F10 ; 62F15.

1 Introduction

The Italian civil engineer and economist Vilfredo Pareto gave the Pareto distribution (1843-1900) in the field of a wealth of societies. The Pareto distribution is beneficial for modeling economics, finance, stock price fluctuation, environmental studies, insurance risk, quenching theory, and other fields. Many scholars studied the Pareto distribution for the actual data, such as Saksena and Johnson [1] discussed the different properties of maximum likelihood and uniformly minimum variance unbiased estimators of unknown parameters. Rytgaard [2] has study the estimation of parameter of Pareto distribution. Dunsmore and Zeinab [3] have study the prediction of missing data using the Gibbs sampling routine method. Habibullah and Ahsanullah [4] have study the estimation of pareto distribution based on generalized order statistics. Donice and Sandra [5] have study the generalized jackknife is used to reduce the bias of an estimator of pareto distribution. The researcher further study the some inferential properties of classical and Bayesian for censoring and record values. Mousa [6] has study the estimation and prediction of parameters using classical and Bayesian method for censored data. Parsi et al. [7] have study the determined the confidence interval of unknown parameter of Pareto distribution under progressive type censoring. Ismaila et al. [8] have obtained the parameter of maximum likelihood estimate using the help of the Optimum constant-stress life test plans model under type-I censoring. Renjini et al.[9] have provide the Bayes estimation of Pareto distribution under different loss function based on record values. Viswakala and Abdul Sathar [10] obtained the maximum likelihood estimates of hazard rate and mean residual function of Pareto distribution further discuss the uniformly minimum variance unbiased estimate and real application. Mehdi and Mehrdad [11] study the inference based on stress strength reliability under the Pareto distribution. Azhad et al. [12] study the inference about multicomponent stress and strength of Pareto distribution under record values. Also, discuss the Bayes estimate under different loss function and constructed the bootstrap confidence intervals.

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In many real-life situations, reliability theory and survival analysis data need a probability distribution with both a descending and an upside-down bathtub (UBT) shaped failure rate function. A patient whose mortality rate reached peak then recovered slowly after some time. Patients with heart transplants face an increasing failure rate of death over the first few days after the heart transplant while the body adapts to the new organ. The failure rate decreases with time as the patient recovers. In such a situation, the (UBT) function be appropriated. There are so many applications of Pareto distribution [13]
On the other hand the Inverse Pareto distribution (IPD) follows both decreasing and downward bathtub shape functions. The hazard function of Inverse Pareto distribution at different values of α is shown in Fig. 1.0.
Many authors studied inverse Pareto distribution, such as Guo and Gui [14] estimate the parameter of inverse Pareto distribution under the stress and strength model. Also, find the exact confidence interval and bootstrap confidence interval. Kumar and Kumar [15,16] discussed the classical and Bayesian estimation of inverse Pareto distribution based on censored data. Kumar et al. [17,18] study the several classical method to estimate the unknown parameters of inverse Pareto distribution. Further, introduced the reliability estimation of unknown parameter using the progressively first failure censored data. Dankunprasert et al. [19] compare the Inverse Pareto distribution to different continuous distributions for the Danish fire data, Nasiru et al. [20] introduced the estimation and regression of Bounded odd inverse Pareto distribution, Single and product moment of order statistics from inverse Pareto distribution were studied by Mustafa et al. [21]. Garg et al. [22] introduced the multicomponent stress and strength reliability from inverse Pareto distribution. Some studies on a different distribution, which is the standard or generalization form of Pareto distribution, such as beta exponentiated Pareto distribution [23] by Zea et al., Nadarajah [24] studied the exponentiated Pareto distribution, Arshad et al. [25] studied the Transmuted exponentiated Moment Pareto distribution, Pareto Type-I Distribution studied by Setiya et al. [26]

In many real-world scenarios, observations of extreme values are of particular interest. Record values are a special type of order statistics that track new highs (or lows) in a sequence of random observations. These values occur naturally in fields such as sports, climate studies, hydrology, industry, weather forecasts, seismology, reliability engineering, athletics, life testing, and financial markets, where tracking maximum or minimum values. For example, record values are crucial in various fields for predicting extreme events and making informed decisions. In climatology and hydrology they help forecast the highest flood levels for designing flood control measures and safety systems. In seismology, tracking the strongest earthquakes aids in risk assessment and preparedness. In sports, analyzing world records helps predict future performances and optimize training. In finance, record-breaking stock prices guide investment strategies and economic forecasting. In engineering and manufacturing, testing material strength ensures the development of stronger materials for industries like construction and aerospace. These applications demonstrate the importance of record values in assessing risks and making data-driven decisions across disciplines. Since several researcher have published the books and paper such as Chandler [27] first study the concept of record values. Resnick [28] and Shorrock [29] have study the concept of record values and inter record times. Balakrishnan and Ahsanullah [30] have discussed the single and product moments based on record values. Pawlas and Szynal [31] obtained the recurrence relation of single and product moments from Pareto, generalized pareto, and Burr distribution. Khan at al.[32] addresses the statistical inference of additive Weibull distribution based on record values . Khan and Arshad [33] obtained the uniformly minimum variance unbiased estimator of reliability function and stress–strength reliability under the lower record values. Singh et al. [34] obtained the expression of single and product moment and illustrate the variance covariance of lower record values. Further, defined the classical and Bayesian estimation of unknown parameters. Anwar et al. [35,36] expresses the single, product moments, and associated statistical inference of different distribution under the record values. Khan et al.[37] studied the single and product moments of inverted Topp-Leone distribution based on record values. Also, computed the best linear unbiased estimator, future record prediction, and maximum likelihood estimation.
Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The j-th order statistic of a sample $(X_1, X_2, \ldots, X_n)$ is denoted by $X_{j:n}$. For fixed $k \geq 1$, we define the sequence $\{L_n^{(k)}, n \geq 1\}$ of k-th lower record times of $\{X_n, n \geq 1\}$ as follows:

$$L_1^{(k)} = 1,$$

$$L_{n+1}^{(k)} = \min\left\{ j > L_n^{(k)} : X_{k:L_n^{(k)}+k-1} > X_{k:j+k-1} \right\}.$$

For $k = 1$, we put $L_n = L_n^{(1)}, n \geq 1$, which are the k-th lower record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$ with $Y_n^{(k)} = X_{k:L_n^{(k)}+k-1}, n = 1, 2, \ldots$, is called the sequence of k-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that for $k = 1$, we have $Y_n^{(1)} = X_{L_n}, n \geq 1$, i.e., the record value of $\{X_n, n \geq 1\}$. Moreover, we see that $Y_1^{(k)} = \max\{X_1, X_2, \ldots, X_k = X_{k:k}.\}$. Let $\{X_n^{(k)}, n \geq 1\}$ be the sequence of k-th lower record values. Then the pdf of $X_{L(n)}^{(k)}$, $n \geq 1$, is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{\Gamma(n)}[-\ln(F(x))]^{n-1}[F(x)]^{k-1}f(x), \quad n \geq 1,$$

and the joint pdf of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, with $1 \leq m < n, n > 2$, is given by

$$f_{X_{L(m)}^{(k)},X_{L(n)}^{(k)}}(x,y) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\ln(F(x))]^{m-1}$$

$$\times[-\ln(F(y)) + \ln(F(x))]^{n-m-1}[F(y)]^{k-1}\frac{f(x)}{F(x)}f(y), \quad x > y.$$

The associated likelihood function of α given the observed lower record values $r = (r_1, r_2, \ldots, r_m)$ can be written as

$$L(\alpha \mid r) = f(r_m;\alpha)\prod_{i=1}^{m-1}\frac{f(r_i;\alpha)}{F(r_i;\alpha)}, \quad r > 0, \quad m = 1,2,\ldots.$$

To the best of our knowledge, the estimation method for inverse Pareto distribution based on record values has not been studied till date. The main aim of this paper is to estimate the parameters of inverse Pareto distribution in both the classical and Bayesian methods. In section 2, the classical method of maximum likelihood estimation is presented. In section 3 Asymptotic confidence intervals are presented. In section 4 Bayes estimation under different loss functions are presented . In section 5 simulation study using the inverse *CDF* method is discussed while actual data are discussed in section 6. Finally, the conclusion of this paper is discussed in section 7.

Suppose X is a random variable that follows the IPD with single parameter α, then probability density function (*CDF*) of IPD(α) is given by,

$$F(x,\alpha) = \left(\frac{x}{x+1}\right)^{\alpha}, \ x > 0, \ \alpha > 0 \tag{1}$$

and (*PDF*), reliability and failure rate of IPD are given by ,

$$f(x,\alpha) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}}; \ x > 0, \alpha > 0 \tag{2}$$

$$R(x,\alpha) = 1 - \left(\frac{x}{1+x}\right)^{\alpha}; \ x > 0, \alpha > 0 \tag{3}$$

$$H(x,\alpha) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}\left[1-\left(\frac{x}{1+x}\right)^{\alpha}\right]}. \tag{4}$$

Fig.a and Fig.b show that the density plot and cumulative distribution function plot of Inverse pareto distribution at different parameters values.The proposed distribution, which can illustrate a number of dataset patterns, is illustrated in the picture. In Fig.c and Fig.d show the Inverse pareto distribution reliability and hazard rate functions. The hazard rate function show both decreasing and upside-downward bathtub shape.

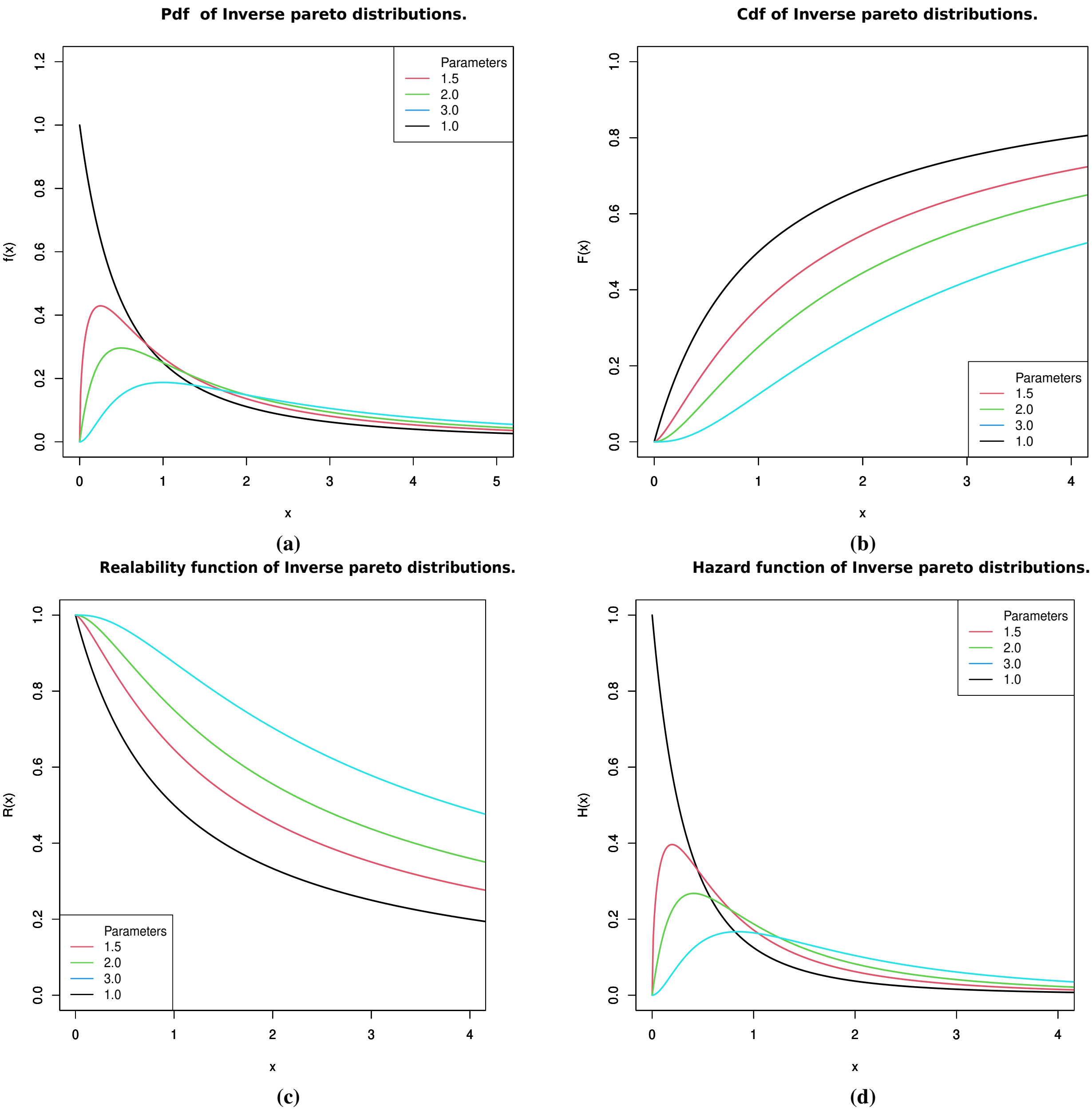

Fig. 1: PDF, CDF, survival, and hazard function.

2 Maximum likelihood Estimates(MLEs)

Let $x=(x_1,\ldots,x_m)$ be a set of m lower record values from inverse pareto distribution with pdf given in (2) and cdf given in (1). The likelihood function is given by Arnold [38] as

$$L(\alpha|\underline{x}) = f(x_m)\prod_{i=1}^{m-1}\frac{f(x_i,\alpha)}{F(x_i,\alpha)} \tag{5}$$

Using (1) and (2) in equation(5),we get

$$= \frac{\alpha x_m^{\alpha-1}}{(1+x_m)^{\alpha+1}}\prod_{i=1}^{m-1}\frac{\alpha x_i^{\alpha-1}(1+x_i)^{\alpha}}{(1+x_i)^{\alpha+1}x_i^{\alpha}}$$

$$= \frac{\alpha x_m^{\alpha-1}}{(1+x_m)^{\alpha+1}} \prod_{i=1}^{m-1} \frac{\alpha}{x_i(1+x_i)}$$

$$L(\alpha|\underline{x}) = \frac{\alpha^m x_m^{\alpha}}{(1+x_m)^{\alpha}} \prod_{i=1}^{m} \frac{1}{x_i(1+x_i)}. \tag{6}$$

Taking log on both sides,we get Log-likelihood function (l) as

$$l(\alpha|\underline{x}) = mln\alpha + \alpha ln\left(\frac{x_m}{1+x_m}\right) + \sum_{i=1}^{m} ln\left(\frac{1}{x_i(1+x_i)}\right). \tag{7}$$

Differentiating equation (7) with respect to α and equating to zero, we get MLE of α, say($\hat{\alpha}$).

$$\frac{d}{d\alpha} = \frac{m}{\alpha} + \ln\left(\frac{x_m}{1+x_m}\right) = 0$$

$$\hat{\alpha} = \frac{m}{ln\left(\frac{1+x_m}{x_m}\right)}$$

We may use R software (using optim function) to find MLE of unkown parameter α.

3 Asymptotic Confidence Intervals

Here, we evaluated the confidence interval for unknown parameter α based on MLE. The Fisher information matrix is defined as

$$I(\hat{\alpha}) = -E\left[\frac{\partial^2 lnL(\alpha|\underline{x})}{\partial^2\alpha}\right]_{\alpha=\hat{\alpha}} = \frac{m}{\hat{\alpha}^2} \tag{8}$$

The inverse of Fisher information matrix is the variance of $\hat{\alpha}$. Thus Var($\hat{\alpha}$)=$\frac{\hat{\alpha}^2}{m}$.
The ML estimate $\hat{\alpha}$ of α follows asymptotic normal distribution.i.e $\hat{\alpha} \sim N(\hat{\alpha}, v(\hat{\alpha}))$. Therefore, $(1-\theta)\times 100\%$ asymptotic CI for α is given by

$$\left(\hat{\alpha} - Z_{\theta/2}\sqrt{Var(\hat{\alpha})}, \hat{\alpha} + Z_{\theta/2}\sqrt{Var(\hat{\alpha})}\right)$$

$$P\left[\left|\frac{\alpha-\hat{\alpha}}{\sqrt{Var(\hat{\alpha})}}\right| \le Z_{\theta/2}\right].$$

$Z_{\theta/2}$ is the upper $(1-\theta)\times 100$ percentile of standard normal distribution.

4 Bayes Estimation

The prior distribution chosen as well as the loss function have a significant impact on Bayesian estimation. For the unknown parameters of a given distribution of interest, numerous prior distributions have been suggested in the literature, such as Jamal and Arshad [39] proposed condtional gamma prior, Calabria and Pulcini [40] proposed non informative prior, Ahmad et.al [41] introduced Jeffreys non-informative prior distribution and a bivariate prior distribution, Raqab and Madi [42], Basu and Ebrahimi [43] gives gamma prior. In the assumption of the above statements, we consider gamma priors on the parameters of the IPD distribution such that, the prior density is the gamma

$$\pi(\alpha|a,b) = \frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha}, a>0, b>0 \tag{9}$$

where $(\Gamma.)$ is the gamma function with hyper-parameters a, b.

The posterior distribution using the Bayes theorem under the prior function can be calculated as,

$$\pi(\alpha|\underline{x}) = \frac{L(\alpha,\underline{x})\pi(\alpha|a,b)}{\int_0^\infty L(\alpha,\underline{x})\pi(\alpha|a,b)d\alpha} \tag{10}$$

substituting the value of equation (6) and (9) in equation (10), we get

$$\pi(\alpha|\underline{x}) = \frac{\frac{\alpha^m x_m^\alpha}{(1+x_m)^\alpha}\prod_{i=1}^m \frac{1}{x_i(1+x_i)}\frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha}}{\int_0^\infty \frac{\alpha^m x_m^\alpha}{(1+x_m)^\alpha}\prod_{i=1}^m \frac{1}{x_i(1+x_i)}\frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha}d\alpha}$$

$$\pi(\alpha|\underline{x}) = \frac{\alpha^{m+a-1}e^{\alpha ln\left(\frac{x_m}{1+x_m}\right)}e^{-b\alpha}\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}}{\Gamma(m+a)} \tag{11}$$

4.1 Bayes estimation based on a squared error loss function(SELF)

The square error loss function is a symmetric function and it was introduced by Legendre (1805). The SEL is defined as

$$L(\hat{\alpha},\alpha) = (\hat{\alpha}-\alpha)^2, \hat{\alpha}\in D, \alpha\in\Theta$$

where D is decision space and Θ is the parameter space. The Bayes estimator under the squared error loss function is the posterior mean ;i.e the Bayes estimator of parameter α is given by

$$\hat{\alpha} = E_\alpha(\alpha|\underline{x})$$

Assuming that $E_\alpha(\alpha|\underline{x})$ exists and is finite. Consider

$$\hat{\alpha} = \int_0^\infty \alpha\pi(\alpha|\underline{x})d\alpha$$

Using equation (11), we have the Bayes estimate of α for SELF is

$$\hat{\alpha} = \frac{(m+a)}{\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]}.$$

4.2 Bayes estimation based on Linear exponential loss function (LINEX)

Varian [44] introduced LINEX loss function it is an asymmetric loss function. It is defined as,

$$L(\hat{\alpha},\alpha) = e^{c_1(\hat{\alpha}-\alpha)-c_1(\hat{\alpha}-\alpha)-1}, c_1\neq 0, \hat{\alpha}\in D, \alpha\in\Theta$$

where $c_1 \neq 0$ is the parameter of loss function. The Bayes estimator of parameter α is given by

$$\hat{\alpha} = -\frac{1}{c_1}ln\left[E(e^{-c_1\alpha}|\underline{x})\right]$$

Assuming that $E_\alpha(\alpha|\underline{x})$ exists and is finite. Consider

$$E(e^{-c_1\alpha}|\underline{x}) = \int_0^\infty e^{-c_1\alpha}\pi(\alpha|\underline{x})d\alpha$$

Using the equation (11), we have

$$E(e^{-c_1\alpha}|\underline{x}) = \int_0^\infty \frac{\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}\alpha^{m+a-1}e^{-\alpha\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]}e^{-c_1\alpha}d\alpha}{\Gamma(m+a)}$$

$$E(e^{-c_1\alpha|\underline{x}}) = \frac{\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}}{\left[c_1+b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}}.$$

Hence, Bayes estimate of α for LINEX is

$$\hat{\alpha} = -\frac{1}{c_1}ln\left\{\frac{\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}}{\left[c_1+b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}}\right\}.$$

4.3 Bayes estimation based on General Entropy loss function (GELF)

A General Entropy Loss Function was developed by Calabria and Pulcinic (1996) as a replacement for the modified LINEX loss function, which is given by

$$L(\hat{\alpha},\alpha)=\left(\frac{\hat{\alpha}}{\alpha}\right)^{k}-kln\left(\frac{\hat{\alpha}}{\alpha}\right)-1$$

where $k\neq 0$ is the parameter of loss function. The Bayes estimator of parameter α is given by

$$\hat{\alpha}=\left[E_{\alpha}\left(\alpha^{-k}|x\right)\right]^{-\frac{1}{k}}$$

Assuming that $E_{\alpha}(\alpha|\underline{x})$ exists and is finite. Consider

$$E_{\alpha}\left(\alpha^{-k}|\underline{x}\right)=\int_{0}^{\infty}\alpha^{-k}\pi(\alpha|\underline{x})d\alpha$$

$$=\int_{0}^{\infty}\alpha^{-k}\alpha^{m+a-1}e^{-\alpha\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]}\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{m+a}d\alpha$$

Hence, Bayes estimate of α for GELF is

$$\hat{\alpha}=\left[\frac{\Gamma(m+a-k)}{\Gamma(m+a)}\left[b-ln\left(\frac{x_m}{1+x_m}\right)\right]^{k}\right]^{-\frac{1}{k}}.$$

5 Simulation

In this section, we study the simulation of IPD using R software. the study consider sample size $(m=6,7,8,9)$ and four parameters combination α=1, 1.5 ,2 ,2.5. Maximum likelihood and Bayesian estimate of the parameter are calculated. For Bayesian estimation gamma informative priors with hyper-parameters $(a=b=1=2)$ are used, the linex loss function is applied with c_1= -0.5 and 0.5, and the general entropy loss function is applied with k_1= 0.5 and 1. The simulation is repeated 1,000 times and performance metrics- average bias (ABs), mean square errors(MSEs), and confidence intervals are calculated. Results are presented in Tables1, Table2, Table3, Table4, and Table5. As a sample size increase average estimates and mean square errors are decreases. Interval length shrink with large sample size at 95% confidence intervals. Therefore, Bayesian estimators are recommended when prior information is available; otherwise, ML estimators are suitable.

Table 1: ABs and MSEs (within bracket) for IPD parameter $\alpha=1$ of the IPD and hyper-parameters $a=1$, $b=2$.

α	m	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LINEX}$		$\hat{\alpha}_{GEL}$	
				$c_1=-0.5$	$c_1=0.5$	$k=0.5$	$k=1$
1	6	0.2075(0.4102)	-0.038(0.0937)	0.00041(0.1088)	-0.07254(0.0847)	-0.1404(0.0934)	-0.1754(0.0986)
1	7	0.1652(0.3083)	-0.0335(0.0870)	3.7053×10^{-6}(0.0994)	-0.0639(0.0792)	-0.1543(0.0896)	-0.1543(0.0896)
1	8	0.1400(0.2555)	-0.02905(0.0860)	0.000796(0.0970)	-0.0564(0.0787)	-0.1369(0.0860)	-0.1369(0.0860)
1	9	0.1255(0.2153)	-0.0222(0.0831)	0.00476(0.0929)	-0.0472(0.0763)	-0.1200(0.0813)	-0.1200(0.0813)

Table 2: ABs and MSEs (within bracket) for IPD parameter $\alpha = 1.5$ of the IPD and hyper-parameters $a = 1$, $b = 2$.

α	m	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LINEX}$		$\hat{\alpha}_{GEL}$	
				$c_1 = -0.5$	$c_1 = 0.5$	$k = 0.5$	$k = 1$
1.5	6	0.3113(0.9231)	-0.2445(0.1773)	-0.1794(0.1768)	-0.3011(0.1884)	-0.3781(0.4239)	0.2368(0.2660)
1.5	7	0.2478(0.6936)	-0.2176(0.1633)	-1589(0.1650)	-0.2695(0.1706)	-0.3371(0.2090)	-0.3779(0.2090)
1.5	8	0.2101(0.5749)	-0.1955(0.1589)	-0.1419(0.1631)	-0.2435(0.1628)	-0.3037(0.1937)	-0.3405(0.2113)
1.5	9	0.1882(0.4845)	-0.1731(0.1519)	-0.1235(0.1576)	-0.2179(0.1533)	-0.2722(0.1785)	-0.3058(0.1923)

Table 3: ABs and MSEs (within bracket) for IPD parameter $\alpha = 2$ of the IPD and hyper-parameters $a = 1$, $b = 2$.

α	m	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LINEX}$		$\hat{\alpha}_{GEL}$	
				$c_1 = -0.5$	$c_1 = 0.5$	$k = 0.5$	$k = 1$
2	6	0.4151(1.641)	-0.5152(0.3945)	-0.4240(0.3443)	-0.5926(0.4556)	-0.6732(0.5563)	-0.7273(0.6238)
2	7	0.3305(1.233)	-0.4646(0.3487)	-0.3807(0.3102)	-0.5373(0.3980)	-0.6077(0.4785)	-0.6565(0.5327)
2	8	0.2801(1.021)	-0.4228(0.3323)	-0.3446(0.2941)	-0.4915(0.3615)	-0.5536(0.4272)	-0.5981(0.4711)
2	9	0.2510(0.8614)	-0.3829(0.2960)	-0.3093(0.2755)	-0.4481(0.3272)	-0.5036(0.3816)	-0.5446(0.4176)

Table 4: ABs and MSEs (within bracket) for IPD parameter $\alpha = 2.5$ of the IPD and hyper-parameters $a = 1$, $b = 2$.

α	m	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LINEX}$		$\hat{\alpha}_{GEL}$	
				$c_1 = -0.5$	$c_1 = 0.5$	$k = 0.5$	$k = 1$
2.5	6	0.5188(2.564)	-0.830(0.8220)	-0.7160(0.6855)	-0.9265(0.9633)	-1.008(1.1220)	-1.0689(1.2390)
2.5	7	0.4131(1.9268)	-0.7566(0.7133)	-0.6485(0.6005)	-0.8487(0.8330)	-0.919(0.9606)	-0.9745(1.0575)
2.5	8	0.3501(1.5968)	-0.6950(0.6398)	-0.5926(0.5476)	-0.7835(0.7420)	-0.8447(0.8453)	-0.8956(0.9259)
2.5	9	0.3138(1.3460)	-0.6371(0.5731)	-0.5394(0.4974)	-0.72231(0.6600)	-0.7762(0.7456)	-0.8234(0.4176)

Table 5: Asymptotic confidence intervals at 95%.

m	α=1.0		α=1.5		α=2.0		α=2.5	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
6	0.0193	2.3958	0.0289	3.5937	0.0386	4.7916	0.0482	5.9895
7	0.1257	2.2040	0.1886	3.3070	0.2515	4.4094	0.31442	5.5118
8	0.1876	2.094	0.2815	3.1387	0.3750	4.1840	0.4692	5.231
9	0.2493	2.0016	0.3740	3.0024	0.4988	4.0032	0.6235	5.0040

6 Real Data

Here, we uses two realistic data of covid-19 mortality rates from Italy and the Netherlands [see https://covid19.who.int/][45]. The first set of data shows 59-day COVID-19 death rates for Italy, which were collected between 27 February and 27 April 2020. Italy data set : 4.571 7.201 3.606 8.479 11.410 8.961 10.919 10.908 6.503 18.474 11.010 17.337 16.561 13.226 15.137 8.697 15.787 13.333 11.822 14.242 11.273 14.330 16.046 11.950 10.282 11.775 10.138 9.037 12.396 10.644 8.646 8.905 8.906 7.407 7.445 7.214 6.194 4.640 5.452 5.073 4.416 4.859 4.408 4.639 3.148 4.040 4.253 4.011 3.564 3.827 3.134 2.780 2.881 3.341 2.686 2.814 2.508 2.450 1.518.
The second data shows 30 days covid-19 deaths rates for Netherlands, which were collected between 31 March to 30 April 2020. Netherlands data set: 14.918 10.656 12.274 10.289 10.832 7.099 5.928 13.211 7.968 7.584 5.555 6.027 4.097 3.611 4.960 7.498 6.940 5.307 5.048 2.857 2.254 5.431 4.462 3.883 3.461 3.647 1.974 1.273 1.416 4.235.
The negative log-likelihood value, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the K-S statistics are used to determine whether this inverse Pareto distribution model is appropriate for the data sets. The better distribution corresponds to lower K-S, log-like, AIC, and BIC values.
From Table 6. for both the Covid data set indicate that Inverse Pareto (IPD) distribution is better fit than inverse power Lindley distribution (IPLD). The lower record values of first data set are 4.571, 3.606, 3.148, 3.134, 2.780, 2.686, 2.508, 2.450, 1.518. Similarly, the lower record values of second data set are 14.918, 10.656, 10.289, 7.099, 5.928, 5.555,

Table 6: ML estimates, log-like, AIC, BIC, and K-S test for Covid data.

Data set	Model	Estimation	loglike	AIC	BIC	K-S Statistics
I	IPD	6.31	-186.91	374.82	377.89	0.2828
	IPLD	0.4562, 0.2904	-271.24	546.49	550.64	0.4446
II	IPD	4.92	-87.81	176.22	177.62	0.2733
	IPLD	0.5175, 0.2810	-125.356	254.71	257.51	0.4093

4.097, 3.611, 2.857, 2.254, 1.974, 1.273. Using these record values we estimates MLE, SEL, LINEX, and GEL. Now, we

Table 7: Estimates MLE, SEL, LINEX, and GEL with lower record values for both data sets.

Data Set	m	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{LINEX}$		$\hat{\alpha}_{GEL}$	
				$c_1 = -0.5$	$c_1 = 0.5$	$k = 0.5$	$k = 1.0$
I	9	17.784	6.6397	4.450	3.630	3.690	3.590
II	12	20.705	8.2300	5.6016	4.6060	4.7495	4.6516

obtain the confidence intervals at 95% levels of significance based on these records. The calculated values of 95% confidence intervals of α are, respectively, [6.1653, 29.4026] and [8.9902, 32.4197].

7 Conclusion

In this study, we have considered the Inverse Pareto distribution for the estimation based on Lower record values. Here, classical and Bayesian estimation methods are considered to find unknown parameters.The asymptotic confidence interval also obtained from it. Further different loss functions such as GELF, SELF, and LINEX are also compared. From the simulations study the Bayes estimators performs better than frequentist estimators, it is expected that Bayes estimators performs better than the frequentist approach because of using different models by incorporating more information via the prior distribution in the Bayesian approach. At last, two real data sets of covid-19 mortality rates are considered,and they shows that Inverse pareto distribution provides a better fit than the inverse power Lindley distribution.

Declarations

Competing interests: There is no conflict of interest among authors.
Authors' contributions: Authors contribution equally.
Funding: Not applicable.
Availability of data and materials: The data presented in this study are openly available.
Acknowledgments: The authors are grateful to the Editor and anonymous reviewer for their helpful comments and suggestions to improve quality of paper.

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