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Hopf Bifurcation Analysis in the Forest Pest System

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Abstract: In this paper, we analyze dynamical behaviors of the non-even-aged forests affected by insect pest system. This system is described by a cubic system of three ordinary nonlinear differential equations with five real parameters. We confirm that the forest pest system displays local Hopf bifurcations under certain conditions. Moreover, we show that a Hopf bifurcation occurs at four equilibrium points for the system. Also, we obtain sufficient conditions for supercritical and subcritical bifurcations via the normal form theory. More precisely, we show that the forest pest system admits limit cycles. Numerical examples are given to validate the theoretical analysis

Keywords: Stability equilibrium point; Hopf bifurcation; limit cycles; Forest pest system.

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1 Introduction

Characterizing whether periodic solutions exist or not is one of the most basic problems in the qualitative theory of autonomous differential systems depending on parameters [12, 8]. Normally, there are no ways to study the periodic solution in a three-dimensional system. A bifurcation of a system is a qualitative change in its dynamical behaviors yielded by varying parameters. The bifurcation can be divided into a local bifurcation and a global bifurcation. The Hopf bifurcation is a kind of typical local bifurcation. It is the birth of a limit cycle from an equilibrium point for the system. The Hopf bifurcation Theorem [16, 19, 21] provides the simplest criterion for a family of periodic solutions to bifurcate from a known family of equilibrium solutions of a dynamical system. Then, there exists a family of periodic solutions bifurcating at Hopf bifurcation. However, the aim of this paper is to use Hopf bifurcation and normal form theory to investigate this phenomenon.
The simplest mathematical models of non-even-aged forests affected by insect pests have been proposed by Antonovsky et al. in [1, 2]. In dimensionless and performing some simplifications by a linear change of variables, parameters, and time, the forest pest system takes the form

$$\begin{aligned} \dot{x} &= by-(y-1)^2x-ax-xz,\\ \dot{y} &= x-dy,\\ \dot{z} &= -ez+cxz, \end{aligned} \tag{1}$$

where x, y are densities of old and young trees, z is insect density and the terms with xz and yz represent the insect forest interaction, with a, b, c, d and e are real parameters. The model is studied and analyzed in [2]. Moreover, in [2, 3] the authors used analytical methods such as bifurcation theory and numerical methods to study qualitative behaviors and dynamics of a nonlinear forest pest system. In [23] the authors analyzed the one-parameter transcritical bifurcation diagram for the forest pest system 1. Some detailed investigations of the forest pest system have been carried out in references [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 15, 18, 19, 20].
Since systems 1 in general cannot be solved explicitly, the qualitative information provided by the theory of dynamical systems is the best that one can expect to obtain in general. There is no general analytical approach to finding an analytical

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solution. The stability of equilibrium points and the bifurcation theory such as the Hopf bifurcation of dynamical systems play quite important roles in studying the dynamics of many differential systems. One of the most classical problems of the qualitative theory of polynomial differential systems depending on parameters is characterizing the existence or not of limit cycles.
A Hopf bifurcation is a local bifurcation in which an equilibrium point of the dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point crosses the imaginary axis of the complex plane. To find the limit cycle (isolated closed orbits), we will use the Hopf bifurcation theorem [16,21] of the forest pest model applying the normal form theory introduced by Hassard et al. [10].
The remainder of this paper is organized as follows. In Section 2, we study the local stability of equilibrium points and the existence of local Hopf bifurcations. Using the normal form theory, the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions are analyzed in detail in Section 3. Finally, in Section 4, numerical examples are presented to illustrate the main results.

2 Stability equilibrium points and Hopf bifurcations analysis

The Hopf bifurcation and the analysis of the equilibrium points of a forest pest model are the topics of this section. The local stability of every equilibrium point is investigated by analyzing the characteristic equation. After that, the Hopf bifurcation around the equilibrium points is found. By simple analysis, it is easy to obtain the following statements for the system 1.

I) The system has only two equilibrium points, which are the origin and

$$E_1 = \left(\frac{e}{c}, \frac{e}{dc}, \frac{-(ad^2c^2 - bdc^2 + d^2c^2 - 2dec + e^2)}{c^2d^2}\right), \quad \text{where } d, c \neq 0.$$

II) If $\Gamma > 0$ and $d \neq 0$, then the system has two other equilibrium points

$$E_{2,3} = \left(d \mp \Gamma, 1 \mp \frac{\Gamma}{d}, 0\right), \quad \text{where } \Gamma = \sqrt{bd - ad^2}.$$

III) If $b = d = 0$ and $e \neq 0$, the system has infinitely many equilibrium point $E_4 = (0, y, 0)$.
IX) If $b = d = 0$ and $e = 0$, the system has infinitely many equilibrium point $E_5 = (0, y, z)$.

Note that the Hopf bifurcation does not occur at the equilibrium points E_4 and E_5 of system 1, because the eigenvalues of the Jacobian matrix of system 1 at E_4 and E_5 are $(0, -e, -y^2 - a + 2y - 1)$ and $(0, 0, -y^2 - a + 2y - 1 - z)$, respectively.

2.1 Hopf bifurcation analysis at $E_0(0,0,0)$

The Jacobian matrix for system 1 at $E_0(0,0,0)$ is given by

$$J_{E_0} = \begin{bmatrix} -(a+1) & b & 0 \\ 1 & -d & 0 \\ 0 & 0 & -e \end{bmatrix}.$$

The characteristic equation of the matrix J_{E_0} is

$$|\lambda I - J_{E_0}| = \lambda^3 + T_1\lambda^2 + K_1\lambda + D_1 = 0, \tag{2}$$

where $T_1 = a + d + e + 1$, $K_1 = ((a + e + 1)d + (a + 1)e - b)$, $D_1 = e((a + 1)d - b)$. Therefore, the eigenvalues of the Jacobian matrix J_{E_0} are $\lambda_{1,2} = \frac{-(a+d+1) \mp \sqrt{(a-d+1)^2 + 4b}}{2}$ and $\lambda_3 = -e$.

Proposition 1. *The equilibrium point $E_0(0,0,0)$ is a Hopf bifurcation if and only if $a = a_{h1} = -(d+1)$, $d^2 + b < 0$ and $e \neq 0$, which satisfy $\mathrm{Re}(\lambda'(a_{h1})) \neq 0$, i.e., the system 1 undergoes a Hopf bifurcation.*

*Proof.*The origin is a Hopf bifurcation point if and only if $T_1K_1 + D_1 = 0, K_1 < 0$ and $T_1 \neq 0$, since $T_1K_1 + D_1 = (a+d+1)((a+e+1)d + (a+1)e + e^2 - b) = 0$, this implies that $a = a_{h1} = -(d+1)$, then equation 2 can be rewritten into

$$(\lambda + e)(-\lambda^2 + d^2 + b) = 0, \tag{3}$$

Obviously, equation 3 has a pair of purely imaginary conjugate roots, and a real root $\lambda_3 = -e \neq 0$, where $\omega = \sqrt{-d^2 - b}, (d^2 + b < 0)$.
Let $\lambda = \lambda(a)$, define the relation from the characteristic equation 2

$$f(\lambda(a), a) = \lambda(a)^3 + T_1\lambda(a)^2 + K_1\lambda(a) + D_1 = 0, \tag{4}$$

Differentiation of equation 4 with respect to a yields, $\frac{\partial f}{\partial \lambda}\frac{d\lambda}{da} + \frac{\partial f}{\partial a} = 0$, implies

$$\frac{d\lambda(a)}{da} = -\frac{\partial f}{\partial a}\left(\frac{\partial f}{\partial \lambda}\right)^{-1} = -\frac{\lambda^2 + (e+d)\lambda + ed}{3\lambda^2 + 2(a+d+e+1)\lambda + a(d+e) + de + d + e - b}. \tag{5}$$

Taking the root $\lambda(a) = \lambda(a_{h1}) = i\omega$, evaluating $a = a_{h1}$, and substituting it into equation 5, we have

$$\alpha'(0) = \left(\frac{d\,\mathrm{Re}(\lambda(a))}{da}\right)_{a=a_{h1}} = -\frac{1}{2} \neq 0, \tag{6}$$

and

$$\omega'(0) = \left(\frac{d\,\mathrm{Im}(\lambda(a))}{da}\right)_{a=a_{h1}} = \frac{d}{2\omega}. \tag{7}$$

Obviously, the first and second conditions for the Hopf bifurcation Theorem hold. Hence, by [8], we know that the the system 1 undergoes a Hopf bifurcation equilibrium point at $E_0(0,0,0)$ when $a = a_{h1}$.

2.2 Hopf bifurcation analysis at E_1

Now, we shift the equilibrium point E_1 of the system 1 to the origin under the following linear transformation $x_1 = x - \frac{e}{c}$, $y_1 = y - \frac{e}{dc}$ and $z_1 = z + \frac{ad^2c^2 - bdc^2 + d^2c^2 - 2dec + e^2}{c^2d^2}$, which transforms system 1 into the following:

$$\begin{aligned} \dot{x}_1 &= -x_1y_1^2 + by_1 - \frac{e}{c}y_1^2 + \frac{2(dc-e)}{dc}x_1y_1 - x_1z_1 - \frac{b}{d}x_1 + \frac{2e}{c}y_1 - \frac{e}{c}z_1 - \frac{2e^2}{dc^2}y_1, \\ \dot{y}_1 &= x_1 - dy_1, \\ \dot{z}_1 &= -\frac{ad^2c^2 - bdc^2 + d^2c^2 - 2dec + e^2}{cd^2}x_1 + cx_1z_1. \end{aligned} \tag{8}$$

Then the stability analysis reduces to that of the equilibrium point $E_1(0,0,0)$ of system 8. The Jacobian matrix at the $E_1(0,0,0)$ is

$$J_{E_1} = \begin{bmatrix} -\frac{b}{d} & b + \frac{2e}{c}(1 - \frac{e}{dc}) & -\frac{e}{c} \\ 1 & -d & 0 \\ -\frac{ad^2c^2 - bdc^2 + d^2c^2 - 2dec + e^2}{cd^2} & 0 & 0 \end{bmatrix}.$$

Obviously, the following characteristic equation of J_{E_1} is

$$|\lambda I - J_{E_1}| = \lambda^3 + T_2\lambda^2 + K_2\lambda + D_2 = 0, \tag{9}$$

where $T_2 = \frac{(d^2+b)}{d}, K_2 = \frac{(-e(ad^2c^2 - bdc^2 + d^2c^2 + 2d^2c - 2dec - 2de + e^2))}{(d^2c^2)}, D_2 = \frac{(-e(ad^2c^2 - bdr^2 + d^2c^2 - 2dec + e^2))}{(dc^2)}$.

Proposition 2.*The equilibrium point E_1 in the system 8 is a Hopf bifurcation if and only if $a = a_{h2} = \frac{(b^2dc^2 - bd^2c^2 - 2d^4c - 2bd^2c + 2bdec + 2d^3e + 2bde - be^2)}{(d^2c^(2)b)}, \frac{(2ed(dc-e)}{(bc^2)} > 0)$ and $-\frac{(d^2+b)}{d}, b, c, d, e \neq 0$. Also, in the equation 9, which satisfies $(\frac{dRe(\lambda(a)}{da})_{a=a_{h2}} \neq 0$, then the forest pest system 1 displays a Hopf bifurcation.*

Proof. The equilibrium point E_1 is a Hopf bifurcation point if and only if $T_2K_2+D_2=0, K_2<0$ and $T_2\neq 0$, since $T_2K_2+D_2=\frac{-e(abd^2c^2-b^2c^2d-d^2c^2b+2d^4c+2bd^2c-2bdec-2d^3e-2bde+be^2)}{c^2d^3}=0$, and $e\neq 0$ by $K_2<0$, then must be $a=a_{h2}=\frac{(b^2dc^2-bd^2c^2-2d^4c-2bd^2c+2bdec+2d^3e+2bde-be^2)}{(d^2c^2b)}$, so equation 9 becomes

$$\left(\lambda+\frac{(d^2+b)}{d}\right)\left(\lambda^2+\frac{2ed(dc-e)}{bc^2}\right)=0. \tag{10}$$

Equation 10 has a pair of purely imaginary conjugate roots $\lambda_{1,2}=\mp i\omega$, where $\omega=\sqrt{\frac{2ed(dc-e)}{bc^2}}$ and a real root $\lambda_3=-\frac{d^2+b}{d}$. Suppose that $\lambda=\lambda(a)$, define the relation from the characteristic equation 9

$$f(\lambda(a),a)=\lambda(a)^3+T_2\lambda(a)^2+K_2\lambda(a)+D_2=0. \tag{11}$$

Differentiation of 11 with respect to a yields $\frac{\partial f}{\partial\lambda}\frac{d\lambda}{da}+\frac{\partial f}{\partial a}=0$, implies

$$\frac{d\lambda(a)}{da}=-\frac{\partial f}{\partial a}\left(\frac{\partial f}{\partial\lambda}\right)^{-1}=-\frac{\lambda d^2c^2e+d^3c^2e}{3\lambda^2d^2c^2+(2bdc^2+2d^3c^2)\lambda+dec(2e+bc-dc-2d-adc)+2de^2-e^3}. \tag{12}$$

Taking the root $\lambda(a)=\lambda(a_{h2})=i\omega$, evaluating $a=a_{h2}$, and substituting it into 12, we have

$$\alpha'(0)=\left(\frac{d\,\mathrm{Re}(\lambda(a))}{da}\right)_{a=a_{h2}}=-\frac{b^2c^2de}{2b(d^2+b)^2c^2+4d^4ec-4d^3e^2}\neq 0, \tag{13}$$

and

$$\omega'(0)=\left(\frac{d\,\mathrm{Im}(\lambda(a))}{da}\right)_{a=a_{h2}}=-\frac{1}{4}\frac{\sqrt{(2bde(dc-e))}dc(bd^2c^2+b^2c^2+2d^2ec-2de^2)}{(b(d^2+b)^2c^2+2d^4ec-2d^3e^2)(dc-e)}. \tag{14}$$

So, first and second conditions for Hopf bifurcation are met and the Hopf bifurcation Theorem holds. Hence, by [8], we know that the system 8 undergoes a Hopf bifurcation at E_1 when $a=a_{h2}$.

2.3 Hopf bifurcation analysis at $E_{2,3}$

First translation of the equilibrium point E_2 to the origin by $x_1=x-d+\Gamma, y_1=y-1+\frac{\Gamma}{d}, z_1=z$, which transforms system 1 into the following:

$$\begin{aligned}\dot{x}&=b(y_1+(1+\Gamma/d))-S,\\ \dot{y}&=x_1-dy_1,\\ \dot{z}&=c(x_1+(d+\Gamma))z_1-ez_1,\end{aligned} \tag{15}$$

where $S=(y_1+\frac{\Gamma}{d})^2(x_1+(d+\Gamma))+a(x_1+(d+\Gamma))+(x_1+(d+\Gamma))z_1$.
The Jacobian matrix for system 15 at $E_2(0,0,0)$ is

$$J_{E_2}=\begin{bmatrix}-\frac{b}{d} & 2ad-b-2\Gamma & -(d+\Gamma)\\ 1 & -d & 0\\ 0 & 0 & cd-e+c\Gamma\end{bmatrix},$$

and its corresponding characteristic equation of $|\lambda I-J_{E_0}|=0$ is

$$\lambda^3+T_3\lambda^2+K_3\lambda+D_3=0, \tag{16}$$

where $T_3=-c(d+\Gamma)+d+e+\frac{b}{d}$, $K_3=-dc(d+\Gamma)-bc(1+\frac{\Gamma}{d})-2ad+de+2\Gamma+2b+\frac{be}{d}$, and $D_3=2(c(d+\Gamma)-e)(-ad+\Gamma+b)$. The eigenvalues of the Jacobian matrix of system 15 at E_2 are $\lambda_{1,2}=\frac{(-(d+\frac{b}{d})\mp\sqrt{(8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2})})}{2}$ and $\lambda_3=cd-e+c\Gamma$.

Next, we will use the system 15 at the equilibrium point E_2 to apply the Hopf bifurcation Theorem.

Proposition 3. *The origin of system 15 is a Hopf bifurcation if and only if $b = b_{h3} = -d^2$. Moreover, which satisfy $Re(\lambda(b_{h3})) \neq 0$ in equation 16, i.e. system 15 undergoes a Hopf bifurcation.*

Proof. When $b = b_{h3}$, equation 16 at a point E_2 can be rewritten into

$$(\lambda - (c\Lambda + dc - e))\left(\lambda^2 + \sqrt{-2ad - 2d^2 + 2\Lambda}\right) = 0, \tag{17}$$

when $\Lambda = \sqrt{-ad^2 - d^3}$. Obviously, equation 17 has a pair of purely imaginary conjugate roots and a real root $\lambda_3 = c\Lambda + dc - e \neq 0$, where $\omega = \sqrt{-2ad - 2d^2 + 2\Lambda}(-2ad - 2d^2 + 2\Lambda > 0)$.
Then a root $\lambda(b)$ of the characteristic polynomial satisfies the relation

$$f(\lambda(b), b) = \lambda(b)^3 + T_3\lambda(b)^2 + K_3\lambda(b) + D_3 = 0, \tag{18}$$

Differentiation of 18 with respect to b yields $\frac{\partial f}{\partial \lambda}\frac{d\lambda}{db} + \frac{\partial f}{\partial b} = 0$ implies

$$\frac{d\lambda(b)}{db} = -\frac{\partial f}{\partial b}\left(\frac{\partial f}{\partial \lambda}\right)^{-1} = \frac{(w_1 + w_2)}{(w_3\Gamma + w_4)}, \tag{19}$$

where $w_1 = d(8d^2c + (2\lambda c - 4e - 4\lambda)d - 2\lambda(e+\lambda))\Gamma, w_2 = -(c(6a - \lambda - 2)d^2 + ((2a\lambda - \lambda^2 - 6b)c + 2e + 2\lambda)d - 3bc\lambda), w_3 = (-2d^3c + (-4r\lambda - 4a + 2e + 4\lambda)d^2 + (-2bc + 4e\lambda + 6\lambda^2 + 4b)d + 2b(e + 2\lambda))$ and $w_4 = 2d(ad - b)(d^2c + (2\lambda c - 2)d + bc)$.
Taking the root $\lambda(b) = \lambda(b) = i\omega$, evaluating $b = b_{h3}$, and substituting it into 19, we have

$$\alpha'(0) = \left(\frac{d\,\mathrm{Re}(\lambda(b))}{db}\right)_{b=b_{h3}} = -\frac{1}{2d} \neq 0, \tag{20}$$

and

$$\omega'(0) = \left(\frac{d\,\mathrm{Im}(\lambda(b))}{db}\right)_{b=b_{h3}} = -\frac{m_1}{m_2}, \tag{21}$$

where $m_1 = \omega(d^5c^2 + 2d^4(1 + c^2(a-1)) + ((a^2 - 2a - 5/2\Lambda)c^2 + 3ce + 4a - 1)d^3 + (-5/2\Lambda(a - 1/5)c^2 + ce(3a + 2\Lambda) + 2a^2 - e^2 - a - 3\Lambda)d^2 + (\Lambda ce(2a-1) - a(e^2 + 3\Lambda))d + 1/2\Lambda e^2)$ and $m_2 = ((d^5c^2 + d^4(2 + (2a-3)c^2) + ((a^2 - 3a - 3\Lambda)c^2 + 4ce + 4a - 2)d^3 + ((-3a\Lambda + \Lambda)c^2 + ce(4a + 2\Lambda) + 2a^2 - e^2 - 2a - 4\Lambda)d^2 + ((2ce\Lambda(a-1) - a(e^2 + 4\Lambda))d + \Lambda e^2)d(a+d))$.
Evidently, the first and second requirements for the Hopf bifurcation theorem [8] are satisfied. That is, system 15 undergoes a Hopf bifurcation at the equilibrium point at E_2 when $b = b_{h3}$.

We can classify the local stability analysis of equilibrium points and types from the results in [6] and the Routh-Hurwitz criteria. Table 1 shows how equilibrium points E_0, E_1 and E_2 can be categorized based on the roots of equations 2, 9 and 16. So, we have the following conclusions.

3 Supercritical and subcritical Hopf bifurcations

In this section, we apply the normal form theory [10] to study the direction, stability, and period of bifurcating periodic solutions for the system 1. In the remainder of this part, we use the three-dimensional classical Hopf bifurcation theory and symbolic computing to analyze parametric changes in the system 1.

Theorem 1. *The system 1 exhibits a Hopf bifurcation when a passes through a_{h1} at equilibrium E_0.*
1. If $d^2 + b + 4d > 0$, the Hopf bifurcation is supercritical,and bifurcating periodic solutions exist for $a > a_{h1}$, with the bifurcating periodic solution is stable.
2. If $d < 0$, the Hopf bifurcation is subcritical and bifurcating periodic solutions exist for $a < a_{h1}$, with the bifurcating periodic solution is orbitally unstable.

Table 1: Local stability of all equilibrium points.

Equilibria	Conditions		Stability
E_0	$(a-d+1)^2+4b<0$	$a+d+1>0, e>0$	stable spiral (focus)
		If $a+d+1<0, e<0$	unstable spiral (focus)
		$a+d+1<0, e>0$, or $(a+d+1)>0, e<0$	unstable saddle-spiral (focus)
	$(a-d+1)^2+4b>0$	$\sqrt{(a-d+1)^2+4b}>(a+d+1)$ or $-\sqrt{(a-d+1)^2+4b}>(a+d+1))$ and $e<0$	unstable node
		$\sqrt{(a-d+1)^2+4b}<(a+d+1)$ or $-\sqrt{(a-d+1)^2+4b}<(a+d+1)$ and $e>0$	Stable node
		$\sqrt{(a-d+1)^2+4b}>a+d+1$ or $(-\sqrt{(a-d+1)^2+4b})>a+d+1$ and $e>0$	saddle node
	$(a-d+1)^2=-4b$	$a+d+1<0$ and $e<0$	unstable node
		$a+d+1>0$ and $e>0$	Stable node
		$a+d+1<0, e>0$, or $a+d+1>0, e<0$	Saddle node
E_1	$\frac{(\mathrm{d}^2+\mathrm{b})}{d}>0$, $\frac{-e(ad^2c^2-bdc^2+d^2c^2-2dec+e^2)}{d\,c^2}>0$ and $\frac{2dec-2d^2c}{ad^2c^2-bdc^2-2de+d^2c^2}>1+\frac{d^2}{d^2+b}$.		asymptotically stable
E_2	$(8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2})<0$	$\left(d+\frac{b}{d}\right)>0, cd-e+c\Gamma<0$	stable spiral (focus)
		$\left(d+\frac{b}{d}\right)<0, cd-e+c\Gamma>0$	unstable spiral (focus)
		$\left(d+\frac{b}{d}\right)<0, cd-e+c\Gamma<0$, or $\left(d+\frac{b}{d}\right)>0, cd-e+c\Gamma>0$	unstable saddle-spiral (focus)
	$\left(8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2}\right)>0$	$-(d+\frac{b}{d})\mp\sqrt{8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2}}>0$ and $cd-e+c\Gamma>0$	unstable node
		$-(d+\frac{b}{d})\mp\sqrt{8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2}}<0$ and $cd-e+c\Gamma<0$	Stable node
		$-(d+\frac{b}{d})\mp\sqrt{8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2}}>0$ and $cd-e+c\Gamma<0$, or $-(d+\frac{b}{d})\mp\sqrt{8ad+d^2-8\Gamma-6b+\frac{b^2}{d^2}}<0$ and $cd-e+c\Gamma>0$	saddle node
	$8ad+d^2-8\Gamma=6b-\frac{b^2}{d^2}$	$d+\frac{b}{d}<0$, and $cd-e+c\Gamma>0$	Unstable node
		$d+\frac{b}{d}>0$, and $cd-e+c\Gamma<0$	Stable node
		$d+\frac{b}{d}<0$, and $cd-e+c\Gamma>0$, or $d+\frac{b}{d}>0$, and $cd-e+c\Gamma<0$	Saddle node

*Proof.*We know that it can display Hopf bifurcation at the point E_0 as $a = a_{h1}$. We consider the linear change of variables $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, where $P = \begin{pmatrix} d & -\sqrt{-d^2-b} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the differential system 1 will be changed to

$$\begin{aligned} \dot{x_2} &= -\sqrt{-d^2-b}y_2, \\ \dot{y_2} &= \sqrt{-d^2-b}x_2 + F_2(x_2,y_2,z_2), \\ \dot{z_2} &= -ez_2 + F_3(x_2,y_2,z_2), \end{aligned} \tag{22}$$

where $F_2(x_2,y_2,z_2) = 2x_2y_2 - x_2z_2 - \frac{2d}{\sqrt{-d^2-b}}y_2^2 + \frac{d}{\sqrt{-d^2-b}}x_2z_2 - x_2^2y_2 + \frac{d}{\sqrt{-d^2-b}}x_2^3$ and $F_3(x_2,y_2,z_2) = -\sqrt{-d^2-b}cy_2z_2 + dcx_2z_2$.
Next, we follow the procedures proposed by Hassard et al. [10] to work out the necessary quantities. We can get

$$g_{11} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial y_2^2} + i\left(\frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right) = -\frac{id}{\sqrt{-d^2-b}}, \tag{23}$$

$$g_{02} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} + i\left(\frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial x_2 \partial y_2}\right)\right) = -1 - \frac{id}{\sqrt{-d^2-b}}, \tag{24}$$

$$g_{20} = \frac{1}{4}\left(\frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} + i\left(\frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial x_2 \partial y_2}\right)\right) = 1 - \frac{id}{\sqrt{-d^2-b}}, \tag{25}$$

$$G_{21} = \frac{1}{4}\left(\frac{\partial^3 F_1}{\partial x_2^3} + \frac{\partial^3 F_1}{\partial x_2 \partial y_2^2} + \frac{\partial^3 F_2}{\partial x_2^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_2^3} + i\left(\frac{\partial^3 F_2}{\partial x_2^3} + \frac{\partial^3 F_2}{\partial x_2 \partial y_2^2} - \frac{\partial^3 F_1}{\partial x_2^2 \partial y_2} - \frac{\partial^3 F_1}{\partial y_2^3}\right)\right) = -\frac{1}{4} + \frac{3i}{4} \cdot \frac{d}{\sqrt{-d^2-b}}. \tag{26}$$

Next, we calculate

$$h_{11} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial y_2^2}\right) = 0, \tag{27}$$

$$h_{20} = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2i\frac{\partial^2 F_3}{\partial x_2 y_2}\right) = 0. \tag{28}$$

By solving the following equations

$$\lambda_3 \phi_{11} = -h_{11},$$
$$(\lambda_3 - 2i\omega)\phi_{20} = -h_{20},$$

the solution is

$$\phi_{11} = \phi_{20} = 0. \tag{29}$$

Furthermore, we have

$$G_{110} = \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x_2 z_2} + \frac{\partial^2 F_2}{\partial y_2 z_2} + i\left(\frac{\partial^2 F_2}{\partial x_2 z_2} - \frac{\partial^2 F_1}{\partial y_2 z_2}\right)\right) = -\frac{1}{2} + \frac{i}{2}\frac{d}{\sqrt{-d^2-b}} \tag{30}$$

$$G_{101} = \frac{1}{2}\left(\frac{\partial^2 F_1}{\partial x_2 z_2} - \frac{\partial^2 F_2}{\partial y_2 z_2} + i\left(\frac{\partial^2 F_2}{\partial x_2 z_2} + \frac{\partial^2 F_1}{\partial y_2 z_2}\right)\right) = \frac{1}{2} + \frac{i}{2}\frac{d}{\sqrt{-d^2-b}} \tag{31}$$

From 26, 29, 30 and 31, we get

$$g_{11} = G_{21} + (2G_{110}\phi_{11} + G_{101}\phi_{20}) = -\frac{1}{4} + \frac{3i}{4}\frac{d}{\sqrt{-d^2-b}} \tag{32}$$

We now can compute the direction and stability of periodic orbits of system 1 at the origin via μ_2 and period of periodic solution and its characteristic exponent from β_2 and τ_2 respectively.
From the above analysis, one can compute the following quantities:

$$M_1(0) = \frac{i}{2\omega}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{1}{2}g_{21} = -\frac{1}{8}\frac{d^2+b+4d}{d^2+b} + \frac{i}{24}\frac{4b-9d^3-36d^2-9bd}{(\sqrt{-d^2-b})^3},$$

$$\mu_2 = -\frac{\mathrm{Re}(M_1(0))}{\alpha'(0)} = -\frac{1}{4}\frac{d^2+b+4d}{d^2+b}, \tag{33}$$

$$\beta_2 = 2\,\mathrm{Re}(M_1(0)) = -\frac{1}{4}\frac{d^2+b+4d}{d^2+b}, \tag{34}$$

$$\tau_2 = -\frac{\mathrm{Im}(M_1(0))+\mu_2\,\omega'(0)}{\omega_0(0)} = -\frac{1}{12}\frac{3d^3+3bd+12d^2-2b}{(d^2+b)^2}. \tag{35}$$

From the system 1 exhibits a Hopf bifurcation at the equilibrium E_0 as a passes through $a_{h1} = -d-1$. Since μ_2 determines the type of Hopf bifurcation and the direction of bifurcating periodic solutions; β_2 determines the stability of the bifurcating periodic solutions; τ_2 determines the increases (decreases) of the period of bifurcating periodic solutions. If $d^2+b+4d > 0$, from equations 33 and 34, then $\mu_2 = \beta_2 = ((-1)/4(d^2+b+4d)/(d^2+b)) > 0$. Hence the Hopf bifurcation is supercritical and the bifurcating periodic solution is orbitally stable with bifurcating periodic solutions exist for $a > a_{h1}$. But if $d < 0$, from equations 33 and 34 then $\mu_2 = \beta_2 = ((-1)/4\ (d^2+b+4d)/(d^2+b)) < 0$, so Hopf bifurcations is subcritical and bifurcating periodic solutions exist for $a < a_{h1}$ and the bifurcating periodic solutions are orbitally unstable. If $\tau_2 > 0$, in equation 35, the period of bifurcating closed orbits increases. whereas, if $\tau_2 < 0$, in 35, the period of bifurcating closed orbits decreases.

Now we use the normal form theory to determine the direction of Hopf bifurcations and the stability of bifurcating limit cycles at equilibrium points E_1 and E_2. However, the calculations are not challenging, just large. They were carried out using Maple.
Analogously to the above analysis at E_0, we have the following results for the equilibrium point E_1 of system 8:

$$\mu_2 = -\frac{\mathrm{Re}(M_1(0))}{\alpha'(0)} = \frac{4(b(d^2+b)^2c^2+2d^4ec-2d^3e^2)(dc-e)}{16c^2(d^2+\omega^2)^4(d^4+d^2\omega^2+2bd^2+b^2)^2\sqrt{(2bde(dc-e))}dc(bd^2c^2+b^2c^2+2d^2ec-2de^2)}(h_1)+$$
$$\frac{1}{8\omega^2bdc^2(d^4+4d^2\omega^2-2bd^2+b^2)(d^4+d^2\omega^2+2bd^2+b^2)^2(d^2+\omega^2)^3}+(\sum_{i=2}^{13}h_i), \tag{36}$$

$$\beta_2 = 2\,\mathrm{Re}(M_1(0)) = \frac{1}{8c^2(d^2+\omega^2)^4(d^4+d^2\omega^2+2bd^2+b^2)^2}(h_1)+$$
$$\frac{1}{4\omega^2bdc^2(d^4+4d^2\omega^2-2bd^2+b^2)(d^4+d^2\omega^2+2bd^2+b^2)^2(d^2+\omega^2)^3}\sum_{i=2}^{13}h_i), \tag{37}$$

$$\tau_2 = -\frac{1}{\omega b^2c^2(d^4+4d^2\omega^2+2bd^2+b^2)\Delta^2}\left(\frac{1}{-24\omega^3d^2}\sum_{i=1}^{5}B_i+\frac{1}{\Omega}\sum_{i=1}^{5}c_i\right), \tag{38}$$

where $\Delta = (d^4+d^2\omega^2+2bd^2+b^2)(d^2+\omega^2)$,
$\Omega = 4\omega(\omega^2(d^2c^4(4b^2d^2+9b^2\omega^2+8b^3)-12bd^4c^3e+4b^4c^4+12bd^3e^2c^2)+4d^6c^2e^2-8d^5e^3c+4d^4e^4)$. Moreover, $\sum_{i=2}^{13}h_i$,$\sum_{i=1}^{5}B_i$ and $\sum_{i=1}^{5}c_i$ can be found in the Appendix. If $\mu_2 > 0(< 0)$, then the Hopf bifurcation is supercritical (subcritical). Notice that it is not easy to analyze equation 36 to find supercritical and subcritical. Now, we consider some special cases of 36.
1.Take $e = b = -c^2$ and $d = c$, then $\omega = 2\sqrt{c} > 0$ and $\mu_2 = \frac{-(10c^3+40c^2+c+64)}{4c^3(c+4)^2}$, or if $e = b = c^2$ and $d = -e$, then $\omega = 2\sqrt{2c^2(c+1)} > 0$ and
$\mu_2 = \frac{-(6c^13+28c^12+58c^11+44c^10+32c^9+352c^8+1340c^7+2581c^6+3134c^5+2755c^4+1674c^3595c^2+60c+21)}{4c^2(c^4+8c^3+10c^2+1)(c^2+2c+2)^2(c^4+2c^3+4c^2+1)}$, or if $e = -d, b = -d^2$ and $c = 1$, then $\omega = 2\sqrt{d} > 0$ and $\mu_2 = \frac{-(10d^2+41d+64)}{4d^2(d+4)^2}$. For all above cases μ are negative, then Hopf bifurcation is subcritical.
2. Take $e = -c \neq 0, b = -1$ and $d = 1$,then $\mu_2 = -\frac{13+10c}{20c}$, if $c \in (-1.3, 0)$ so Hopf bifurcation is supercritical and if $c \in (-\infty, -1.3) \cup (0, \infty)$ Hopf bifurcation is subcritical. In addition, take $e = -c \neq 0, b = -3$ and $d = 3$, then $\mu_2 = \frac{9248c^2-62220c-261819}{187272c}$, if $c \in (-2.93104141701..., 0)$ the Hopf bifurcation is supercritical and if $c \in (-\infty, -2.9310414170 10904...) \cup (0, \infty)$ Hopf bifurcation is subcritical.
Furthermore, for the equilibrium point E_2 of system 15, we have the following results:

$$\mu_2 = -\frac{\mathrm{Re}(M_1(0))}{\alpha'(0)} = -\frac{d^2(a+d)}{\omega^2}, \tag{39}$$

$$\beta_2 = 2\,\mathrm{Re}(M_1(0)) = -\frac{d^2(a+d)}{\omega^2}, \tag{40}$$

$$\tau_2 = -\frac{\mathrm{Im}(M_1(0)) + \mu_2\,\omega'(0)}{\omega_0(0)} = -\frac{1}{24\omega}\left(\frac{\Pi_1}{\Pi_1} + \frac{486ad^2+486d^3-648d\Lambda+4ad-212d^2-4\Lambda}{12\omega^3}\right), \tag{41}$$

where $\Pi_1 = (-33\omega ad - 33d^2\omega + 21\omega\Lambda)(-2d^2d^3c^2 - 4ad^4c^2 - 2d^5c^2 + 4ad^3c^2 + 4d^4c^2 - 6d^3ec - 4a^2d^2 - 8ad^3 + 2ade^2 - 4d^4 + 2d^2e^2 + 2ad^2 + 2d^3 + \Lambda(5ad^2c^2 + 5d^3c^2 - 4adec - 4d^2ec - d^2c^2 + 2dec + 6ad + 6d^2 - e^2)$, and $\Pi_2 = \omega^2(-2a^3d^2 - 4a^2d^2ec - 8ad^3ec - 4d^4ec - d^6c^2 + d^3e^2 + 4ad^3 - 6a^2d^3 - 6ad^4 + 3d^5c^2 + 2a^2d^2 - 2d^5 + a^2de^2 + 2ad^2e^2 - a^3d^3c^2 - 3a^2d^4c^2 - 3ad^5c^2 + 6ad^4c^2 + 3a^2d^3c^2 + 2d^4 + \Lambda(2d^2ec - ad^2c^2 + 3a^2d^2c^2 + 6ad^3c^2 - 2d^3ec - ae^2 + 4a^2d - de^2 + 8ad^2 - d^3c^2 + 3d^4c^2 + 4d^3 + 2adec - 2a^2dec - 4ad^2ec))$.

From the above analysis, the system 15 exhibits a Hopf bifurcation at the equilibrium $E_2(0,0,0)$ as a passes through $a = a_{h3}$. When $d > 0$ then $\alpha'(0) < 0$, hence the signs of μ_2 and β_2 are the same according to equation 39 and equation 40. Also, from equation 41 the period of bifurcating closed orbits increases $\tau_2 > 0$ and decreases $\tau_2 < 0$. Also, if $d > 0$, then $\beta_2 > 0$. Likewise, if $d < 0$, then $\beta_2 < 0$. In short, we have proved the next result.

Theorem 2.*System 15 exhibits a Hopf bifurcation at the equilibrium point E_2. When $a+d < 0$, the bifurcation is supercritical and the direction of the the bifurcation is$b < b_{h3}$.*

Remark 1. The results for the equilibrium point E_3 are analogous to those of Proposition 3 and Theorem 2 for E_2. As a result, the formulation of the conclusion for the equilibrium E_3 and its proof are omitted. This is due to the fact that when we perform the appropriate computations for the equilibrium point E_3, we obtain the same results as when we perform the identical computations for the equilibrium point E_2.

4 Numerical examples

In the present section 3, we make some numerical simulations with help of Maple to support our analytical results.
1. The stability of equilibrium point E_0 is changed from one side to other of a_{h1}.Hence, there is Hopf bifurcation at $a = a_{h1}$.When $b = -5, c = 1, d = -1.1$, and $e = 2$ with initial conditions: $x(0) = 0, y(0) = z(0) = 0.1$. The theoretical analysis suggests that

$$\omega = 1.946792233 \quad \text{and} \quad a_{h1} = 0.1.$$

It follows from the results that

$$\mu_2 = \beta_2 = -1.410949868 \quad \text{and} \quad \tau_2 = 0.7269142625.$$

In the light Theorem 1, since $\mu_2 < 0$, the Hopf bifurcation is subcritical which means that equilibrium point E_0 of the system 1 is stable when $a > a_{h1}$ where $a = 0.2$, and the equilibrium point losses its stability and a Hopf bifurcation occurs when a decreases past a_{h1},where $a = -0.1$ i.e., a family of periodic solutions bifurcates from the equilibrium point, as shown in Fig. 1. Since $\beta_2 < 0$, each individual closed solution is stable. Since $\tau_2 > 0$, the period of bifurcating periodic solutions increases with a increasing.
2. The stability of equilibrium point E_1 is changed from one side to the other of a_{h2}.Hence, there is Hopf bifurcation at $a = a_{h2}$. When $b = 1, c = 1, d = 1.9$, and $e = 1$ with initial conditions: $x(0) = 0, y(0) = z(0) = 1$. The theoretical analysis suggests that

$$\omega = 1.849324201 \quad \text{and} \quad a_{h2} = -4.065429363.$$

It follows from the results that

$$\mu_2 = 1.499683545, \beta_2 = -0.08480781953 \quad \text{and} \quad \tau_2 = -0.1280863068.$$

Since $\mu_2 > 0$, the Hopf bifurcation is supercritical which means that equilibrium point $E_1(0,0,0)$ of the system 8 is stable when $a < a_{h2}$, a Hopf bifurcation occurs when a increases past a_{h2}, i.e., a family of periodic solutions bifurcate from the equilibrium point, as shown in Fig. 2. Since $\beta_2 < 0$, each individual periodic solution is stable. Since $\tau_2 < 0$, periods of bifurcating periodic solutions decrease with decreasing a.
3. The stability of equilibrium point E_3 is changed from one side to other of b_{h3}. Hence, there is Hopf bifurcation at $b = b_{h3}$. When $a = 1, c = 5, d = -1.2$, and $e = 5$ with initial conditions: $x(0) = y(0) = z(0) = 0.1$. The theoretical analysis suggests that

$$\omega = 0.7702678943 \quad \text{and} \quad b_{h3} = -1.44.$$

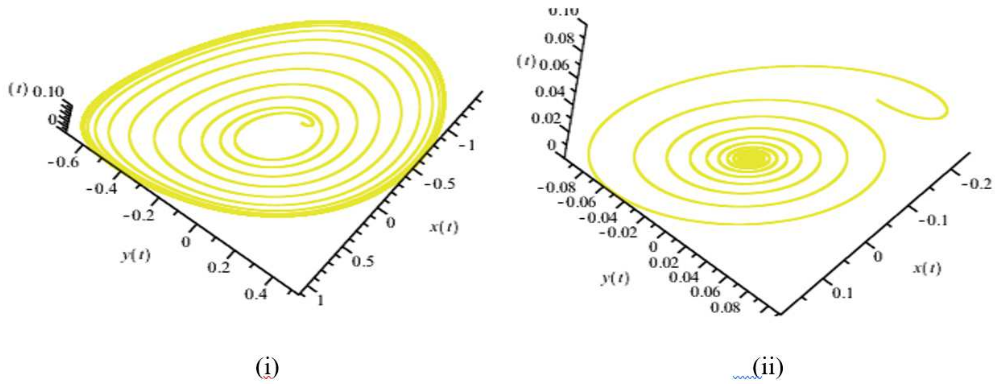


Fig. 1: A trajectory of forest pest system 1 at equilibrium E_0. (i) the bifurcating periodic solution when $a = 0.2 > a_{h1} = 0.1$. (ii) a stable when $a = -0.1 < a_h 1 = 0.1$.

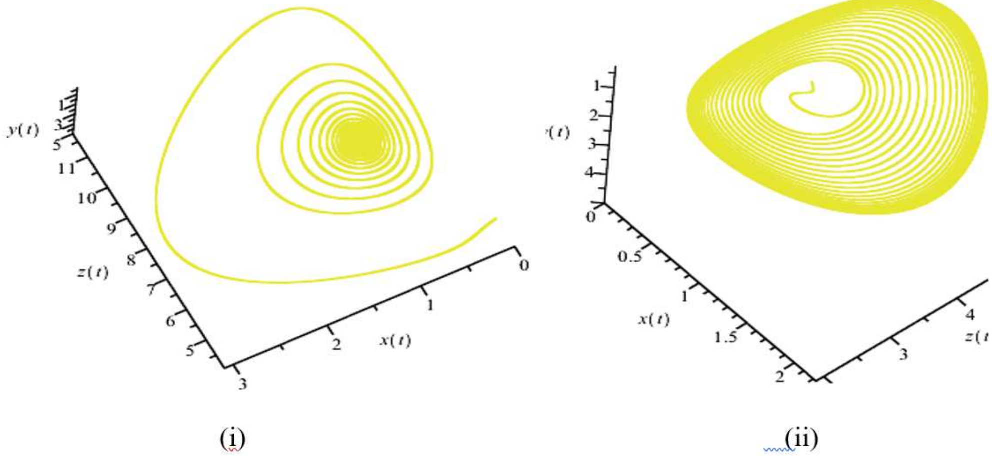


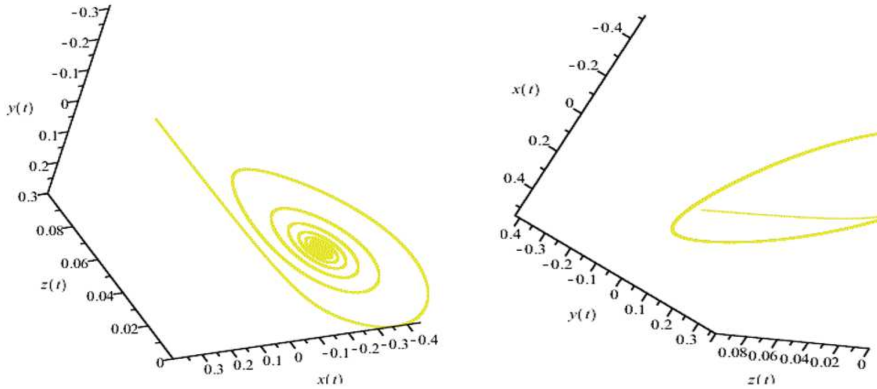
Fig. 2: A trajectory of forest pest system 8 at equilibrium E_1 (i) a stable when $a = -7.06542 < a_{h2} = -4.065429363$. (ii) the bifurcating periodic solution when $a = -3.06542 > a_{h2} = -4.065429363$.

It follows from the results that

$$\mu_2 = 0.4854101966, \beta_2 = -0.4045084972 \quad \text{and} \quad \tau_2 = 1.181523057.$$

In the light Theorem 2, since $\mu_2 > 0$, the Hopf bifurcation is supercritical which means that equilibrium point $E_2(0,0,0)$ of the system 15 is stable when $b < b_{h3}$ and the equilibrium point losses its stability and a Hopf bifurcation occurs when a increases past b_{h3}, i.e., a family of periodic orbits bifurcates from the equilibrium point, as shown in Fig. 3. Since $\beta_2 < 0$, each individual closed orbit is stable. Since $\tau_2 > 0$, the periods of bifurcating periodic orbits increase with b increasing.

Remark 2. An essential tool for studying time-periodic nonlinear differential systems with small parameters is the averaging approach, which has been used in several research domains, see [5,13]. In addition, the zero-Hopf bifurcation and averaging Theory [3,13,14,17,19] and normal form Theory [22] plays a central role in the periodic solutions of the polynomial differential models. Hence, to find all possible limit cycles of the system 1, one may use the averaging and normal form methods to get information about another periodic solution of this system. This leads us to believe that there should be a zero Hopf bifurcation point on the parameters, as will be demonstrated in a future study.



(i) (ii)

Fig. 3: A trajectory of forest pest system 15 at equilibrium E_2 (i) a stable when $b = -1.55 < b_{h3} = -1.44$. (ii) the bifurcating periodic solution when $b = -1.38 > b_{h3} = -1.44$.

Conclusion

New insights into the forest pest differential system are presented. In summary, the periodic solutions of the forest pest system have been studied in this paper. First, the stability of four equilibrium points E_0, E_1, and $E_{2,3}$ are analyzed. The Hopf bifurcation analysis of system 1 has also been studied using normal form theory. We thoroughly analyzed the Hopf bifurcation direction as well as the stability of bifurcating periodic solutions. Then, by employing the normal form theory of the Hopf bifurcation, its normal form is derived, and some sufficient conditions guarantee the occurrence of Hopf bifurcation. Furthermore, a numerical result is presented to illustrate that Hopf bifurcation can take place.

Appendix

$h_1 = ed^2(d^2+\omega^2+b)(c^2d^8-2cd^7+(3\omega^2c^2+2bc^2+2e)d^6-2c(2\omega^2+b)d^5+(3c^2\omega^4+(5bc^2+4e)\omega^2+10bc\omega+b(bc^2+2e))d^4+(-2\omega^4c-2b\omega^2c-15be\omega)d^3+(c^2\omega^4+2(2bc^2+e)\omega^2+10be\omega+2b^2c^2)\omega^2d^2-15be\omega^3d+bc^2\omega^4(\omega^2+b))$
$h_2 = 4d^20c^2-8d^19ec+((-2\omega c^3+12c^2)b+16c^2\omega^2+4e^2)d^18+2c((-2/3c\omega^2+c(e+2)\omega-12e)b-14e\omega^2)d^17+((-6\omega c^3+8c^2)b^2+(-8\omega^3c^3+36\omega^2c^2-8e\omega c+12e^2)b+24c^2\omega^4+12e^2\omega^2)d^16,$
$h_3 = ((c^4\omega^3+(2c^3-3c^2)\omega^2+6c^2(e+4/3)\omega-16ce)b^2+4(-11/2c^2\omega^3+2c^2(e+3/2)\omega^2-10ec\omega+e^2)\omega b-36ec\omega^4)d^15+((-4c^3\omega-8c^2)b^3+(-c^4\omega^4-18c^3\omega^3-2c^2(e-8)\omega^2-16ec\omega+8e^2)b^2-2(6c^3\omega^3+c^2(e-18)\omega^2+14ec\omega+e^2)b\omega^2+16c^2\omega^6+12e^2\omega^4)d^14,$
$h_4 = ((2c^4\omega^3+(4c^3+6c^2)\omega^2+4ec^2\omega+16ce)b^3+(4c^4\omega^5+(2c^3-39c^2)\omega^4+14c^2(e+4/7)\omega^3-2ce\omega^2+8e^2\omega)b^2+(-52c^2\omega^6+12c^2(e+1)\omega^5+44ec\omega^4+16e^2\omega^3)b-20ec\omega^6)d^13+((4\omega c^3-12c^2)b^4+(-4c^4\omega^4+2c^3\omega^3+(8+(-4e-48)c^2)\omega^2-8e^2)b^3-12(1/3c^4\omega^4+3/2c^3\omega^3+1/6c^2(e-18)\omega^2+ec\omega+e^2)\omega^2b^2+(-8c^3\omega^7-6c^2(e-2)\omega^6-32ce\omega^5-108e^2\omega^4)b+4c^4\omega^8+4e^2\omega^6)d^12,$
$h_5 = (-c(-6c\omega^2+4c(e+2)\omega-24e)b^4-c(-7c^3\omega^3+(-6c^2+19c)\omega^2+12c(e+2)\omega-102e)\omega^2b^3+(bc^4\omega^7+(-6c^3-87c^2)\omega^6+(10e-8)c^2\omega^5+24ec\omega^4+4e^2\omega^3)b^2+(-54c^2\omega^8+4c^2(2e+1)\omega^7+112ec\omega^6+20e^2\omega^5)b-4ec\omega^8)d^11,$
$h_6 = 6b((c^3\omega-2/3c^2)b^4+(5c^3\omega^3+(-8/3c^2-8/3)\omega^2+8/3ec\omega-2e^2)b^3-28/3(1/7c^4\omega^4-13/28r^3\omega^3+(-5/7+(5/56e+15/14)c^2)\omega^2-8/7ec\omega+e^2)\omega^2b^2-35/3(3/25c^4\omega^4+3/35c^3\omega^3-2/35c^2(e+14)\omega^2-16/35ec\omega+e^2)\omega^4b-1/3c^3\omega^9-ec^2\omega^8-2ec\omega^7-83/3e^2\omega^6)d^10,$
$h_7 = -6b(c(1/3c^3\omega^3+(2/3c^2+1/2c)\omega^3+c(2/3+e)\omega-4/3e)b^4+4/3(1/8c^4\omega^4+(11/4c^3+3/8c^2)\omega^3+9/2c^2(2/3+e)\omega^2-39/4ec\omega+e^2)\omega b^3+(-2/3c^4\omega^7+(-2c^3+61/6c^2)\omega^6+8c^2(e+3/4)\omega^5-47/3ec\omega^4+20/3e^2\omega^3)b^2+16/3(-1/8c^4\omega^4+(5/16c^3+69/32c^2)\omega^3-1/16c^2(e-4)\omega^2+23/16ec\omega+e^2)\omega^5b-4/3(1/4ec^2\omega^2-25/8c^2\omega^3+13/2ec\omega+e^2)\omega^7)d^9$, $h_8 = 2b(b^5c^3\omega+(c^4\omega^4+11c^3\omega^3+(4+(2e-2)c^2)\omega^2+4ec\omega-2e^2)b^4+$

$(-1/2c^4\omega^6 + 28c^3\omega^5 + (-8 + (14e + 6)c^2)\omega^4 + 26ec\omega^3 - 24e^2\omega^2)b^3 + (-6c^4\omega^8 + 15c^3\omega^7 + (16 + (-9e - 10)c^2)\omega^6 + 50ce\omega^5 - 33e^2\omega^4)b^2 + (-2c^4\omega^10 + 3c^2(e + 14/3)\omega^8 + 14ec\omega^7 - 13e^2\omega^6)b - ec^2\omega^10 - 36e^2\omega^8)d^8$,
$h_9 = -2(c^2(1/2c^2\omega^2(c + 3/2)\omega + e)b^5 + (7/2c^4\omega^4 + (7c^3 + 7/2c^2)\omega^3 + 10(e + 1/5)c^2\omega^2 + 7ec\omega + 2e^2)b^4 + (3/2c^4\omega^6 + (22c^3 + 12c^2)\omega^5 + c^2(35e + 8)\omega^4 - 12ec\omega^3 + 14e^2\omega^2)b^3 + (-5/2c^4\omega^8 + (-11c^3 + 41/2c^2)\omega^7 + c^2(23e + 6)\omega^6 - 4ec\omega^5 + 34e^2\omega^4)b^2 + (-1/2c^4\omega^10 + (2c^3 + 9c^2)\omega^9 + 28ec\omega^7 + 12e^2\omega^6)b + 2c^2\omega^11)b\omega d^7$,
$h_{10} = 2(c^2(1/2c^2\omega^2 + 2c\omega + e)b^4 + (3c^4\omega^4 + 13c^3\omega^3 + (11/2ec^2 + 4)\omega^2 + 2ec\omega + 5e^2)b^3 + (-3/2c^4\omega^6 + 19c^3\omega^5 + c^2(29e + 8)\omega^4 + 18ec\omega^3 - 7e^2\omega^2)b^2) + (-4c^4\omega^8 + 5c^3\omega^7 - 37/2ec^2\omega^6 + 20ce\omega^5 - 13e^2\omega^4)b - 1/2c^4\omega^10 + ec^2\omega^8 + 12e^2\omega^6)b^2\omega^2 d^6$,
$h_{11} = -2((3/2c^2\omega^2 + (2c + 3)\omega + e)c^2b^3 + (9/2c^4\omega^4 + (8c^3 + 5/2c^2)\omega^3 + 12ec^2\omega^2 + 7ec\omega)b^2 + (3/2c^4\omega^6 + (11c^3 + 15/2c^2)\omega^5 + 25ec^2\omega^4 + 19ec\omega^3 + 8e^2\omega^2)b - 1/2c^4\omega^8 + (-6c^3 + 5/2c^2)\omega^7 + 7ec^2\omega^6 + 16e^2\omega^4)b^3\omega^3 d^5$,
$h_{12} = 2b^3(c^2(3/2c^2\omega^2 + c\omega + e)b^3 + (3\omega^4c^4 + 5\omega^3c^3 + 7e\omega^2c^2 + 3e^2)b^2 + (-3/2c^4\omega^6 + 4c^3\omega^5 + 15ec^2\omega^4 + 11e^2\omega^2)b - c^4\omega^8 - 10ec^2\omega^6)\omega^4d^4 - 3c^2b^4((c^2\omega + 2/3c + 1)b^2 + 10/3(1/2c^2\omega^2 + (3/5c + 1/10)\omega + e)\omega b + 1/3c^2\omega^5 + 4e\omega^3)\omega^6d^3$,
$h_{13} = (3c^4\omega^8b^6 + (2\omega^10c^4 + 7e\omega^8c^2)b^5 - b^4c^4\omega^12)d^2 - b^5c^4\omega^9(\omega^2 + b)d + b^6c^4\omega^10$,
$B_1 = b^2d^18\omega^2c^4 + 8b^2d^16\omega^4c^4 + 22b^2d^14\omega^6c^4 + 28b^2d^12\omega^8c^4 + 17b^2d^10\omega^10c^4 + 4b^2d^8\omega^12c^4 + 6b^3d^16\omega^2c^4 + 35b^3d^14\omega^4c^4 + 71b^3d^12\omega^6c^4 + 63b^3d^10\omega^8c^4 + 23b^3d^8\omega^10c^4 + 2b^3d^6\omega^12c^4 + 15b^4d^14\omega^2c^4 + 63b^4d^12\omega^4c^4 + 91b^4d^10\omega^6c^4 + 51b^4d^8\omega^8c^4 + 6b^4d^6\omega^10c^4 - 2b^4d^4\omega^12c^4 - 4b^2d^17\omega^2c^3 - 52b^2d^15\omega^4c^3 - 132b^2d^13\omega^6c^3 - 124b^2d^11\omega^8c^3 - 40b^2d^9\omega^10c^3 + 4d^22c^2 + 32d^20\omega^2c^2 + 88d^18\omega^4c^2 + 112d^16\omega^6c^2 + 68d^14\omega^8c^2 + 16d^12\omega^10c^2 + 20b^5d^12\omega^2c^4 + 62b^5d^10\omega^4c^4 + 63b^5d^8\omega^6c^4 + 20b^5d^6\omega^8c^4 - b^5d^4\omega^10c^4 - 20b^3d^15\omega^2c^3 + 148b^3d^13\omega^4c^3$,
$B_2 = -228b^3d^11\omega^6c^3 - 92b^3d^9\omega^8c^3 + 8b^3d^7\omega^10c^3 + 9b^2d^16e\omega^2c^2 + 89b^2d^14e\omega^4c^2 + 207b^2d^12e\omega^6c^2 + 183b^2d^10e\omega^8c^2 + 56b^2d^8e\omega^10c^2 + 24bd^20c^2 + 168bd^18\omega^2c^2 + 360bd^16\omega^4c^2 + 312bd^14\omega^6c^2 + 96bd^12\omega^8c^2 - 8d^21ec - 64d^19e\omega^2c - 176d^17e\omega^4c - 224d^15e\omega^6c - 136d^13e\omega^8c - 32d^11e\omega^10c + 15b^6d^10\omega^2c^4 + 38b^6d^8\omega^4c^4 + 30b^6d^6\omega^6c^4 + 6b^6d^4\omega^8c^4 - b^6d^2\omega^10c^4 - 20b^4d^13\omega^2c^3 - 122b^4d^11\omega^4c^3 - 22b^4d^9\omega^6c^3 + 60b^4d^7\omega^8c^3 + 45b^3d^14e\omega^2c^2 + 274b^3d^12e\omega^4c^2 + 385b^3d^10e\omega^6c^2 + 140b^3d^8e\omega^8c^2 - 16b^3d^6e\omega^10c^2 + 60b^2d^18c^2 + 396b^2d^16\omega^2c^2 + 864b^2d^14\omega^4c^2 + 780b^2d^12\omega^6c^2 + 252b^2d^10\omega^7c^2$,
$B_3 = -48bd^19ec - 340bd^17e\omega^2c - 720bd^15e\omega^4c - 612bd^13e\omega^6c - 184bd^11e\omega^8c + 4d^20e^2 + 32d^18e^2\omega^2 + 88d^16e^2\omega^4 + 112d^14e^2\omega^6 + 68d^12e^2\omega^8 + 16d^10e^2\omega^10 + 6b^7d^8\omega^2c^4 + 15b^7d^6\omega^4c^4 + 12b^7d^4\omega^6c^4 + 3b^7d^2\omega^8c^4 - 40b^5d^11\omega^2c^3 + 2b^5d^9\omega^4c^3 + 118b^5d^7\omega^6c^3 + 76b^5d^5\omega^8c^3 + 90b^4d^12e\omega^2c^2 + 278b^4d^10e\omega^4c^2 + 101b^4d^8e\omega^6c^2 - 78b^4d^6e\omega^8c^2 + 80b^3d^12c^2 + 544b^3d^14\omega^2c^2 + 1112b^3d^12\omega^4c^2 + 648b^3d^10\omega^6c^2 - 120b^2d^17ec - 840b^2d^15e\omega^2c - 1880b^2d^13e\omega^4c - 1744b^2d^11e\omega^6c - 584b^2d^9e\omega^8c + 24bd^18e^2 + 172bd^16e^2\omega^2 + 360bd^14e^2\omega^4 + 300bd^12e^2\omega^6 + 88bd^10e^2\omega^8 + b^8d^6\omega^2c^4 + 3b^8d^4\omega^4c^4$,
$B_4 = 3b^8d^2\omega^6r^4 + b^8\omega^8c^4 - 20b^6d^9\omega^2c^3 + 38b^6d^7\omega^4c^3 + 58b^6d^5\omega^6c^3 + 90b^5d^10e\omega^2c^2 + 80b^5d^8e\omega^4c^2 - 126b^5d^6e\omega^6c^2 - 116b^5d^4e\omega^8c^2 + 60b^4d^14c^2 + 456b^4d^12\omega^2c^2 + 1032b^4d^10\omega^4c^2 + 1716b^4d^8\omega^6c^2 - 160b^3d^15ec - 1240b^3d^13e\omega^2c - 2560b^3d^11e\omega^4c - 1372b^3d^9e\omega^6c + 60b^2d^16e^2 + 448b^2d^14e^2\omega^2 + 1016b^2d^12e^2\omega^4 + 952b^2d^10e^2\omega^6 + 324b^2d^8e^2\omega^8 - 4b^7d^7\omega^2c^3 + 10b^7d^5\omega^4c^3 + 14b^7d^3\omega^6c^3 + 45b^6d^8e\omega^2c^2 - 23b^6d^6e\omega^4c^2 - 68b^6d^4e\omega^6c^2 + 24b^5d^12c^2 + 216b^5d^10\omega^2c^2 + 912b^5d^8\omega^4c^2 - 120b^4d^13ec - 1120b^4d^11e\omega^2c$,
$B_5 = -2654b^4d^9e\omega^4c - 4912b^4d^7e\omega^6c + 80b^3d^14e^2 + 712b^3d^12e^2\omega^2 + 1438b^3d^10e^2\omega^4 + 644b^3d^8e^2\omega^6 + 9b^7d^6e\omega^2c^2 - 10b^7d^4e\omega^4c^2 - 19b^7d^2e\omega^6c^2 + 4b^6d^10c^2 + 44b^6d^8\omega^2c^2 + 400b^6d^6\omega^4c^2 - 48b^5d^11ec - 564b^5d^9e\omega^2c - 2604b^5d^7e\omega^4c + 60b^4d^12e^2 + 688b^4d^10e^2\omega^2 + 1687b^4d^8e^2\omega^4 + 3516b^4d^6e^2\omega^6 - 8b^6d^9ec - 120b^6d^7e\omega^2c - 1174b^6d^5e\omega^4c + 24b^5d^10e^2 + 364b^5d^8e^2\omega^2 + 1852b^5d^6e^2\omega^4 + 4b^6d^8e^2 + 80b^6d^6e^2\omega^2 + 859b^6d^4e^2\omega^4$,
$c_1 = (2bc^2d^2 + 3bc^2\omega^2 + 2b^2c^2 - 2ced^2 + 2de^2)(bc^2d^4 + 2b^2c^2d^2 + 2cd^4e + b^3c^2 - 2d^3e^2) - b^2d^14c^4\omega^4 - 3b^2d^10c^4\omega^6 - 3b^2d^8c^4\omega^8 - b^2d^6c^4\omega^10 - 2b^3d^10c^4\omega^4 - 4b^3d^8c^4\omega^6 - 2b^3d^6c^4\omega^8 + 4bd^15c^3\omega^2 + 12bd^13c^3\omega^4 + 12bd^11c^3\omega^6 + 4bd^9c^3\omega^8 + b^4d^6c^4\omega^6$,
$c_2 = 2b^4d^4c^4\omega^8 + b^4d^4c^4\omega^10 + 22b^2d^13c^3\omega^2 + 84b^2d^11c^3\omega^4 + 102b^2d^9c^3\omega^6 + 40b^2d^7c^3\omega^8 + 3bd^15c^2\omega^2 - 4bd^14ec^2\omega^2 + 19bd^13c^2\omega^4 - 10bd^12ec^2\omega^4 + 33bd^11c^2\omega^6 - 8bd^10ec^2\omega^6 + 21bd^9c^2\omega^8 - 2bd^8ec^2\omega^8 + 4bd^7\omega^10c^2 + 16d^16c^2\omega^2 + 48d^14c^2\omega^4 + 48d^12c^2\omega^6 + 16d^10c^2\omega^8 + 2b^5d^6c^4\omega^4 + 4b^5d^4c^4\omega^8 + 2b^5d^2c^4\omega^8 + 48b^3d^11c^3\omega^2 + 150b^3d^9c^3\omega^4 + 122b^3d^7c^3\omega^8 + 20b^3d^5c^3\omega^8 + 15b^2c^2d^13\omega^2 - 26b^2d^12ec^2\omega^2 + 64b^2d^11c^2\omega^4$,
$c_3 = -102b^2d^10ec^2\omega^4 + 71b^2d^9c^2\omega^6 - 130b^2d^8ec^2\omega^6 + 22b^2d^7c^2\omega^8 - 54b^2d^6ec^2\omega^8 + 40bd^14c^2\omega^2 + 80bd^12c^2\omega^4 + 40bd^10c^2\omega^6 - 2d^17ce - 50d^15e\omega^2c - 134d^13ce\omega^4 - 126d^11ce\omega^6 - 40d^9ce\omega^8 + b^6d^4c^4\omega^4 + 2b^6d^2c^4\omega^6 + b^6c^4\omega^8 + 52b^4d^9c^3\omega^2 + 102b^4d^7c^3\omega^4 + 50b^4d^5c^3\omega^6 + 30b^3d^11c^2\omega^2 - 64b^3d^10ec^2\omega^2 + 81b^3d^9c^2\omega^4 - 199b^3d^8c^2e\omega^4 + 56b^3d^7c^2\omega^6 - 163b^3d^9c^2e\omega^6 + 5b^3d^5c^2\omega^8 - 28b^3d^4c^2e\omega^8 + 24b^2d^12c^2\omega^2 + 32b^2d^10c^2\omega^4 + 8b^2d^8c^2\omega^6 - 10bd^15ce$,
$c_4 = -148bd^13ce\omega^2 - 258bd^11ec\omega^4 - 120bd^9ce\omega^6 + 2d^16e^2 + 34d^14e^2\omega^2 + 86d^12e^2\omega^4 + 78d^10e^2\omega^6 + 24d^8e^2\omega^8 + 28b^5d^7c^3\omega^2 + 30b^5d^5c^3\omega^4 + 2b^5d^3c^3\omega^6 + 30b^4d^9c^2\omega^2 - 76b^4d^8ec^2\omega^2 + 49b^4d^7c^2\omega^4 - 142b^4d^6ec^2\omega^4 + 19b^4d^5\omega^6c^2 - 66b^4d^4ec^2\omega^6 - 8b^3d^10c^2\omega^2 - 8b^3d^8c^2\omega^4 - 20b^2d^13ce - 152b^2d^11ce\omega^2 - 178b^2d^9ce\omega^4 - 10b^2d^7ce\omega^6 + 10bd^14e^2 + 112bd^12e^2\omega^2 + 168bd^10e^2\omega^4 + 84bd^8e^2\omega^6 + 6b^6d^5c^3\omega^2 + 6b^6d^3c^3\omega^4 + 15b^5d^7c^2\omega^2 - 44b^5d^6c^2e\omega^2 + 16b^5d^5c^2\omega^4 - 45b^5d^4c^2e\omega^4 + b^5d^3c^2\omega^6 - b^5d^2c^2e\omega^6 - 8b^4d^8c^2\omega^2$
and $c_5 =$
$-8b^4d^6c^2\omega^4 - 20b^3d^11ce - 68b^3d^9ce\omega^2 - 42b^3d^7ce\omega^4 + 20b^2d^12e^2 + 143b^2d^10e^2\omega^2 + 171b^2d^8e^2\omega^4 - 6b^2d^6e^2\omega^6 +$

$3b^6d^5c^2\omega^2 - 10b^6d^4c^2e\omega^2 + 3b^6d^3c^2\omega^4 - 10b^6d^2c^2e\omega^4 - 10b^4d^9ce - 22b^4d^7ce\omega^2 + 12b^4d^5ce\omega^4 + 20b^3d^10e^2 + 97b^3d^8e^2\omega^2 + 68b^3d^6e^2\omega^4 - 2b^5d^7ce - 8b^5d^5ce\omega^2 + 10b^4d^8e^2 + 43b^4d^6e^2\omega^2 - 3b^4d^4e^2\omega^4 + 2b^5d^6e^2 + 11b^5d^4e^2\omega^2.$

Declarations

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