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Dynamical Analysis of a Pest-Natural Enemy- Predator Model with Local Bifurcation Analysis

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Abstract: Crop protection is vital for agriculture-dependent economies, where pest infestations significantly threaten crop yields. This study develops a pest–natural enemy–predator model based on the Smith growth function, which more accurately represents ecological dynamics than the traditional logistic model. We analyze the system's equilibrium stability and bifurcation behaviour with respect to the pest-capturing rate (γ), establishing the existence of a Hopf bifurcation. Numerical simulations support our analytical results, while MATCONT-based bifurcation analysis identifies limit points, Hopf bifurcations, and branch points, offering deeper insights into the system's dynamic behaviour.

Keywords: Pest Population; Natural Enemy; Smith Model; Local Bifurcation; Branch point; limit point.

2010 Mathematics Subject Classification. 32G34; 34C23.

1 Introduction

To nourish the ever-growing global population, we must produce more food while adapting to limited fertile land and accessible water per capita. Ensuring global food security has become one of the most pressing challenges of the 21st century, particularly with the increasing threats posed by climate change, land degradation, and pest outbreaks. Supplying sufficient food to the population is not just about increasing agricultural output but also about ensuring sustainability and safety. One of the significant obstacles to achieving food security is crop damage caused by pests and plant diseases, which results in substantial yield losses worldwide.

In recent times, food losses due to pests and plant infections have become significant hazards to food security, especially in developing regions. Approximately 69,000 species of pests affect crop populations, causing direct damage to plants and disrupting the environmental balance of the biosphere. Traditional pest control strategies, such as chemical pesticides, have been widely employed to mitigate these threats. However, their excessive use has led to serious environmental and health concerns, including pesticide resistance, biodiversity loss, and soil and water contamination. These challenges highlight the urgent need for sustainable pest management approaches that balance effectiveness with ecological safety.

Farmers utilize all viable strategies to control pests to preserve plants, human health, income, and the nation's economy. Pesticides are widely used on a large scale to eradicate pests from crops. However, excessive pesticide use negatively impacts farm crops and the environment. The overuse of pesticides not only harms beneficial organisms but also leaves harmful chemical residues in plants, leading to environmental pollution and risks to human health [1]. Consequently, alternative and environmentally friendly pest control methods have been explored to reduce reliance on chemical pesticides.

To address these challenges, biological control has emerged as a promising alternative. Natural enemies, such as parasites, predators, or disease-causing organisms, are introduced into the pest's environment or, if already present, are

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encouraged to multiply and become more effective in reducing pest numbers [2]. For example, ants serve as beneficial natural enemies by preying on insect pests such as caterpillars and termites. Ladybugs are important natural enemies that prey on aphids, while stem borers, a dominant pest species in maize fields in Kenya [3], are attacked by natural enemies such as spiders and ants. These natural enemies do not affect plant populations but specifically target pests that feed on plants, making biological control an effective method. However, biological control is unpredictable. Over time, natural enemies may increase to levels where they begin affecting plant populations. Therefore, it is crucial to regulate them. To ensure a balanced ecological approach, higher-order predators (such as birds and lizards) are introduced to control populations of natural enemies, thereby preventing them from disrupting plant ecosystems. For example, stem borers are considered highly damaging pests in rice fields at all growth stages, from seedling to maturity. Spiders, as natural enemies, prey on stem borers and help control their population. However, spiders are themselves preyed upon by various natural predators, such as lizards and birds [4].

Despite the success of biological control, a major challenge remains: understanding the complex interactions between pests, natural enemies, and predators. An effective pest management strategy requires a quantitative understanding of these relationships to predict population dynamics under different conditions. One of the essential components of successful biological control implementation is a fundamental understanding of the interactions between different elements of the agricultural environment, including plants, pests, natural enemies, and predators. Mathematical models serve as decision-support tools by simulating these interactions and predicting long-term ecological outcomes.

Recent research has refined mathematical models to better capture predator-prey dynamics. For instance, [5] developed a model for managing the eggplant fruit and shoot borer using biological control, while many studies explore combined biological and chemical control, often lacking real-world case studies. However, a significant limitation of these models is their reliance on simplified growth assumptions that fail to capture dynamic environmental factors.

Advancements like the delay differential model of the one-predator two-prey system [6] improve traditional models by incorporating realistic prey consumption rates. Similarly, the fractional-order delay differential model of the prey-predator system [7] introduces fractional dynamics for more accurate ecological predictions. These studies emphasize the importance of incorporating delay and fractional calculus to enhance the realism of pest control models.

To bridge this gap, our study develops a pest-natural enemy-predator model based on the Smith growth function, which better represents population dynamics than traditional logistic models. The Smith model accounts for varying food availability and is more suitable for ecological systems where population interactions are complex and dynamic. Compared to traditional models, the Smith growth function captures nonlinear growth patterns more effectively, making it relevant for pest control applications that require adaptive strategies. In 2019, Mandal studied the predator-prey model from a pest management perspective [4]. His study explored pest control management by eliminating natural enemies through super-predators or human intervention. Abdul Satar [8] proposed a prey-predator-scavenger food web model, examining how two distinct biological factors influence different species. In 2018, Vijay Kumar [9] developed a food chain system (plant-pest-natural enemy) incorporating gestation delays for both natural enemies and pest populations. His findings indicated that if the gestation delay of the pest population is sufficiently low, the natural-enemy-free steady state remains stable. Sasmal [10] investigated the interaction between the strong Allee effect in pest populations and biological control methods. He found that providing supplementary food to predator populations alone could be an effective pest control strategy.

In 2019, Xinyu [11] also examined the Allee effect. His model, which included two interacting species with a non-monotonic functional response and the Allee effect on the second species, revealed that introducing the Allee effect did not impact the final species density but increased the time required to reach steady-state equilibrium. Yaning Li [12] developed the Smith predator-prey model to study the balance between chemical and biological control. Various studies have explored integrated pest management (IPM) strategies, combining biological and chemical methods within traditional predator-prey models to investigate their interactions [13,14,15].

The optimal control problem in pest management has also gained research attention in recent years. For example, Diva Amalia [16] examined a predator-prey model incorporating infection and harvesting in the prey population. The purpose of introducing infection in prey was to increase the susceptible prey population while reducing the number of infected individuals. Toaha [17] studied the optimal harvesting of predator-prey populations, proposing that the ecosystem be divided into two zones: a free-fishing zone and a prohibited zone. The pest population was allowed to move between these zones, while the predator and prey populations in the free-fishing zone were harvested with constant effort. The primary aim of the study [18] was to regulate the pest population so that it does not harm the environment. The authors divided the optimal control problem into two periods: the first focused on pest eradication, while the second introduced a natural enemy population.

The Lotka-Volterra problem was formulated in [19], where optimal control measures, such as hunting, were applied to both prey and predator species. Sadiq and Al-Nassir [20] examined the dynamics and optimal control of a prey-predator system. Furthermore, several theoretical results on the predator-prey model [21,22,23] have been obtained using impulsive differential equations. However, predator-prey models often rely on the logistic model, which is the simplest and most effective model in biomathematics. Some researchers believe that the logistic equation accurately describes population growth under limited food supply conditions. However, it is primarily applicable to low-level biological populations.

In 1963, Smith [24] carefully observed a species of algae called Daphnia in the laboratory and found that its population dynamics did not conform to the linear growth law. Smith proposed that the relative growth rate of a population depends on the sum of the remaining food at a given time rather than the total initial food supply. This concept forms the basis of the present study [25].

This paper deals with the rich dynamics of the pest-natural enemy-predator model using the Smith model, which assumes that the growth rate of the population is relative to the sum of food remaining at that time instead of the whole amount of food. The study provides significant insight into the local stability analysis of steady-state solutions, validated through numerical simulations. The variation in system dynamics is analyzed with respect to key parameters r, D, and initial conditions. Furthermore, qualitative bifurcation analysis using MATCONT by continuation of equilibrium points and branch points has shown the existence of a Limit point (fold; saddle-node) bifurcation, multiple Branch points, supercritical Hopf bifurcation points, and Neutral saddle equilibrium. These mathematical analyses provide a deeper understanding of pest population control strategies and their long-term sustainability.

1.1 Structure of the paper

The paper is organised as follows: Using the Smith model, a mathematical model of a pest, a natural enemy, and a predator is created in section 2. The stability and positivity of the equilibrium points will be covered in section 3 along with more details on their dynamics. subsection 3.1 talks about the local stability of the equilibrium points. In section 4, local bifurcation utilising Sotomayor's theorem based on the bifurcation parameter γ at the positive interior equilibrium point is discussed. Finally, in section 5, validation of all the theoretical results is done and followed by a discussion of the sensitivity analysis on critical parameters like the relative growth rate of the pest in subsection 5.1 and Hopf bifurcation analysis using MATCONT in subsection 5.2.
The following section will deal with the Model Formulation of the pest-predator population using the Smith model.

2 Model Formulation

The mathematical model describes the interactions between the pest population $x(t)$, natural enemy $y(t)$, and predator population $z(t)$.
1. Pest Population $x(t)$:
The growth of the pest population follows a Smith Model-based logistic growth rather than a traditional logistic equation. The Smith Model, introduced by Smith (1963) [24], suggests that population growth is limited not only by carrying capacity but also by food availability. This modification incorporates a density-dependent correction factor, $D = \frac{r}{c}$, as derived in [5]. Additionally, pest mortality due to predation by natural enemies follows a Holling Type II functional response [26]. Thus, the governing equation is formulated as follows:

$$\frac{dx}{dt} = x(t)\left(\frac{r(K - x(t))}{K + Dx(t)} - \frac{\gamma y(t)}{b + x(t)}\right) \tag{1}$$

2. Natural enemy $y(t)$:
The natural enemy population follows predator-prey dynamics similar to the Rosenzweig-MacArthur model [27], consuming pests at a rate of $\frac{\gamma y(t)}{b+x(t)}$. However, unlike traditional predator-prey models, this framework introduces an additional regulatory mechanism: when the natural enemy population grows excessively, it becomes prey to a higher-level predator. This higher-order predation follows a Lotka-Volterra-type interaction [28], helping to maintain ecological balance. As a result, the governing equation is formulated as follows:

$$\frac{dy}{dt} = \frac{\gamma y(t)x(t)}{b + x(t)} - dy(t) - \delta y(t)z(t) \tag{2}$$

3. Predator population $z(t)$:
The top predator $z(t)$ hunts natural enemies at a capture rate δ experiencing a natural death rate d. This predator-prey

Table 1: Parameters Description

Parameters	Meaning
γ	Capturing rate of pest population by natural enemy
b	Saturation in the amount of pest consumed
δ	Capturing rate of natural enemy by predators
d_1	Death rate of predator population
d	Death rate of natural enemy
K	Maximum carrying capacity of prey population
D	Rate of food consumption
r	Growth rate of pest population

relationship reflects the food chain dynamics studied by Mandal (2019) [4], where higher-level predators help regulate the population of intermediate consumers. Based on this interaction, the governing equation is formulated as follows:

$$\frac{dz}{dt} = \delta y(t)z(t) - d_1 z(t) \tag{3}$$

Combining all three equations, we have formulated the final equation as:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = x(t)\left(\frac{r(K-x(t))}{K+Dx(t)} - \frac{\gamma y(t)}{b+x(t)}\right) \\ \frac{dy}{dt} = \frac{\gamma y(t)x(t)}{b+x(t)} - dy(t) - \delta y(t)z(t) \\ \frac{dz}{dt} = \delta y(t)z(t) - d_1 z(t) \end{array} \right\} \tag{4}$$

where, $x, y, z > 0$.

The flow chart of the model is shown as

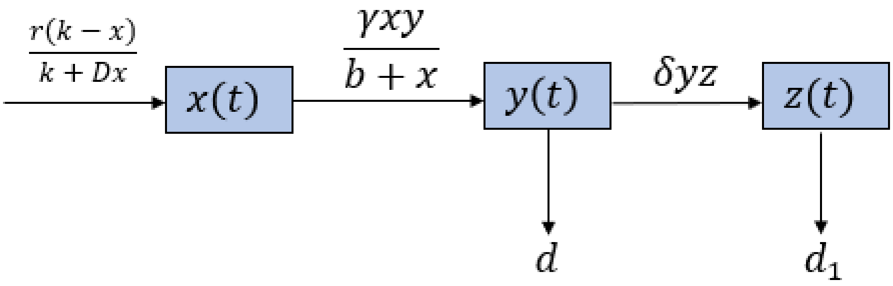


Fig. 1: *Schematic Diagram of the dynamical system*

3 Positivity and stability of Equilibrium Points

This section will discuss the positivity and local stability of the boundary and interior equilibrium points.

The equilibrium point $E_0(0,0,0)$ is trivial.

Theorem 1.*The system admits positive equilibrium points under the following conditions:*

1.The equilibrium point $E_1(K,0,0)$ *is always positive.*
2.The equilibrium point $E_2(x_2,y_2,0)$ *is positive for* $\gamma > \frac{d(b+K)}{K}$.
3.The equilibrium point $E^*(x^*,y^*,z^*)$ *is positive if* $x^* > \max\left\{0, \frac{db}{\gamma-d}\right\}$.

Proof. Equilibrium point $E_2(x_2,y_2,0)$ in the absence of predator is defined as

$$x_2 = \frac{db}{\gamma - d}$$

which exists if the capturing rate of the pest population is greater than the death rate of a natural enemy, that is, $\gamma > d$.

$$y_2 = \frac{br(K\gamma - Kd - bd)}{(\gamma - d)(K\gamma - Kd + Dbd)} \tag{5}$$

y_2 is positive if $\gamma \in \left(d\left(1 - \frac{bD}{K}\right), d\right) \cup \left(d\left(1 + \frac{b}{K}\right), \infty\right)$ and hence implies that equilibrium point E_2 is positive for $\gamma > d\left(1 + \frac{b}{K}\right)$.
Defining the interior equilibrium point $E^*(x^*,y^*,z^*)$ as

$$x^* = \frac{\sqrt{w^2 - a} - w}{2r\delta} > 0$$

where $w = r\delta b - Kr\delta + D\gamma d_1$ and $a = 4r\delta K(\gamma d_1 - r\delta b)$. Further, in specific for the parameter γ, x^* is real if $\gamma \in (-\infty, u_1) \cup (u_2, \infty)$ where
$u_{1,2} = \frac{r\delta}{D^2 d_1}\left[2K + DK - Db \mp 2\sqrt{K(D+1)(K-Db)}\right]$
$x^* > 0$ if $(\gamma \in (-\infty, u_1] \cup [u_2, \infty)) \cap \left(\gamma < \max\left\{\frac{r\delta(K-b)}{Dd_1}, \frac{r\delta b}{Dd_1}\right\}\right)$

$$y^* = \frac{d_1}{\delta}$$

$$z^* = \frac{\gamma x^*}{\delta(b + x^*)} - \frac{d}{\delta}$$

Further, $z^* > 0$ if $x^* > \frac{db}{\gamma - d}$.

Next, we will discuss the local stability of the equilibrium points.

3.1 Local Stability of the Equilibrium points

The general Jacobian Matrix of (4) is as follows:

$$J = \begin{bmatrix} \dfrac{r\left(K^2 - 2Kx - Dx^2\right)}{(K + Dx)^2} - \dfrac{\gamma by}{(b+x)^2} & -\frac{\gamma x}{b+x} & 0 \\ \dfrac{\gamma by}{(b+x)^2} & \frac{\gamma x}{b+x} - d - \delta z & -\delta y \\ 0 & \delta z & \delta y - d_1 \end{bmatrix}$$

The eigenvalues for trivial equilibrium point $E_0(0,0,0)$ are $\lambda_1 = r, \lambda_2 = -d, \lambda_3 = -d_1$. Therefore, $E_0(0,0,0)$ is a saddle point.

Theorem 2.*The equilibrium points are stable as per the following conditions:*

1.The axial equilibrium point $E_1(K,0,0)$ *is stable for* $\gamma < \frac{d(b+K)}{K}$.
2.The predator free equilibrium point $E_2^*(x_2^*, y_2^*, 0)$ *is stable if* $\frac{\delta br(K\gamma - Kd - bd)}{d_1(\gamma - d)(K\gamma - Kd + Dbd)} < 1$
and $\frac{Db^2d^2 - K^2d^2 - K^2\gamma^2 + Kb\gamma^2 - Kbd^2 + 2K^2\gamma d + DKb\gamma^2 + DKbd^2 - 2DKb\gamma d}{\gamma(K\gamma - Kd + Dbd)^2} > 0.$

3.The interior equilibrium point $E^*(x^*,y^*,z^*)$ *is stable if* $\frac{\gamma b d_1}{\delta(b+x^*)^2} - \frac{r\left(K^2-2Kx^*-Dx^{*2}\right)}{(K+Dx^*)^2} > 0$ *and* $x^* > \frac{db}{\gamma-d}$ *and possess Hopf bifurcation at* $\gamma^* = \frac{\delta r\left(2K-Db-2\sqrt{K(D+1)(K-Db)}+DK\right)}{D^2 d_1}$.

Proof. The eigenvalues corresponding to axial equilibrium point $E_1(K,0,0)$ are given by $\lambda_1 = \frac{-r}{(1+D)}, \lambda_2 = \frac{\gamma K}{(b+K)} - d < 0$ if $\gamma < \frac{d(b+K)}{K}, \lambda_3 = -d_1$ which shows that E_1 and E_2 cannot co-exists.
The characteristic equation for predator-free equilibrium point $E_2^*(x_2^*, y_2^*, 0)$ is given by :

$$(\delta y_2^* - d_1 - \lambda)\left(\lambda^2 - a_{11}\lambda + \frac{a_{21}\gamma x_2^*}{b+x_2^*}\right) = 0 \tag{6}$$

where $a_{11} = \frac{r\left(K^2-2Kx_2^*-Dx_2^{*2}\right)}{\left(K+Dx_2^*\right)^2} - \frac{\gamma b y_2^*}{(b+x_2^*)^2}$ and $a_{21} = \frac{\gamma b y_2^*}{(b+x_2^*)^2}$.
Therefore, the system is stable if $y_2^* < \frac{d_1}{\delta}$ that is, $\frac{\delta b r(K\gamma-Kd-bd)}{d_1(\gamma-d)(K\gamma-Kd+Dbd)} < 1$ and $a_{11} < 0$ if $\frac{Db^2d^2-K^2d^2-K^2\gamma^2+Kb\gamma^2-Kbd^2+2K^2\gamma d+DKb\gamma^2+DKbd^2-2DKb\gamma d}{\gamma(K\gamma-Kd+Dbd)^2} > 0$.
The characteristic equation corresponding to interior equilibrium point $E^*(x^*,y^*,z^*)$ is given by:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{7}$$

where,
$a_1 = \frac{\gamma b d_1}{\delta(b+x^*)^2} - \frac{r\left(K^2-2Kx^*-Dx^{*2}\right)}{(K+Dx^*)^2}, \quad a_2 = d_1\left(\frac{\gamma x^*}{(b+x^*)} - d\right) + \frac{\gamma^2 b d_1 x^*}{\delta(b+x^*)^3}$
$a_3 = d_1\left(\frac{\gamma x^*}{(b+x^*)} - d\right)\left(\frac{\gamma b d_1}{\delta(b+x^*)^2} - \frac{r\left(K^2-2Kx^*-Dx^{*2}\right)}{(K+Dx^*)^2}\right)$
By using Routh-Hurwitz criteria [29], the eigenvalues are negative if $\frac{\gamma b d_1}{\delta(b+x^*)^2} - \frac{r\left(K^2-2Kx^*-Dx^{*2}\right)}{(K+Dx^*)^2} > 0$ and $x^* > \frac{db}{\gamma-d}$.
Further, we will capsulate the Hopf Bifurcation of equilibrium point E^* at parameter $\gamma = \gamma^*$. Hopf Bifurcation is the appearance or the disappearance of a periodic orbit through a local change in the stability properties of a fixed point. At $\gamma^* = \frac{\delta r\left(2K-Db-2\sqrt{K(D+1)(K-Db)}+DK\right)}{D^2 d_1}$ we have $a_1(\gamma^*) = 0$ which implies $a_3(\gamma^*) = 0$. Thus, our characteristic equation reduces to $\lambda^3 + a_2\lambda = 0$. Since $a_2 > 0$ for $x^* > \max\left\{0, \frac{db}{\gamma-d}\right\}$ implies at $\gamma = \gamma^*$ the system undergoes Hopf bifurcation as one eigenvalue is zero and the other two eigenvalues are imaginary.

4 Local Bifurcation Analysis at E_1, E_2

In this section, local bifurcation analysis of bifurcation parameter γ, will be done on different equilibrium points E_1, E_2 of a dynamical system (4). Also, Sotomayor's theorem is applied to specify the type of bifurcation to different equilibrium points.

Theorem 3.***Sotomayor's Theorem*** *[30, 31]*
Let $\dot{x} = f(x,\mu)$ *be a system of differential equations in* R^n *depending on the single parameter* μ. *When* $\mu = \mu_0$ *assume that there is an equilibrium p for which the following hypotheses are satisfied:*

–$D_X f_{\mu_0}(p)$ *has simple eigenvalue 0 with right vector v and left eigenvector w.* $D_X f_{\mu_0}(p)$ *has k eigenvalues with negative real parts and* $n-k-1$ *eigenvalues with positive real parts(counting multiplicity).*
–$w^T\left(\frac{\partial f}{\partial \mu_0}\right) \neq 0$
–$w^T\left(D_X^2(p,\mu_0)(v,v)\right) \neq 0$

Then the system $\dot{x} = f(x,\mu)$ *experience a saddle node bifurcation at the equilibrium point p as the parameter* μ *varies through the bifurcation value* $\mu = \mu_0$. *Moreover, if the bifurcation condition changes to*

–$w^T\left(\frac{\partial f}{\partial \mu_0}\right) = 0,\ w^T\left(\frac{\partial^2 f}{\partial \mu \partial x}(p,\mu_0)(v)\right) \neq 0$

the system has a transcritical bifurcation of the equilibrium point p and if the condition $(2),(3)$ *are changed to*

-$w^T\left(\frac{\partial f}{\partial \mu}(p,\mu_0)\right)=0, w^T\left(\frac{\partial^2 f}{\partial \mu \partial x}(p,\mu_0)(v)\right)\neq 0$
-$w^T\left(D_X^2(p,\mu_0)(v,v)\right)=0, w^T\left(D_X^3(p,\mu_0)(v,v,v)\right)\neq 0$

Then the system has a pitchfork bifurcation at p.

Theorem 4. *Let parameter $\gamma=\gamma^*=\frac{d(b+K)}{K}$ then the dynamical behaviour of the system (4) at $E_1(K,0,0)$ exhibits :*
1. *No Saddle-Node bifurcation.*
2. *A transcritical bifurcation as the bifurcation parameter γ crosses γ^*.*

Proof. Let the vector function $F:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$ be defined as: $F(X,\gamma)=\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{bmatrix}=\begin{bmatrix}x(t)\left(\frac{r(K-x(t))}{K+Dx(t)}-\frac{\gamma y(t)}{b+x(t)}\right)\\ \frac{\gamma y(t)x(t)}{b+x(t)}-dy(t)-\delta y(t)z(t)\\ \delta y(t)z(t)-d_1 z(t)\end{bmatrix}$,
where $X=(x,y,z)^T$ represents the populations of pest, natural enemy, and predator respectively.
Given that at equilibrium point $E_1(K,0,0)$ having zero eigenvalues $(\lambda_2=0)$ with bifurcation parameter as $\gamma=\gamma^*$ with all the other parameters as fixed and positive, then the matrix is given by

$$J_1(\gamma^*)=\begin{bmatrix}\frac{-r}{1+D} & -\frac{\gamma^* K}{b+K} & 0\\ 0 & 0 & 0\\ 0 & 0 & -d_1\end{bmatrix}$$

Let $\phi=(\phi_1,\phi_2,\phi_3)^T$ be the right eigenvector. On solving $(J_1-\lambda I)\phi=0$ we will get

$$\phi=(-\gamma K(1+D),\quad r(b+K),\quad 0)^T$$

and on solving $(J_1^T-\lambda I)\psi=0$ for left eigenvector we obtain $\psi=(\psi_1,\psi_2,\psi_3)^T$ as

$$\psi=(0,\quad \psi_2,\quad 0)^T$$

where, ψ_2 is any nonzero real number.
Consider,

$$\frac{\partial F}{\partial \gamma}=F_\gamma=\left(\frac{-xy}{b+x},\quad \frac{xy}{b+x},\quad 0\right)^T$$

hence we obtain that $F_\gamma(E_1,\gamma^*)=(0,\ 0,\ 0)^T$. Now, evaluating the condition $\psi^T[F_\gamma(E_1,\gamma^*)]$ we get,

$$\psi^T[F_\gamma(E_1,\gamma^*)]=0$$

Therefore, by Sotomayor's Theorem (3) saddle node bifurcation does not exist. Further, evaluating the condition for transcritical bifurcation
Now, we have

$$\psi^T[DF_\gamma(E_1,\gamma^*)\phi]=Kr\psi_2\neq 0$$

where, DF_γ is the derivative of F with respect to $X=(x,y,z)^T$, Also

By the above conditions, we can say that system (4) experiences a Transcritical bifurcation at equilibrium point E_1. Hence, the theorem is proved.

Theorem 5. *Let parameter $\gamma=\gamma^*$ then the dynamical behaviour of the system (4) near $E_2(x_2,y_2,0)$ neither has saddle-node bifurcation nor transcritical bifurcation.*

Proof. Given that at equilibrium point $E_2(x_2,y_2,0)$ having zero eigen value $(\lambda_3=0)$ with bifurcation parameter as

$$\gamma^*=\frac{2Kdd_1-Dbdd_1+K\delta br\pm b\sqrt{D^2d^2{d_1}^2-2DK\delta dd_1 r+K^2\delta^2r^2-4K\delta dd_1 r}}{2Kd_1}$$

with all the other parameter as fixed and positive, then the matrix is given by

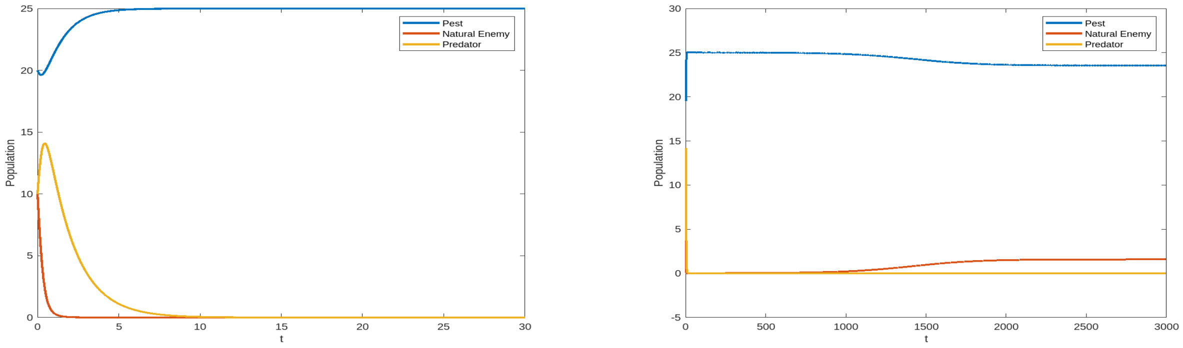
$$J_2^* = J_2^*(E_2^*, \gamma^*) = \begin{bmatrix} \dfrac{r(K^2 - 2Kx_2^* - Dx_2^{*2})}{\left(K + Dx_2^*\right)^2} - \dfrac{\gamma^* b y_2^*}{(b + x_2^*)^2} & -d & 0 \\ \dfrac{\gamma^* b y_2^*}{(b + x_2^*)^2} & 0 & -\delta y_2^* \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\Phi = (\phi_1, \phi_2, \phi_3)^T$ be the right eigen vector. On solving $(J_1 - \lambda I)\Phi = 0$ we will get $\Phi = \left(\frac{b\delta\gamma}{(\gamma-d)^2}, \left(\frac{r(K^2-2Kx_2^*-Dx_2^{*2})}{\left(K+Dx_2^*\right)^2} - \frac{\gamma^* b y_2^*}{(b+x_2^*)^2} \right) \frac{b\delta\gamma}{d(\gamma-d)^2}, 1 \right)^T$

Let left eigenvector of J_2^{*T} is $\Psi = (\psi_1, \psi_2, \psi_3)^T$. On calculating we get the vector as $\Psi = (0,\ 0,\ \psi_3)^T$ where, ψ_3 is any nonzero real number.
Consider,

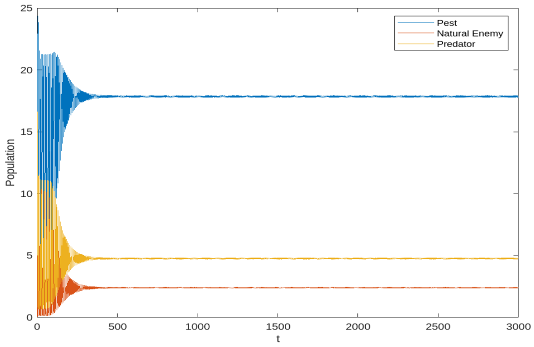
$$\frac{\partial F}{\partial \gamma} = F_\gamma = \left(\frac{-xy}{b+x}, \quad \frac{xy}{b+x}, \quad 0 \right)^T$$

hence we obtain that $F_\gamma(E_2, \gamma^*) = (0,\ 0,\ 0)^T$, where F is vector function, defined in Theorem (4).
Since, $\psi^T[F_\gamma(E_2, \gamma^*)] = 0$ and $\psi^T[DF_\gamma(E_2, \gamma^*)\phi] = 0$, where, DF_γ is the derivative of F with respect to $X = (x, y, z)^T$
Hence, E_2 has neither saddle-node bifurcation nor transcritical bifurcation at $\gamma = \gamma^*$.



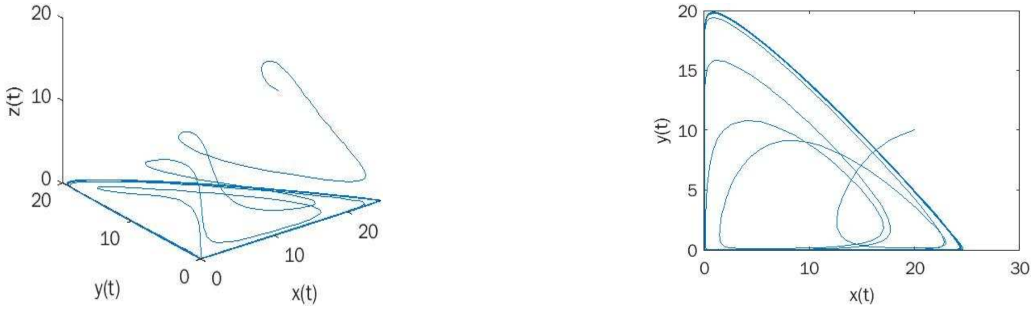
(a) $E_1(24.9995, 0, 0)$ *at* $\gamma = 0.84$

(b) $E_2(23.5257, 1.5579, 0)$ *at* $\gamma = 0.885$



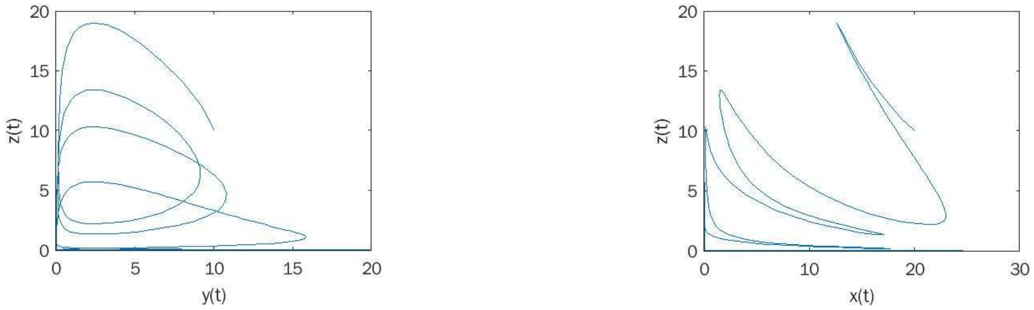
(c) $E^*(17.8805, 2.3667, 4.7676)$ *at* $\gamma = 2.27$

Fig. 2: Dynamics of axial(E_1), predator-free(E_2) and interior(E^*) equilibrium Points



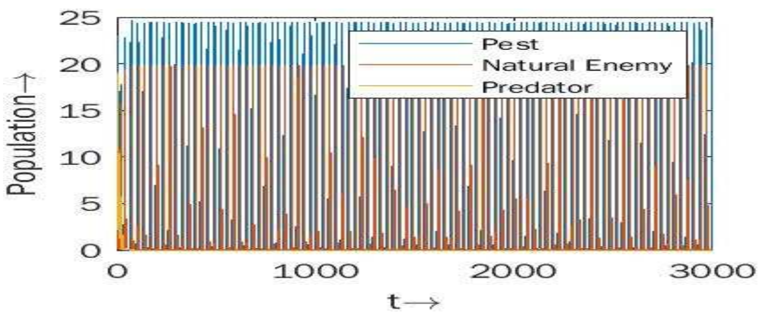
(a) 3D Plot for Unstable system

(b) Unstable behaviour in $x-y$ plane



(c) Unstable behaviour in $y-z$ plane

(d) Unstable behaviour in $x-z$ plane



(e) Time series for Unstable behaviour of the system

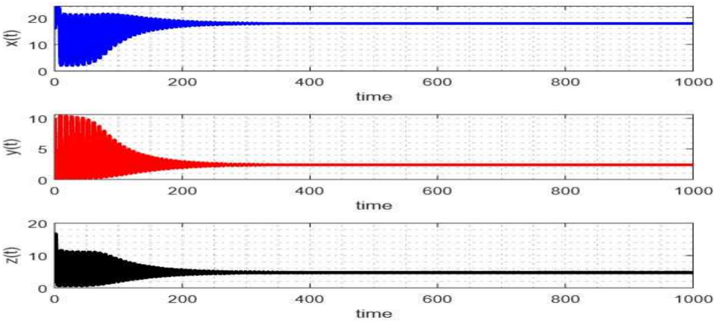
Fig. 3: Behaviour of the system for $\gamma = 3.101$ at initial $[20\,10\,10]$

(a) 3D Plot for stable system

(b) Stable behaviour in y-z plane

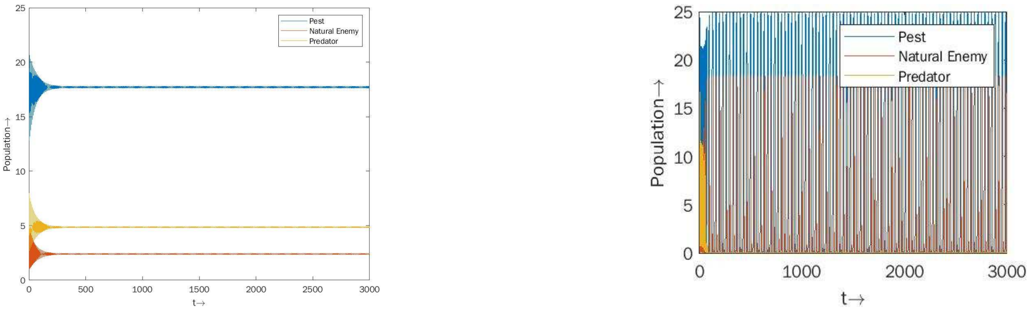
(c) Stable behaviour in x-z plane

(d) Stable behaviour in x-y plane



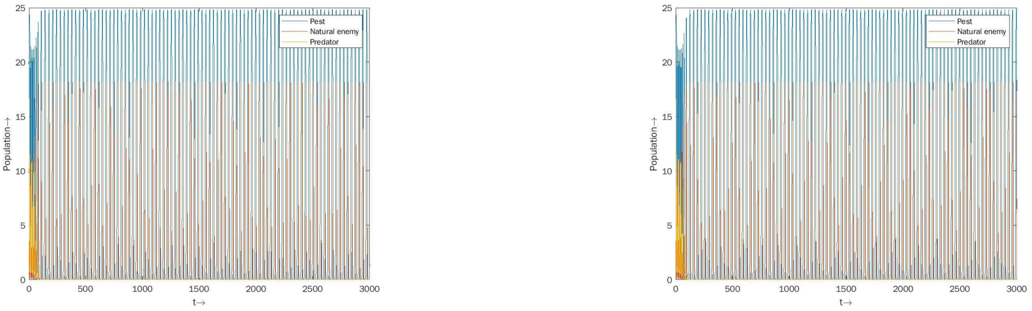
(e) Time series for stable behaviour of the system

Fig. 4: Behaviour of the system for $\gamma = 2.27$ at initial $[20\,10\,10]$



(a) Initial condition [15 5 5]

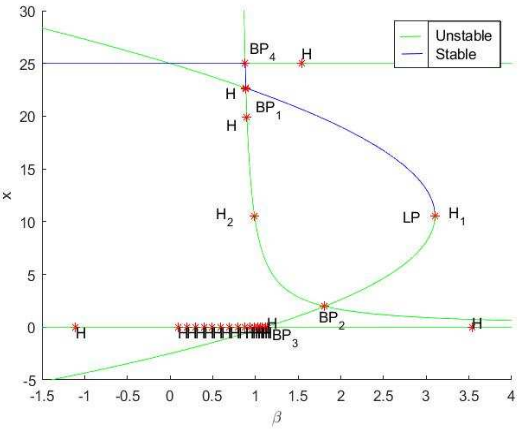
(b) Initial condition [20 10 10]



(c) when $r = 1.06$

(d) when $D = 0.22$

Fig. 5: Dynamical system with initial condition and parameters r and D



(c)

Fig. 6: *Bifurcations and stability*

5 Numerical section

This section presents the numerical validation of the theoretical results derived earlier, using MATLAB simulations. The parameter values are set as follows:

$r = 1.07, K = 25, \gamma = 2.27, b = 2.5, D = 0.2, \delta = 0.25, d = 0.8, d_1 = 0.6$. All the graphs in (Figure 2) are plotted for initial condition $[20,10,10]$.The axial equilibrium point $E_1(24.9995,0,0)$ has eigenvalues $(-0.8916,-0.0364,-0.6)$ and is positive for $\gamma < 0.88$ (Figure 2a). The predator free equilibrium point $E_2(23.5257,1.5579,0)$ has eigenvalues $(-0.8027,-0.0051,-0.2106)$ which is positive for $\gamma > 0.88$ and remains stable for $0.6491 < 1$. The interior equilibrium point $E^* = (17.8805,2.3667,4.7676)$ has eigenvalues $(-0.4377,-0.0198+0.8694i,-0.0198-0.8694i)$, exist for $x^* > 1.3605$ and is stable for $0.4672 > 0$ (Figure 2). These findings align with the stability conditions outlined in Theorems(1,2). Further, as per the analytic proof and numerical results E^* undergoes Hopf bifurcation as the bifurcation parameter γ reaches 3.101. Before this threshold, the system remains stable at the initial condition [20 10 10], but beyond this point, it transitions to oscillatory behaviour. This illustrates how small changes in γ can significantly alter system dynamics, leading to stability loss. The numerical simulations reinforce the analytical results, revealing how sensitive the system is to parameter variations. The stability of equilibrium points hinges on initial conditions and key parameters γ, r and D, where even slight shifts can trigger dramatic changes in system behaviour. This insight is particularly crucial for ecological and epidemiological models, where minor fluctuations can alter long-term outcomes.

5.1 Sensitivity based on initial conditions, r and D.

The system's behavior is highly sensitive to initial conditions and parameter variations, particularly r and D. For $\gamma = 2.30$, the system stabilizes at [15,5,5] (5a), but a slight perturbation to [20,10,10] (5b) results in instability, demonstrating its dependence on initial states. For $\gamma = 2.27$, stability is observed when $r = 1.07$ or $D = 0.2$ (2c), whereas reducing r to 1.06 (5d) or increasing D to 0.22 leads to instability. These findings underscore the critical influence of small parameter variations on system stability and long-term behaviour. Such sensitivity is a well-known property of nonlinear dynamical models, where small perturbations can significantly impact system trajectories. This phenomenon is widely documented in ecological and epidemiological models, emphasizing the necessity of precise parameter estimation [32,33]. Future research could integrate bifurcation analysis and extensive parameter-space exploration to enhance the understanding of the system's stability and transition dynamics.

5.2 Bifurcation Theory (Continuation of Equilibrium point and branch points)

This section finds a few qualification theories of bifurcation depending upon one free parameter γ. We would be checking the continuation of the equilibrium and branch points for Hopf bifurcations (H_i), branch point (BP_i), limit point (fold;

saddle-node) bifurcation(LP) and Neutral Saddle Equilibrium (H). We would be using MATCONT to obtain these points. The theoretically obtained equilibrium point $(17.8525, 2.4, 4.7647)$ for the parametric values mentioned above is taken as an initial equilibrium and continuation of this equilibrium resulted in the branch points BP_1, BP_2 and BP_3, supercritical-Hopf bifurcation point H_1 as first Lyapunov coefficient is negative and Limit point(fold; saddle-node) bifurcation LP. Next, the continuation of branch point BP_2 resulted in branch points BP_1 and BP_4, a Hopf bifurcation point H_2 and two Neutral saddle equilibriums denoted by H. Further, the continuation of branch points BP_3 and BP_4 resulted in Neutral saddle equilibrium points as shown in (Figure 6). Table [2] infers the values at bifurcation points.

Table 2: Boundary equilibrium Point (E_1) without control

Bifurcation	$[x, y, z]$	γ	Coefficient
H_1	$[10.5544 \quad 2.4 \quad 6.8290]$	3.1011	$l_1 = -1.3130 * e^{02}$
H_2	$[10.5544 \quad 7.5218 \quad 0]$	0.9895	$l_1 = -2.3497 * e^{-03}$
BP_1	$[22.6616 \quad 2.4 \quad 0]$	0.8883	-
BP_2	$[1.9796 \quad 2.4 \quad 0]$	1.8103	-
BP_3	$[0 \quad 2.4 \quad -3.2]$	1.1146	-
BP_4	$[25 \quad 0 \quad 0]$	0.88	-

6 Discussion and Conclusion

This paper has discussed the dynamical analysis of the pest-natural enemy-predator model. The growth rate of the pest population is considered as per the Smith model. From the analytical results, we have captured the dynamics of the system. We have discussed the positivity of four equilibrium points. Further, local stability of $E_0(0,0,0)$, axial equilibrium point $E_1(K,0,0)$, predator-free equilibrium point E_2 and interior equilibrium point E^* are discussed with the help of Routh-Hurwitz criteria. Further, Sotomayor's theorem is used to determine the dynamical system's local bifurcation. It is observed that the dynamical system (4) does not undergo saddle-node bifurcation but experiences transcritical bifurcation near the axial equilibrium point when a specific bifurcation parameter γ passes a critical value γ^*. It was also observed that the dynamical system (4) neither experiences saddle-node bifurcation nor transcritical bifurcation near predator-free equilibrium when bifurcation parameter $\gamma = \gamma^*$. However, the dynamical system undergoes the Hopf bifurcation at E^* when parameter γ passes γ^*. Finally, the analytical results are validated numerically and changes in the system's dynamics with variation in initial conditions, r and D are also show The positivity, stability and bifurcation theory are explored using MATCONT to show Hopf bifurcation, branch point, limit point and limit point cycle. The results of the continuation of the equilibrium point and branch points have shown the existence of Limit point (fold; saddle-node) bifurcation, multiple Branch points, supercritical Hopf bifurcation points and Neutral saddle equilibrium. The results analysed in this paper attempt to show the rich dynamics of the system.

Declarations

Competing interest: The authors declare no conflict of interest.

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