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Bi-Univalent Characteristics For a Particular Class of Bazilevič Functions Generated By Convolution Using Balancing Polynomials

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Abstract: Bi-univalent functions, which are analytic and univalent in the open unit disk, have been a significant area of research in complex analysis due to their rich geometric properties. This study introduces a novel subclass of bi-univalent functions, called the Bazilevič functions class, which is created via the convolution of Balancing polynomials. We calculated the bounds for the Fekete-Szegö inequality as well as the initial coefficients $|a_2|$ and $|a_3|$. We have also included several relevant corollaries.

Keywords: Analytic function; Bi-Univalent function; Bazilevič functions; Convolution; Balancing polynomials.

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1 Introduction and preliminaries

A function that is complex-valued and depends on one or more complex variables is considered analytic if it is differentiable at every point within its domain. Any such analytic function that is normalized can be represented as a series in the form

$$\mathtt{f}(z) = z + \sum_{t=2}^{\infty} a_t z^t . \tag{1}$$

This series is in the complex variable z and converges within the set $\mathscr{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. The set $\mathscr{A}$ is composed of all such functions. Indicate by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of univalent functions.

Let $\mathtt{c}$ be a Schwarz function defined as

$$\mathtt{c}(z) = \sum_{n=1}^{\infty} \mathtt{c}_n z^n \quad (\mathtt{c}(0) = 0,\ |\mathtt{c}(z)| < 1).$$

Consider two functions, $\mathtt{f}_1$ and $\mathtt{f}_2$, which are part of the class $\mathscr{A}$. We say that $\mathtt{f}_1$ is *subordinate* to $\mathtt{f}_2$ if there exists a function $\mathtt{c}(z)$ that is analytic in $\mathscr{U}$, and makes $\mathtt{f}_1(z) = \mathtt{f}_2(\mathtt{c}(z))$ hold true. This relationship is denoted as $\mathtt{f}_1(z) \prec \mathtt{f}_2(z)$. Here, it is known that for the Schwarz function $\mathtt{c}$, the coefficient inequality $|\mathtt{c}_n| < 1$ holds [8], and specifically, we have

$$|\mathtt{c}_1| \le 1, \quad |\mathtt{c}_n| \le 1 - |\mathtt{c}_1|^2 \quad (n \in \mathbb{N} \setminus \{1\}).$$

Comprehensive information on the subordination concept is available in monographs authored by Miller and Mocanu [18].

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The Hadamard product of $\mathtt{f}(z)$ and $\mathtt{k}(z)$, where $\mathtt{k}(z)$ is an element of the set $\mathscr{A}$ and

$$\mathtt{k}(z) = z + \sum_{t=2}^{\infty} \mathtt{k}_t z^t$$

is expressed by

$$(\mathtt{f} * \mathtt{k})(z) = z + \sum_{t=2}^{\infty} a_t \mathtt{k}_t z^t.$$

Consider $\mathtt{f}$ as a function in $\mathscr{S}$. The function $\mathtt{f}(z)$ is classified as *bi-univalent* if its inverse, denoted by $\mathtt{f}^{-1}(w)$, extends analytically to $|w| < 1$ within the w-plane. That is, according to the *Koebe-One Quarter Theorem* [8], it provides that the image of $\mathscr{U}$ under every univalent function $\mathtt{f} \in \mathscr{A}$ contains a disc of radius $1/4$. Thus, clearly, every such univalent function $\mathtt{f} \in \mathscr{A}$ has an inverse $\mathtt{f}^{-1}$ satisfying $\mathtt{f}^{-1}(\mathtt{f}(z)) = z$ and $\mathtt{f}\left(\mathtt{f}^{-1}(w)\right) = w \ \left(|w| < r_0(f),\ r_0(\mathtt{f}) \geq \frac{1}{4}\right)$, where

$$\mathtt{f}^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

We use σ to represent the collection of all such bi-univalent functions present in $\mathscr{U}$. Here, we present several examples of functions belonging to the class σ which have greatly renewed interest in the study of bi-univalent functions:

$$\mathtt{f}(z) = \frac{z}{1-z}, \quad \mathtt{g}(z) = -\log(1-z), \quad \mathtt{h}(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

with their respective inverses

$$\mathtt{f}^{-1}(w) = \frac{w}{1+w}, \quad \mathtt{g}^{-1}(w) = \frac{e^w - 1}{e^w}, \quad \mathtt{h}^{-1}(w) = \frac{e^{2w} - 1}{e^{2w} + 1}.$$

The concept was first introduced by Lewin [17] in 1967, who also provided an estimate for the second coefficient of these functions, stating that $|a_2| < 1.51$. This estimation was later refined by Brannan and Clunie [5], who proposed that $|a_2| \leq \sqrt{2}$. Over the years, the initial coefficients of bi-univalent functions have been extensively studied and estimated, contributing to a rich body of literature on the subject. Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only limited works determining the general coefficient bounds. Hence, the coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} = \{1,2,3,\cdots\})$ is still an open problem.

Table 1: Summary of findings on coefficients a_n for functions in Σ.

Researchers	**Findings**	**Estimates**
Lewin 1967	For all functions in Σ	$\lvert a_2\rvert \leq 1,51$ [17]
Brannan and Clunie 1980	For all functions in Σ	$\lvert a_2\rvert \leq \sqrt{2}$ [5]
Netanyahu 1969	Proven maximum for $\lvert a_2\rvert$	$maks\lvert a_2\rvert = \frac{4}{3}$ [19]
Tan 1984	The best estimate for $\lvert a_2\rvert$	$\lvert a_2\rvert \leq 1,485$ [29]
Various researchers	Estimates for Maclaurin coefficients $\lvert a_2\rvert$ and $\lvert a_3\rvert$	See [16,24,26,27,28,31,32,33,34,35]
Open Problem	Estimating the coefficient $\lvert a_n\rvert$	Unresolved [2,6,12,25]

These studies not only enrich the theoretical foundation of complex analysis but also have practical implications in solving problems related to differential equations, approximation theory, and other applied mathematical disciplines.

The process of determining precise bounds for $|a_3 - \kappa a_2^2|$ within any compact function family is referred to as the *Fekete-Szegö* problem. Specifically, when $\kappa = 1$, the function represents the Schwarzian derivative. The Schwarzian derivative plays a significant role in the theory of geometric functions [9].

The literature is rich with various integer number sequences, including well-known ones like Fibonacci, Lucas, Pell, and more. Recently, a new sequence known as Balancing numbers was introduced by Behera and Panda [4]. Over the past quarter-century, this sequence has been extensively studied, revealing many of its unique properties. Research has been conducted, and generalizations have certainly been made. For those interested in delving deeper into Balancing numbers, comprehensive information can be found in the references [7,10,11,13,14,15,20,21,22].

An interesting extension of Balancing numbers is the concept of Balancing polynomials. These polynomials, defined and explored in [23], represent a natural progression from Balancing numbers and exhibit some fascinating characteristics.

Definition 1. *The Balancing polynomials, for any* $\mathtt{x}$ *in the complex plane* $\mathbb{C}$, *are expressed by the recurrence relation:*

$$\mathscr{B}_n(\mathtt{x}) = 6\mathtt{x}\mathscr{B}_{n-1}(\mathtt{x}) - \mathscr{B}_{n-2}(\mathtt{x}).$$

Here, $\mathscr{B}_0(\mathtt{x}) = 0$ *and* $\mathscr{B}_1(\mathtt{x}) = 1$ *serve as the initial conditions.*

Similar to other number polynomials, Balancing polynomials can also be derived from certain generating functions. Here is one such example:

Lemma 1([3]). *The ordinary generating function for Balancing polynomials is expressed as follows:*

$$\mathscr{B}(\mathtt{x}, z) = \sum_{n=0}^{\infty} \mathscr{B}_n(\mathtt{x}) z^n = \frac{z}{1 - 6\mathtt{x}z + z^2}.$$

This paper contributes to the understanding of Balancing polynomials by offering insights into their structural properties and their implications across various values of $\mathtt{x}$ in the complex plane. We establish a novel class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k})$ and compute bounds for the initial coefficients $|a_2|$ and $|a_3|$ for functions belonging to this newly defined class. Additionally, we explore the Fekete-Szegö inequality, enhancing our comprehension of the analytic properties and bounds of Balancing polynomials and their applications in mathematical analysis.

2 Coefficient Bounds For The Function Class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k})$

In this section, we establish a novel class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k})$, and calculate the bounds for the initial coefficients $|a_2|$ and $|a_3|$ for functions that belong to $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k})$.

Definition 2. *Assume* $\mathtt{f}, \mathtt{k} \in \sigma$, $\mu \in \mathbb{C} \setminus \{0\}$, *and* $\upsilon \geq 0$. *A function* $(\mathtt{f} * \mathtt{k})(z) \in \sigma$ *is said to belong to the class* $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k})$ *if it satisfies the following conditions:*

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon} + \mu\left(\frac{z(\mathtt{f}*\mathtt{k})'(z)}{(\mathtt{f}*\mathtt{k})(z)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon} \prec \frac{\mathscr{B}(x,z)}{z} = \mathrm{I}(x,z) \tag{2}$$

and

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon} + \mu\left(\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon} \prec \frac{\mathscr{B}(\mathtt{x},w)}{w} = \mathrm{I}(x,w). \tag{3}$$

Example 1. We show that the class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f} * \mathtt{k}))$ is non-empty. We consider the function

$$\mathtt{f}(z) = z + \frac{a}{4}z^2, \quad z \in \mathscr{U},$$

where

$$a = \frac{\sqrt{2}|6\mathtt{x}|\sqrt{|6\mathtt{x}|}}{|\mathtt{k}_2|\sqrt{|36\mathtt{x}^2(\upsilon+2\mu)(\upsilon+1) - 2(36\mathtt{x}^2-1)(\mu+\upsilon)^2|}}.$$

By some simple calculation, we can obtain its inverse as follows

$$\mathtt{f}^{-1}(w) = \frac{2\left(-1+\sqrt{1+aw}\right)}{a}, \quad w \in \mathscr{U}.$$

For $x = \frac{1}{2}$, $\mu = 0$, $\upsilon = 1$ and $\mathtt{k}(z) = \frac{z}{1-z}$, we see that $\frac{(\mathtt{f}*\mathtt{k})(z)}{z} \prec \mathrm{I}(\frac{1}{2}, z)$ and $\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w} \prec \mathrm{I}(\frac{1}{2}, w)$. Figures 1 and 2 show that the transformation of $\mathscr{U}$ under $\mathtt{f}(z)$ and $\mathtt{f}^{-1}(w)$, respectively.

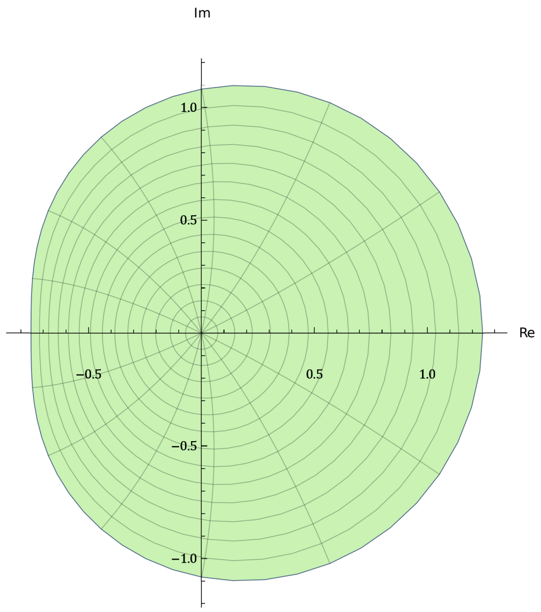

Fig. 1: The transformation of $\mathscr{U}$ under $\mathtt{f}(z)$.

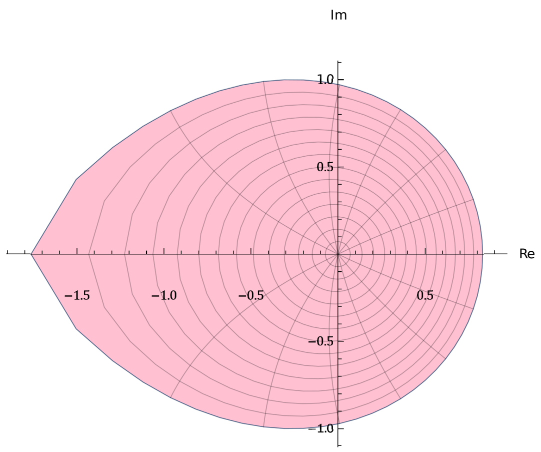

Fig. 2: The transformation of $\mathscr{U}$ under $\mathtt{f}^{-1}(w)$.

Remark.(i) For $\upsilon = 1$, we define the new class $\mathfrak{BM}^{\mathtt{x}}_{\mu,1}(\mathtt{f} * \mathtt{k})$ consisting of functions $(\mathtt{f} * \mathtt{k})(z) \in \sigma$ that satisfy

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)+\mu(\mathtt{f}*\mathtt{k})'(z) \prec \mathrm{I}(\mathtt{x},z) \quad (z \in \mathscr{U})$$

and

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)+\mu((\mathtt{f}*\mathtt{k})^{-1}(w))' \prec \mathrm{I}(\mathtt{x},w) \quad (w \in \mathscr{U}).$$

(ii) For $\mu = 1$, we get the new class $\mathfrak{B}\mathscr{M}^{\mathtt{x}}_{1,\upsilon}(\mathtt{f} * \mathtt{k})$ as follows:

$$\left(\frac{z(\mathtt{f}*\mathtt{k})'(z)}{(\mathtt{f}*\mathtt{k})(z)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon} \prec \mathrm{I}(\mathtt{x},z) \quad (z \in \mathscr{U})$$

and

$$\left(\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

(*iii*) For $\mu=1$ and $\upsilon=1$, the new class $\mathfrak{BM}^{\mathtt{x}}_{1,1}(\mathtt{f}*\mathtt{k})$ is defined as follows:

$$(\mathtt{f}*\mathtt{k})'(z)\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$((\mathtt{f}*\mathtt{k})^{-1}(w))'\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

(*iv*) For $\mu=1$ and $\upsilon=0$, the new class $\mathfrak{BM}^{\mathtt{x}}_{1,0}(\mathtt{f}*\mathtt{k})$ is defined as follows:

$$\frac{z(\mathtt{f}*\mathtt{k})'(z)}{(\mathtt{f}*\mathtt{k})(z)}\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

Remark.(v) We define the new class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f}*\frac{z}{1-z})$ as follows:

$$(1-\mu)\left(\frac{\mathtt{f}(z)}{z}\right)^{\upsilon}+\mu\left(\frac{z\mathtt{f}'(z)}{\mathtt{f}(z)}\right)\left(\frac{\mathtt{f}(z)}{z}\right)^{\upsilon}\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$(1-\mu)\left(\frac{\mathtt{f}^{-1}(w)}{w}\right)^{\upsilon}+\mu\left(\frac{w(\mathtt{f}^{-1}(w))'}{\mathtt{f}^{-1}(w)}\right)\left(\frac{\mathtt{f}^{-1}(w)}{w}\right)^{\upsilon}\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

(*vi*) For $\upsilon=1$, the new class $\mathfrak{BM}^{\mathtt{x}}_{\mu,1}(\mathtt{f}*\frac{z}{1-z})$ is defined as follows:

$$(1-\mu)\left(\frac{\mathtt{f}(z)}{z}\right)+\mu\mathtt{f}'(z)\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$(1-\mu)\left(\frac{\mathtt{f}^{-1}(w)}{w}\right)+\mu(\mathtt{f}^{-1}(w))'\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

(*vii*) For $\mu=1$ and $\upsilon=1$, the new class $\mathfrak{BM}^{\mathtt{x}}_{1,1}(\mathtt{f}*\frac{z}{1-z})$ is defined as follows:

$$\mathtt{f}'(z)\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$(\mathtt{f}^{-1}(w))'\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

(*viii*) For $\mu=1$ and $\upsilon=0$, the new class $\mathfrak{BM}^{\mathtt{x}}_{1,0}(\mathtt{f}*\frac{z}{1-z})$ is defined as follows:

$$\frac{z\mathtt{f}'(z)}{\mathtt{f}(z)}\prec \mathrm{I}(\mathtt{x},z) \quad (z\in\mathscr{U})$$

and

$$\frac{w(\mathtt{f}^{-1}(w))'}{\mathtt{f}^{-1}(w)}\prec \mathrm{I}(\mathtt{x},w) \quad (w\in\mathscr{U}).$$

This class was introduced and studied by Aktaş and Karaman [1].

Theorem 1. *Assume that* $\mathtt{f}$*, as defined in* (1)*, belongs to the class* $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f}*\mathtt{k})$*. Then*

$$|a_2| \leq \frac{\sqrt{2}|6\mathrm{x}|\sqrt{|6\mathrm{x}|}}{|\mathrm{k}_2|\sqrt{\left|36\mathrm{x}^2(\upsilon+2\mu)(\upsilon+1)-2(36\mathrm{x}^2-1)(\mu+\upsilon)^2\right|}}$$

and

$$|a_3| \leq \frac{1}{|\mathrm{k}_3|}\left(\left|\frac{6\mathrm{x}}{\upsilon+2\mu}\right|+\left|\frac{36\mathrm{x}^2}{(\mu+\upsilon)^2}\right|\right).$$

Proof. If $(\mathtt{f}*\mathtt{k}) \in \mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f}*\mathtt{k})$, then from (2) and (3), there exist Schwarz functions $l(z)$ and $m(w)$, mapping $\mathscr{U}$ to $\mathscr{U}$, such that

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}+\mu\left(\frac{z(\mathtt{f}*\mathtt{k})'(z)}{(\mathtt{f}*\mathtt{k})(z)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}=\mathrm{I}(\mathtt{x},l(z))$$

and

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}+\mu\left(\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}=\mathrm{I}(\mathtt{x},m(w)).$$

The functions $l(z)$ and $m(w)$ are defined as follows:

$$l(z)=l_1z+l_2z^2+l_3z^3+\cdots$$

and

$$m(w)=m_1w+m_2w^2+m_3w^3+\cdots.$$

These functions are analytic within $\mathscr{U}$, with $l(0)$=0 and $m(0)=0$. Additionally, the absolute values of $l(z)$ and $m(w)$ are less than 1 for all $z,w\in\mathscr{U}$. It is important to note that if the following conditions are met:

$$|l(z)|=|l_1z+l_2z^2+l_3z^3+\cdots|<1 \quad (z\in\mathscr{U})$$

and

$$|m(w)|=|m_1w+m_2w^2+m_3w^3+\cdots|<1 \quad (w\in\mathscr{U}),$$

then, the absolute values of the coefficients l_t and m_t (for $t=1,2,3,...$) are less than or equal to 1 [8]:

$$|l_t|\leq 1, \quad |m_t|\leq 1 \quad (t=1,2,3,...).$$

Since

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}+\mu\left(\frac{z((\mathtt{f}*\mathtt{k})(z))'}{(\mathtt{f}*\mathtt{k})(z)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}$$
$$=1+(\upsilon+\mu)a_2\mathrm{k}_2z+\left((\upsilon+2\mu)a_3\mathrm{k}_3+\left(\frac{(\upsilon+2\mu)(\upsilon-1)}{2}\right)a_2^2\mathrm{k}_2^2\right)z^2+\cdots$$

and

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}+\mu\left(\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}$$
$$=1-(\upsilon+\mu)a_2\mathrm{k}_2w+\left((\upsilon+2\mu)(2a_2^2\mathrm{k}_2^2-a_3\mathrm{k}_3)+\left(\frac{(\upsilon+2\mu)(\upsilon-1)}{2}\right)a_2^2\mathrm{k}_2^2\right)w^2+\cdots$$

or, equivalently

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}+\mu\left(\frac{z((\mathtt{f}*\mathtt{k})(z))'}{(\mathtt{f}*\mathtt{k})(z)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})(z)}{z}\right)^{\upsilon}=\mathscr{B}_1(\mathtt{x})l_1z+\left(\mathscr{B}_1(\mathtt{x})l_2+\mathscr{B}_2(\mathtt{x})l_1^2\right)z^2+\cdots$$

and

$$(1-\mu)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}+\mu\left(\frac{w((\mathtt{f}*\mathtt{k})^{-1}(w))'}{(\mathtt{f}*\mathtt{k})^{-1}(w)}\right)\left(\frac{(\mathtt{f}*\mathtt{k})^{-1}(w)}{w}\right)^{\upsilon}=\mathscr{B}_1(\mathtt{x})m_1w+\left(\mathscr{B}_1(\mathtt{x})m_2+\mathscr{B}_2(\mathtt{x})m_1^2\right)w^2+\cdots,$$

we get following equations

$$(\upsilon+\mu)a_2\mathtt{k}_2=\mathscr{B}_2(\mathtt{x})l_1, \tag{4}$$

$$(\upsilon+2\mu)a_3\mathtt{k}_3+\frac{(\upsilon+2\mu)(\upsilon-1)}{2}a_2^2\mathtt{k}_2^2=\mathscr{B}_2(\mathtt{x})l_2+\mathscr{B}_3(\mathtt{x})l_1^2, \tag{5}$$

$$-(\upsilon+\mu)a_2\mathtt{k}_2=\mathscr{B}_2(\mathtt{x})m_1 \tag{6}$$

and

$$(\upsilon+2\mu)(2a_2^2\mathtt{k}_2^2-a_3\mathtt{k}_3)+\frac{(\upsilon+2\mu)(\upsilon-1)}{2}a_2^2\mathtt{k}_2^2=\mathscr{B}_2(\mathtt{x})m_2+\mathscr{B}_3(\mathtt{x})m_1^2. \tag{7}$$

By adding equations (4) and (6), we obtain the following result

$$l_1=-m_1. \tag{8}$$

Additionally, squaring and adding (4) and (6), we derive

$$2(\mu+\upsilon)^2\mathtt{k}_2^2a_2^2=\mathscr{B}_2^2(\mathtt{x})(l_1^2+m_1^2). \tag{9}$$

The addition of (5) and (7) yields

$$(\upsilon+2\mu)(\upsilon+1)a_2^2\mathtt{k}_2^2=\mathscr{B}_2(\mathtt{x})(l_2+m_2)+\mathscr{B}_3(\mathtt{x})(l_1^2+m_1^2).$$

From these equations, we can deduce

$$\left[\mathscr{B}_2^2(\mathtt{x})(\upsilon+2\mu)(\upsilon+1)-2\mathscr{B}_3(\mathtt{x})(\mu+\upsilon)^2\right]a_2^2\mathtt{k}_2^2=\mathscr{B}_2^3(\mathtt{x})(l_2+m_2). \tag{10}$$

A minor computation leads to the following upper bound for $|a_2|$:

$$|a_2|\le\frac{\sqrt{2}|\mathscr{B}_2(\mathtt{x})|^{\frac{3}{2}}}{|\mathtt{k}_2|\sqrt{\left|\mathscr{B}_2^2(\mathtt{x})(\upsilon+2\mu)(\upsilon+1)-2\mathscr{B}_3(\mathtt{x})(\mu+\upsilon)^2\right|}}.$$

To find the bound for $|a_3|$, we subtract equation (7) from (5), resulting in

$$2a_3\mathtt{k}_3(\upsilon+2\mu)-2(\upsilon+2\mu)a_2^2\mathtt{k}_2^2=\mathscr{B}_2(\mathtt{x})(l_2-m_2)+\mathscr{B}_3(\mathtt{x})(l_1^2-m_1^2). \tag{11}$$

By substituting equations (8) and (9) into (11), we obtain

$$a_3\mathtt{k}_3=\frac{\mathscr{B}_2(\mathtt{x})(l_2-m_2)}{2(\upsilon+2\mu)}+\frac{\mathscr{B}_2^2(\mathtt{x})(l_1^2+m_1^2)}{2(\mu+\upsilon)^2}.$$

After applying the value of $\mathscr{B}_2(\mathtt{x})$ and taking the modulus, we get the desired bound for $|a_3|$:

$$|a_3|\le\frac{1}{|\mathtt{k}_3|}\left(\left|\frac{6\mathtt{x}}{\upsilon+2\mu}\right|+\left|\frac{36\mathtt{x}^2}{(\mu+\upsilon)^2}\right|\right).$$

Corollary 1.*If* $\mathtt{f}$*, given by* (1)*, is in* $\mathfrak{BM}_{\mu,1}^{\mathtt{x}}(\mathtt{f}*\mathtt{k})$*, then*

$$|a_2|\le\frac{|6\mathtt{x}|\sqrt{|6\mathtt{x}|}}{|\mathtt{k}_2|\sqrt{|(36\mathtt{x}^2-1)\mu^2-(2\mu+1)|}}$$

and

$$|a_3|\le\frac{1}{|\mathtt{k}_3|}\left(\left|\frac{6\mathtt{x}}{1+2\mu}\right|+\left|\frac{36\mathtt{x}^2}{(1+\mu)^2}\right|\right).$$

Corollary 2. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,\upsilon}(\mathtt{f}*\mathtt{k})$, *then*

$$|a_2| \le \frac{|6\mathrm{x}|\sqrt{|6\mathrm{x}|}}{|\mathtt{k}_2|\sqrt{|(18\mathrm{x}^2-1)\upsilon^2+(9\mathrm{x}^2-1)2\upsilon-1|}}$$

and

$$|a_3| \le \frac{1}{|\mathtt{k}_3|}\left(\left|\frac{6\mathrm{x}}{\upsilon+2}\right|+\left|\frac{36\mathrm{x}^2}{(1+\upsilon)^2}\right|\right).$$

Corollary 3. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,1}(\mathtt{f}*\mathtt{k})$, *then*

$$|a_2| \le \frac{|3\mathrm{x}|\sqrt{|6\mathrm{x}|}}{|\mathtt{k}_2|\sqrt{|9\mathrm{x}^2-1|}}$$

and

$$|a_3| \le \frac{1}{|\mathtt{k}_3|}\left(2|\mathrm{x}|+9|\mathrm{x}^2|\right).$$

Corollary 4. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,0}(\mathtt{f}*\mathtt{k})$, *then*

$$|a_2| \le \frac{1}{|\mathtt{k}_2|}\left(|6\mathrm{x}|\sqrt{|6\mathrm{x}|}\right)$$

and

$$|a_3| \le \frac{1}{|\mathtt{k}_3|}\left(3|\mathrm{x}|+36|\mathrm{x}^2|\right).$$

Corollary 5. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{\mu,\upsilon}(\mathtt{f}*\frac{z}{1-z})$, *then*

$$|a_2| \le \frac{\sqrt{2}|6\mathrm{x}|\sqrt{|6\mathrm{x}|}}{\sqrt{\left|36\mathrm{x}^2(\upsilon+2\mu)(\upsilon+1)-2(36\mathrm{x}^2-1)(\mu+\upsilon)^2\right|}}$$

and

$$|a_3| \le \left|\frac{6\mathrm{x}}{\upsilon+2\mu}\right|+\left|\frac{36\mathrm{x}^2}{(\mu+\upsilon)^2}\right|.$$

Corollary 6. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{\mu,1}(\mathtt{f}*\frac{z}{1-z})$, *then*

$$|a_2| \le \frac{|6\mathrm{x}|\sqrt{|6\mathrm{x}|}}{\sqrt{|(36\mathrm{x}^2-1)\mu^2-(2\mu+1)|}}$$

and

$$|a_3| \le \left|\frac{6\mathrm{x}}{1+2\mu}\right|+\left|\frac{36\mathrm{x}^2}{(1+\mu)^2}\right|.$$

Corollary 7. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,\upsilon}(\mathtt{f}*\frac{z}{1-z})$, *then*

$$|a_2| \le \frac{|6\mathrm{x}|\sqrt{|6\mathrm{x}|}}{\sqrt{|(18\mathrm{x}^2-1)\upsilon^2+(9\mathrm{x}^2-1)2\upsilon-1|}}$$

and

$$|a_3| \le \left|\frac{6\mathrm{x}}{\upsilon+2}\right|+\left|\frac{36\mathrm{x}^2}{(1+\upsilon)^2}\right|.$$

Corollary 8. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,0}(\mathtt{f}*\frac{z}{1-z})$, *then*

$$|a_2| \le |6\mathrm{x}|\sqrt{|6\mathrm{x}|}$$

and

$$|a_3| \le 3|\mathrm{x}|+36|\mathrm{x}^2|.$$

3 Fekete-Szegö Inequalities For The Function Class $\mathfrak{BM}^{x}_{\mu,\upsilon}(\mathtt{f}*\mathtt{k})$

In this section, we will be determining the Fekete-Szegö inequality, denoted as $|a_3-\kappa a_2^2|$, where $\kappa\in\mathtt{R}$, for functions that belong to the class $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f}*\mathtt{k})$.

Theorem 2. *If the function* $\mathtt{f}$*, as defined in* (1)*, belongs to the class* $\mathfrak{BM}^{\mathtt{x}}_{\mu,\upsilon}(\mathtt{f}*\mathtt{k})$*, then the following inequality holds*

$$|a_3-\kappa a_2^2|\le\begin{cases}\left|\frac{6\mathtt{x}}{\upsilon+2\mu}\right|, & |\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|\le\frac{1}{2|\mathtt{k}_3(\upsilon+2\mu)|}\\ |12\mathtt{x}\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|, & |\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|\ge\frac{1}{2|\mathtt{k}_3(\upsilon+2\mu)|}\end{cases},$$

where

$$\Psi(\mu,\upsilon,\mathtt{k};\mathtt{x})=\frac{36\mathtt{x}^2\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)}{\mathtt{k}_2^2\left(36\mathtt{x}^2(\upsilon+2\mu)(\upsilon+1)-2(36\mathtt{x}^2-1)(\mu+\upsilon)^2\right)}.$$

Proof. For $\kappa\in\mathtt{R}$, it follows from (8) and (11) that

$$a_3-\kappa a_2^2=\frac{1}{\mathtt{k}_3}\frac{\mathscr{B}_2(\mathtt{x})(l_2-m_2)}{2(\upsilon+2\mu)}+\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)a_2^2.$$

By using (10) in the above equality, we obtain

$$\begin{aligned}a_3-\kappa a_2^2&=\mathscr{B}_2(\mathtt{x})\left[\frac{l_2-m_2}{2\mathtt{k}_3(\upsilon+2\mu)}+\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)\frac{\mathscr{B}_2^2(\mathtt{x})(l_2+m_2)}{\mathtt{k}_2^2(\mathscr{B}_2^2(\mathtt{x})(\upsilon+2\mu)(\mu+1)-2\mathscr{B}_3(\mathtt{x})(\mu+\upsilon)^2)}\right]\\&=\mathscr{B}_2(\mathtt{x})\left[\left(\frac{1}{2\mathtt{k}_3(\upsilon+2\mu)}+\Psi(\mu,\upsilon,k;\mathtt{x})\right)l_2+\left(\frac{-1}{2\mathtt{k}_3(\upsilon+2\mu)}+\Psi(\mu,\upsilon,k;\mathtt{x})\right)m_2\right],\end{aligned}$$

where

$$\Psi(\mu,\upsilon,\mathtt{k};\mathtt{x})=\frac{\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)\mathscr{B}_2^2(\mathtt{k})}{\mathtt{k}_2^2\left(\mathscr{B}_2^2(\mathtt{x})(\upsilon+2\mu)(\upsilon+1)-2\mathscr{B}_3(\mathtt{x})(\mu+\upsilon)^2\right)}.$$

Therefore, we have

$$|a_3-\kappa a_2^2|\le\begin{cases}\left|\frac{6\mathtt{x}}{\upsilon+2\mu}\right|, & |\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|\le\frac{1}{2|\mathtt{k}_3(\upsilon+2\mu)|}\\ |12\mathtt{x}\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|, & |\Psi(\mu,\upsilon,\mathtt{k},\mathtt{x})|\ge\frac{1}{2|\mathtt{k}_3(\upsilon+2\mu)|}\end{cases}.$$

Corollary 9. *If* $\mathtt{f}$*, given by* (1)*, is in* $\mathfrak{BM}^{\mathtt{x}}_{\mu,1}(\mathtt{f}*\mathtt{k})$ *and* $\kappa\in\mathtt{R}$*. Then*

$$|a_3-\kappa a_2^2|\le\begin{cases}\left|\frac{6\mathtt{x}}{1+2\mu}\right|, & |\Psi(\mu,1,\mathtt{k};\mathtt{x})|\le\frac{1}{2|\mathtt{k}_3(1+2\mu)|}\\ |12\mathtt{x}\Psi(\mu,1,\mathtt{k},\mathtt{x})|, & |\Psi(\mu,1,\mathtt{k};\mathtt{x})|\ge\frac{1}{2|\mathtt{k}_3(1+2\mu)|}\end{cases},$$

where

$$\Psi(\mu,1,\mathtt{k};\mathtt{x})=\frac{36\mathtt{x}^2\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)}{\mathtt{k}_2^2\left(36\mathtt{x}^2(1+2\mu)(\upsilon+1)-2(36\mathtt{x}^2-1)(\mu+1)^2\right)}.$$

Corollary 10. *If* $\mathtt{f}$*, given by* (1)*, is in* $\mathfrak{BM}^{\mathtt{x}}_{1,\upsilon}(\mathtt{f}*\mathtt{k})$ *and* $\kappa\in\mathtt{R}$*, then*

$$|a_3-\kappa a_2^2|\le\begin{cases}\left|\frac{6\mathtt{x}}{\upsilon+2}\right|, & |\Psi(1,\upsilon,\mathtt{k},\mathtt{x})|\le\frac{1}{2|\mathtt{k}_3(\upsilon+2)|}\\ |12\mathtt{x}\Psi(1,\upsilon,\mathtt{k},\mathtt{x})|, & |\Psi(1,\upsilon,\mathtt{k},\mathtt{x})|\ge\frac{1}{2|\mathtt{k}_3(\upsilon+2)|}\end{cases},$$

where

$$\Psi(1,\upsilon,\mathtt{k};\mathtt{x})=\frac{36\mathtt{x}^2\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3}-\kappa\right)}{\mathtt{k}_2^2\left(36\mathtt{x}^2(\upsilon+2)(\upsilon+1)-2(36\mathtt{x}^2-1)(1+\upsilon)^2\right)}.$$

Corollary 11. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,1}(\mathtt{f}*\mathtt{k})$ *and* $\kappa \in \mathrm{R}$, *then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} 2\,|\mathrm{x}|, & |\Psi(1,1,\mathtt{k},\mathrm{x})| \le \frac{1}{6|\mathtt{k}_3|} \\ |12\mathrm{x}\Psi(1,1,\mathtt{k},\mathrm{x})|, & |\Psi(1,1,\mathtt{k},\mathrm{x})| \ge \frac{1}{6|\mathtt{k}_3|} \end{cases},$$

where

$$\Psi(1,1,\mathtt{k};\mathrm{x}) = \frac{9\mathrm{x}^2\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3} - \kappa\right)}{2\mathtt{k}_2^2(1-9\mathrm{x}^2)}.$$

Corollary 12. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,0}(\mathtt{f}*\mathtt{k})$ *and* $\kappa \in \mathrm{R}$, *then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} 3\,|\mathrm{x}|, & |\Psi(1,0,\mathtt{k},\mathrm{x})| \le \frac{1}{4|\mathtt{k}_3|} \\ |12\mathrm{x}\Psi(1,0,\mathtt{k},\mathrm{x})|, & |\Psi(1,0,\mathtt{k},\mathrm{x})| \ge \frac{1}{4|\mathtt{k}_3|} \end{cases},$$

where

$$\Psi(1,0,\mathtt{k};\mathrm{x}) = \frac{18\mathrm{x}^2\left(\frac{\mathtt{k}_2^2}{\mathtt{k}_3} - \kappa\right)}{\mathtt{k}_2^2}.$$

Corollary 13. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{\mu,\upsilon}(\mathtt{f}*\frac{z}{1-z})$ *and* $\kappa \in \mathrm{R}$, *then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} \left|\frac{6\mathrm{x}}{\upsilon+2\mu}\right|, & |\Psi(\mu,\upsilon,\mathrm{x})| \le \frac{1}{2|\upsilon+2\mu|} \\ |12\mathrm{x}\Psi(\mu,\upsilon,\mathrm{x})|, & |\Psi(\mu,\upsilon,\mathrm{x})| \ge \frac{1}{2|\upsilon+2\mu|} \end{cases},$$

where

$$\Psi(\mu,\upsilon;\mathrm{x}) = \frac{36\mathrm{x}^2(1-\kappa)}{36\mathrm{x}^2(\upsilon+2\mu)(\upsilon+1) - 2(36\mathrm{x}^2-1)(\mu+\upsilon)^2}.$$

Corollary 14. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{\mu,1}(\mathtt{f}*\frac{z}{1-z})$ *and* $\kappa \in \mathrm{R}$, *then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} \left|\frac{6\mathrm{x}}{1+2\mu}\right|, & |\Psi(\mu,1,\mathrm{x})| \le \frac{1}{2|1+2\mu|} \\ |12\mathrm{x}\Psi(\mu,1,\mathrm{x})|, & |\Psi(\mu,1,\mathrm{x})| \ge \frac{1}{2|1+2\mu|} \end{cases},$$

where

$$\Psi(\mu,1,\mathrm{x}) = \frac{36\mathrm{x}^2(1-\kappa)}{36\mathrm{x}^2(1+2\mu)(\upsilon+1) - 2(36\mathrm{x}^2-1)(\mu+1)^2}.$$

Corollary 15. *If* $\mathtt{f}$, *given by* (1), *is in* $\mathfrak{BM}^{\mathrm{x}}_{1,\upsilon}(\mathtt{f}*\frac{z}{1-z})$ *and* $\kappa \in \mathrm{R}$, *then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} \left|\frac{6\mathrm{x}}{\upsilon+2}\right|, & |\Psi(1,\upsilon,\mathrm{x})| \le \frac{1}{2|(\upsilon+2)|} \\ |12\mathrm{x}\Psi(1,\upsilon,\mathrm{x})|, & |\Psi(1,\upsilon,\mathrm{x})| \ge \frac{1}{2|(\upsilon+2)|} \end{cases},$$

where

$$\Psi(1,\upsilon,\mathtt{k};\mathrm{x}) = \frac{36\mathrm{x}^2(1-\kappa)}{36\mathrm{x}^2(\upsilon+2)(\upsilon+1) - 2(36\mathrm{x}^2-1)(1+\upsilon)^2}.$$

Corollary 16. *If* f*, given by* (1)*, is in* $\mathfrak{BM}^{\mathrm{x}}_{1,0}(\mathtt{f}*\mathtt{k})$ *and* $\kappa \in \mathbb{R}$*, then*

$$|a_3-\kappa a_2^2| \le \begin{cases} 3\,|\mathrm{x}|, & |\Psi(1,0,\mathrm{x})| \le \frac{1}{4} \\ |12\mathrm{x}\Psi(1,0,\mathrm{x})|, & |\Psi(1,0,\mathrm{x})| \ge \frac{1}{4} \end{cases},$$

where

$$\Psi(1,0,\mathrm{x}) = 18\mathrm{x}^2\big(1-\kappa\big).$$

4 Conclusion

In this study, we have introduced and thoroughly investigated a novel subclass of bi-univalent functions known as Bazilevič functions, which are constructed through the convolution of Balancing polynomials. Our primary objective was to establish rigorous bounds for the Fekete-Szegö inequality and to compute the initial coefficients $|a_2|$ and $|a_3|$ for functions belonging to this subclass.

Furthermore, the corollaries derived from our analysis offer additional insights and extensions, underscoring the versatility and applicability of Bazilevič functions in mathematical research. These results pave the way for further exploration into the deeper implications and potential applications of Bazilevič functions across different parameters.

In conclusion, our study contributes to the advancement of mathematical analysis by establishing a solid theoretical foundation for Bazilevič functions and by providing concrete computational results. Future research can build upon these findings to explore more intricate aspects of Bazilevič functions and to investigate their role in solving complex mathematical problems.

Declarations

Competing interests: The authors declare that they have no competing interests.
Authors' contributions: Conceptualization, S.G., B.S., B.V and S.A.; methodology, S.G., B.S., B.V and S.A.; software, S.G., B.S and S.A..; validation, S.G., B.S., B.V and S.A..; formal analysis, S.G., B.S., B.V and S.A..; investigation, S.G., B.S. and S.A.; resources, S.G., B.S., B.V and S.A.; writingoriginal draft preparation, S.G., B.S., B.V and S.A.; writingreview and editing, S.G., B.S., B.V and S.A.; visualization, S.G., B.S., B.V and S.A.; supervision, S.A.; All authors have read and agreed to the published version of the manuscript.
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