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Generalizations of p Numerical Radii Inequalities for Operators

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Abstract: Let $T = [T_{kj}]_{k,j=1}^{n}$ be an operator matrix, where T_{kj} is a Schatten $p-$class operator. Then

$$w_p(T) \leq \left(\sum_{j=1}^{n} w_p^p(T_{jj}) + \sum_{\substack{k,j=1 \\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}}$$

for $1 \leq p \leq 2$, and

$$w_p(T) \leq n^{1-\frac{2}{p}} \left(\sum_{j=1}^{n} w_p^p(T_{jj}) + \sum_{\substack{k,j=1 \\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}}$$

for $2 \leq p \leq \infty$. These inequalities generalize a recent $p-$numerical radius inequalities. The first inequality refines another recent $p-$numerical radius inequality.

Keywords: Inequality; Numerical radius; Operator; Norm; Schatten p-norm.

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1 Introduction

Let $B(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathbb{H}$ and $k(\mathbb{H})$ denote the class of compact operators in $B(\mathbb{H})$. For $A \in B(\mathbb{H})$, let $w(A)$ and $||A||$ denote the numerical radius and the usual operator norm respectively, where

$$w(A) = \sup_{||x||=1} |\langle Ax, x\rangle|$$

and

$$||A|| = \sup_{||x||=1} ||Ax||.$$

For $A \in B(\mathbb{H})$, the schatten p-norm of A is given by:

$$||A||_p = (\sum_{j=1}^{n} s_j^p(A))^{\frac{1}{p}} = (tr|A|^p)^{\frac{1}{p}},$$

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where $1 \leq p \leq \infty$. Note that $||A||_\infty = ||A||$ is the usual operator norm of A.

The operator A belongs to the trace class $B_1(\mathbb{H})$ if $||A||_1 = tr|A|$ is finite, it belongs to the Hilbert Schmidt class $B_2(\mathbb{H})$ if $||A||_2 = (tr|A|^2)^{\frac{1}{2}}$ is finite. In general $A \in B(\mathbb{H})$ belongs to the Schatten p-class $B_p(\mathbb{H})$for $1 \leq p \leq \infty$ if $||A||_p = (tr|A|^p)^{\frac{1}{p}}$ is finite.

The singular value of the operator $A \in B(H)$ is the eigenvalue of the absolute value of A, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A. The singular values of A are ordered descendingly as $s_1(A) \geq s_2(A) \geq ....$ For more details and generalizations for singular value inequality, we advise the reader to read [$4-11$].

Note that

$$||A||_p = ||\ |A|\ ||_p. \tag{1}$$

$$||A||_p^2 = ||AA^*||_{p/2}. \tag{2}$$

For more details about Schatten $p-$norms, we refer to [12].

The Cartesian decomposition of an operator $A \in B(H)$ is given by $A = Re(A) + iIm(A)$, where $Re(A) = \frac{A+A^*}{2}$ and $Im(A) = \frac{A-A^*}{2i}$. $Re(A)$ and $Im(A)$ are Hermitian matrices. The operator A is called accretive if $Re(A)$ is positive semidefinite operator and A is called dissipative if $Im(A)$ is positive semidefinite operator.

It is shown that, [18], the numerical radius of $A \in B(\mathbb{H})$ can be written as

$$w(A) = \sup_{\theta \in \mathbb{R}} ||Re(e^{i\theta}A)||. \tag{3}$$

Some remarkable numerical radius inequalities, are given see e.g.[3] and [17].

The authors in [1] define for $A \in B(\mathbb{H})$, a more general setting for numerical radius, if $A \in B(\mathbb{H})$ and $N(.)$ is a norm, the generalized numerical radius of A is defined as

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N(Re(e^{i\theta}A)). \tag{4}$$

If N is the trace norm, then we get $w_1(A)$. If N is the Hilbert Schmidt norm then we get $w_2(A)$ and more general if N is the Schatten p-norm, then we get $w_p(A)$. Note that if $N(.)$ is the usual operator norm then the relation (4) reduces to relation (3.)

The authors in [2], proved that if $T = [T_{kj}]_{k,j=1}^n$, where $T_{kj} \in B_2(\mathbb{H})$, then

$$w_2(T) \leq \left(\sum_{j=1}^n w_2^2(T_{jj}) + \sum_{\substack{k,j=1 \\ k \neq j}}^n ||T_{kj}||_2^2 \right)^{\frac{1}{2}}. \tag{5}$$

The authors in [15], proved that for $T \in B_p(\mathbb{H})$

$$w_p^p(T) \leq ||Re(T)||_p^p + ||Im(T)||_p^p \quad for\ 1 \leq p \leq 2 \tag{6}$$

and

$$w_p^p(T) \leq 2^{\frac{p}{2}-1}(||Re(T)||_p^p + ||Im(T)||_p^p) \quad for\ 2 \leq p < \infty. \tag{7}$$

Recently, the authors in [16], proved that if $T = [T_{kj}]_{k,j=1}^n$, where $T_{kj} \in B_p(\mathbb{H})$, then

$$w_p(T) \leq \sum_{j=1}^n \left(||Re(T_{jj})||_p^p + ||Im(T_{jj}))||_p^p + 2^{2-p} \sum_{\substack{k=1 \\ k \neq j}}^n ||T_{kj}||_p^p \right)^{\frac{1}{p}} \quad for\ 1 \leq p \leq 2 \tag{8}$$

and

$$w_p(T) \leq (\sqrt{2}n)^{1-\frac{2}{p}} \sum_{j=1}^n \left(||Re(T_{jj})||_p^p + ||Im(T_{jj}))||_p^p + 2^{2-p} \sum_{\substack{k=1 \\ k \neq j}}^n ||T_{kj}||_p^p \right)^{\frac{1}{p}} \quad for\ 2 \leq p < \infty. \tag{9}$$

In this paper, we give a generalization of inequality (5), a refinement of inequality (8) and a new numerical radius inequalities for sums, products and direct sums of operators via unitarily invariant norms and Schatten p-norms.

2 Numerical radius inequality for Schatten $p-$norms

In this section, we give a generalization of inequality (5) and a new p-numerical radius inequalities. To prove our results in this paper, we need the following lemmas. The first lemma is given by Kittaneh in [13], the second lemma is well-known. The first theorem in this paper is ready to present.

Lemma 1. *Let $T=[T_{kj}]_{i,j=1}^{n}$ be an operator matrix, where $T_{kj}\in B_p(\mathbb{H})$ for $k,j=1,...,n$. Then*

$$\sum_{k,j=1}^{n}||T_{kj}||_p^p\le||T||_p^p\le n^{p-2}\sum_{k,j=1}^{n}||T_{kj}||_p^p \tag{10}$$

for $2\le p\le\infty$, and

$$n^{p-2}\sum_{k,j=1}^{n}||T_{kj}||_p^p\le||T||_p^p\le\sum_{k,j=1}^{n}||T_{kj}||_p^p \tag{11}$$

for $1\le p\le 2$.

Lemma 2. *Let $a,b\ge 0$. Then*

$$a^p+b^p\le(a+b)^p\le 2^{p-1}(a^p+b^p),\ for\ \ p\ge 1, \tag{12}$$

and

$$2^{p-1}(a^p+b^p)\le(a+b)^p\le a^p+b^p,\ for\ 0\le p\le 1. \tag{13}$$

Theorem 1. *Let $T=[T_{kj}]_{k,j=1}^{3}$ be an operator matrix , where $T_{kj}\in B_p(\mathbb{H})$ for $k,j=1,2,...,3$. Then*

$$w_p(T)\le 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}w_p^p(T_{jj})+\sum_{\substack{k,j=1\\k\neq j}}^{3}||T_{kj}||_p^p\right)^{\frac{1}{p}}\quad for\ 2\le p\le\infty \tag{14}$$

and

$$w_p(T)\le\left(\sum_{j=1}^{3}w_p^p(T_{jj})+\sum_{\substack{k,j=1\\k\neq j}}^{3}||T_{kj}||_p^p\right)^{\frac{1}{p}}\quad for\ 1\le p\le 2. \tag{15}$$

Proof. Throughout this proof, let $A=\dfrac{e^{i\theta}T_{12}+e^{-i\theta}T_{21}^*}{2}$, $B=\dfrac{e^{i\theta}T_{13}+e^{-i\theta}T_{31}^*}{2}$ and $C=\dfrac{e^{i\theta}T_{23}+e^{-i\theta}T_{32}^*}{2}$, where

$$\begin{aligned}2||A||_p^p=2\left|\left|\frac{e^{i\theta}T_{12}+e^{-i\theta}T_{21}^*}{2}\right|\right|_p^p&=2^{1-p}||e^{i\theta}T_{12}+e^{-i\theta}T_{21}^*||_p^p\\&\le 2^{1-p}(||e^{i\theta}T_{12}||_p+||e^{-i\theta}T_{21}^*||_p)^p(by\ the\ triangle\ inequality)\\&=2^{1-p}(||T_{12}||_p+||T_{21}^*||_p)^p\\&=2^{1-p}(||T_{12}||_p+||T_{21}||_p)^p\\&\le 2^{1-p}2^{p-1}(||T_{12}||_p^p+||T_{21}||_p^p)(by\ inequlitiy\ (12))\\&=(||T_{12}||_p^p+||T_{21}||_p^p).\end{aligned} \tag{16}$$

In the same procedure, we can easily show that

$$2||B||_p^p\le||T_{13}||_p^p+||T_{31}||_p^p \tag{17}$$

and

$$2||C||_p^p\le||T_{23}||_p^p+||T_{32}||_p^p. \tag{18}$$

Now, we prove inequality (14):

$$\begin{aligned}
w_p(T) &= \sup_{\theta\in\mathbb{R}} ||Re(e^{i\theta}T)||_p \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left(\left|\left|e^{i\theta}\begin{bmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix} + e^{-i\theta}\begin{bmatrix} T_{11}^* & T_{21}^* & T_{31}^*\\ T_{12}^* & T_{22}^* & T_{32}^*\\ T_{13}^* & T_{23}^* & T_{33}^*\end{bmatrix}\right|\right|_p\right) \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left(\left|\left|\begin{bmatrix} e^{i\theta}T_{11}+e^{-i\theta}T_{11}^* & e^{i\theta}T_{12}+e^{-i\theta}T_{21}^* & e^{i\theta}T_{13}+e^{-i\theta}T_{31}^*\\ e^{i\theta}T_{21}+e^{-i\theta}T_{12}^* & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & e^{i\theta}T_{23}+e^{-i\theta}T_{32}^*\\ e^{i\theta}T_{31}+e^{-i\theta}T_{13}^* & e^{i\theta}T_{32}+e^{-i\theta}T_{23} & e^{i\theta}T_{33}+e^{-i\theta}T_{33}\end{bmatrix}\right|\right|_p\right) \\
&= \sup_{\theta\in\mathbb{R}}\left(\left|\left|\begin{bmatrix} Re(e^{i\theta}T_{11}) & A & B\\ A^* & Re(e^{i\theta}T_{22}) & C\\ B^* & C^* & Re(e^{i\theta}T_{33})\end{bmatrix}\right|\right|_p\right) \\
&\le 0.8\sup_{\theta\in\mathbb{R}} 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}||Re(e^{i\theta}T_{jj})||_p^p + ||A||_p^p + ||B||_p^p + ||C||_p^p + ||A^*||_p^p + ||B^*||_p^p + ||C^*||_p^p\right)^{\frac{1}{p}} \\
&\quad (by\ inequality\ (10)) \\
&= 0.8\sup_{\theta\in\mathbb{R}} 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}||Re(e^{i\theta}T_{jj})||_p^p + 2||A||_p^p + 2||B||_p^p + 2||C||_p^p\right)^{\frac{1}{p}} \\
&\le 0.8\sup_{\theta\in\mathbb{R}} 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}||Re(e^{i\theta}T_{jj})||_p^p + (||T_{12}||_p^p + ||T_{21}||_p^p) + (||T_{13}||_p^p + ||T_{31}||_p^p) + (||T_{23}||_p^p + ||T_{32}||_p^p)\right)^{\frac{1}{p}} \\
&\quad (by\ inequlities\ (16), (17)\ and\ (18)) \\
&= \sup_{\theta\in\mathbb{R}} 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}||Re(e^{i\theta}T_{jj})||_p^p + \sum_{\substack{k,j=1\\ k\neq j}}^{3}||T_{kj}||_p^p\right)^{\frac{1}{p}} \\
&= 3^{1-\frac{2}{p}}\left(\sup_{\theta\in\mathbb{R}}\sum_{j=1}^{3}||Re(e^{i\theta}T_{jj})||_p^p + \sup_{\theta\in\mathbb{R}}\sum_{\substack{k,j=1\\ k\neq j}}^{3}||T_{kj}||_p^p\right)^{\frac{1}{p}} \\
&= 3^{1-\frac{2}{p}}\left(\sum_{j=1}^{3}w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{3}||T_{kj}||_p^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Replacing the same steps in proving inequality (14) and by using inequalities (11), we can give inequality (15).

Using the same techniquefor $T = [T_{kj}]_{k,j=1}^n$, used in the proof of inequality (14), we can give the following generalization of inequalities (14) and (15).

Theorem 2. *Let $T = [T_{kj}]_{k,j=1}^n$ be an operaor matrix, where $T_{kj} \in B_p(\mathbb{H})$ for $k, j = 1, 2, \ldots, n$. Then*

$$w_p(T) \le n^{1-\frac{2}{p}}\left(\sum_{j=1}^{n}w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{n}||T_{kj}||_p^p\right)^{\frac{1}{p}} \quad for\ 2 \le p \le \infty, \tag{19}$$

and

$$w_p(T) \le \left(\sum_{j=1}^{n}w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{n}||T_{kj}||_p^p\right)^{\frac{1}{p}} \quad for\ 1 \le p \le 2. \tag{20}$$

In particular if $p=2$ in inequality (19), then we retain inequality (5).

*Remark.*Inequality (20) is sharper than inequality (8). To show this:

$$w_p(T) \le \left(\sum_{j=1}^{n} w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}}$$

$$w_p(T) \le \left(\sum_{j=1}^{n} ||Re(T_{jj})||_p^p + ||Im(T_{jj})||_p^p + \sum_{\substack{k,j=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}} \quad (by\ inequality\ 6)$$

$$w_p(T) \le \left(\sum_{j=1}^{n} ||Re(T_{jj})||_p^p + ||Im(T_{jj})||_p^p + 2^{2-p} \sum_{\substack{k,j=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}} \quad (since\ 2^{2-p} \ge 1)$$

$$w_p(T) \le \sum_{j=1}^{n} \left(||Re(T_{jj})||_p^p + ||Im(T_{jj}))||_p^p + 2^{2-p} \sum_{\substack{k=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \right)^{\frac{1}{p}} \quad (by\ inequality\ (13)).$$

Corollary 1.*If $T_1, T_2, ..., T_n \in B_p(\mathbb{H})$, then*

$$w_p(T_1 \oplus T_2 \oplus ... \oplus T_n) \le (\sum_{j=1}^{n} w_p^p(T_j))^{\frac{1}{p}}, \tag{21}$$

for $1 \le p \le 2$, and

$$w_p(T_1 \oplus T_2 \oplus ... \oplus T_n) \le n^{1-\frac{2}{p}} (\sum_{j=1}^{n} w_p^p(T_j))^{\frac{1}{p}}. \tag{22}$$

for $2 \le p \le \infty$.

The authors in [19] proved that if $T = [T_{kj}]_{k,j=1}^n$ accretive dissipative matrix , then

$$\sum_{\substack{k,j=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \le (n-1)2^{p-2} \sum_{j=1}^{n} ||T_{jj}||_p^p,\ for\ p \ge 2, \tag{23}$$

$$\sum_{\substack{k,j=1\\ k\neq j}}^{n} ||T_{kj}||_p^p \le (n-1)2^{2-p} \sum_{j=1}^{n} ||T_{jj}||_p^p,\ for\ 0 < p \le 2, \tag{24}$$

and

$$||T||_2^2 \le n \sum_{j=1}^{n} ||T_{jj}||_2^2. \tag{25}$$

Theorem 3.*Let $T = [T_{kj}]_{k,j=1}^n$ be a accretive-dissipative matrix, where $T_{kj} \in B_p(\mathbb{H})$ for $k,j = 1,2,...,n$. Then*

$$w_p(T) \le n^{1-\frac{2}{p}} \left(\sum_{j=1}^{n} w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{n} (n-1)2^{p-2}||T_{jj}||_p^p \right)^{\frac{1}{p}} \quad for\ 2 \le p \le \infty, \tag{26}$$

and

$$w_p(T) \le \left(\sum_{j=1}^{n} w_p^p(T_{jj}) + \sum_{\substack{k,j=1\\ k\neq j}}^{n} (n-1)2^{2-p}||T_{jj}||_p^p \right)^{\frac{1}{p}} \quad for\ 1 \le p \le 2. \tag{27}$$

*Proof.*We reach inequality (26) by using inequalities (19) and (23). Similarly, we give inequality (27) by using inequalities (20) and (24).

Theorem 4.*Let $T=[T_{kj}]_{k,j=1}^{n}$ be an operator matrix , where $T_{kj}\in B_p(\mathbb{H})$ for $k,j=1,2,...,n$ and $e^{i\theta}T$ is accretive. Then*

$$w_2(T)\leq\sqrt{n}(\sum_{j=1}^{n}(w_2^2(T_{jj}))^{\frac{1}{2}}. \tag{28}$$

Proof.

$$\begin{aligned} w_2(T) &= \sup_{\theta\in\mathbb{R}}||Re(e^{i\theta}T)||_2 \\ &= \sup_{\theta\in\mathbb{R}}\left(\left|\left|\begin{bmatrix} Re(e^{i\theta}T_{11}) & \cdots & \frac{e^{i\theta}T_{1n}+e^{-i\theta}T_{n1}^*}{2} \\ \vdots & \ddots & \vdots \\ \frac{e^{i\theta}T_{n1}+e^{-i\theta}T_{1n}^*}{2} & \cdots & Re(e^{i\theta}T_{nn}) \end{bmatrix}\right|\right|_2\right) \\ &\leq \sqrt{n}\sup_{\theta\in\mathbb{R}}(\sum_{j=1}^{n}||Re(e^{i\theta}T_{jj})||_2^2)^{\frac{1}{2}}(by\ inequality\ (25)) \\ &\leq \sqrt{n}(\sum_{j=1}^{n}w_2^2(T_{jj}))^{\frac{1}{2}}. \end{aligned}$$

*Remark.*Inequality (28) is sharper than, in some cases, inequality (5). To show this, we consider the following example.

*Example 1.*Let $T=\begin{bmatrix}4 & 5\\ 5 & 0\end{bmatrix}$. Then the right hand side of inequality (28) is $4\sqrt{2}$while the right hand side of inequality (5) is $\sqrt{66}$.

3 Generalized numerical radius inequalities for operators

In this section, we give a new numerical radius inequalities via unitarily invariant norms and Schatten p-norms inequalities for sums, products and direct sums of operators. To prove the next theorems, we need the following lemmas. The first lemma is proved in [14] and the second lemma is proved in [8].

Lemma 3.*Let $K,L\in B(H)$ and $N(.)$ be any unitarily invariant norm. Then*

$$2N(KL^*)\leq N(K^*K+L^*L). \tag{29}$$

Lemma 4.*If $K,L\in B(H)$, then*

$$2s_j(KL^*+LK^*)\leq s_j^2\left(\begin{bmatrix}K & L\\ L & K\end{bmatrix}\right),\quad for\ j=1,2,...,n. \tag{30}$$

In particular for any unitarily invariant norm

$$2N(KL^*+LK^*)\leq N^2\left(\begin{bmatrix}K & L\\ L & K\end{bmatrix}\right). \tag{31}$$

Lemma 5.*If A,B,X and $Y\in B(H)$ such that X and Y are positive semidefinite and $N(.)$ is any unitarily invariant norm, then*

$$2N(AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^*+BY^{\frac{1}{2}}X^{\frac{1}{2}}A^*)\leq N^2\left(\begin{bmatrix}AX^{\frac{1}{2}} & BY^{\frac{1}{2}}\\ BY^{\frac{1}{2}} & AX^{\frac{1}{2}}\end{bmatrix}\right). \tag{32}$$

*Proof.*Let $K = \begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}$ and $L^* = \begin{bmatrix} Y^{\frac{1}{2}}B^* & 0 \\ X^{\frac{1}{2}}A^* & 0 \end{bmatrix}$ in inequality (29), we give

$$\begin{aligned} 2N\left(\begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y^{\frac{1}{2}}B^* & 0 \\ X^{\frac{1}{2}}A^* & 0 \end{bmatrix}\right) &\le N\left(\begin{bmatrix} X^{\frac{1}{2}}A^* & 0 \\ Y^{\frac{1}{2}}B^* & 0 \end{bmatrix}\begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y^{\frac{1}{2}}B^* & 0 \\ X^{\frac{1}{2}}A^* & 0 \end{bmatrix}\begin{bmatrix} BY^{\frac{1}{2}} & AX^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} X^{\frac{1}{2}}A^*AX^{\frac{1}{2}} + Y^{\frac{1}{2}}B^*BY^{\frac{1}{2}} & X^{\frac{1}{2}}A^*BY^{\frac{1}{2}} + Y^{\frac{1}{2}}B^*AX^{\frac{1}{2}} \\ X^{\frac{1}{2}}A^*BY^{\frac{1}{2}} + Y^{\frac{1}{2}}B^*AX^{\frac{1}{2}} & X^{\frac{1}{2}}A^*AX^{\frac{1}{2}} + Y^{\frac{1}{2}}B^*BY^{\frac{1}{2}} \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} X^{\frac{1}{2}}A^* & Y^{\frac{1}{2}}B^* \\ Y^{\frac{1}{2}}B^* & X^{\frac{1}{2}}A^* \end{bmatrix}\begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & AX^{\frac{1}{2}} \end{bmatrix}\right) \\ &= N\left(\left|\begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & AX^{\frac{1}{2}} \end{bmatrix}\right|^2\right) \\ &\le N^2\left(\begin{bmatrix} AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & AX^{\frac{1}{2}} \end{bmatrix}\right). \end{aligned}$$

We present a generalized numerical radius inequality for products of operators. Several special cases are given.

Theorem 5.*If A, B, X and $Y \in B(H)$ such that X and Y are positive semidefinite and $N(.)$ be any unitarily invariant norm, then*

$$w_N(AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^*) \le \frac{1}{4}N^2((|AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}|) \oplus (|AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}|)). \tag{33}$$

In particular, letting $N(.) = ||.||_p$ in inequality (33), we give

$$w_p(AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^*) \le 4^{\frac{1}{p}-1}|||AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}|||_p^2. \tag{34}$$

Proof.

$$\begin{aligned} w_N(AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^*) &= \sup_{\theta\in\mathbb{R}} N(Re(e^{i\theta}AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^*) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} N(e^{i\theta}AX^{\frac{1}{2}}Y^{\frac{1}{2}}B^* + e^{-i\theta}BY^{\frac{1}{2}}X^{\frac{1}{2}}A^*) \\ &\le \frac{1}{4}N^2\left(\begin{bmatrix} e^{i\theta}AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & e^{i\theta}AX^{\frac{1}{2}} \end{bmatrix}\right) \\ &= \frac{1}{4}N^2\left(\left|\begin{bmatrix} e^{i\theta}AX^{\frac{1}{2}} & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & e^{i\theta}AX^{\frac{1}{2}} \end{bmatrix}\right|\right) \\ &\le \frac{1}{4}N^2\left(\left|\begin{bmatrix} e^{i\theta}AX^{\frac{1}{2}} & 0 \\ 0 & e^{i\theta}AX^{\frac{1}{2}} \end{bmatrix}\right| + \left|\begin{bmatrix} 0 & BY^{\frac{1}{2}} \\ BY^{\frac{1}{2}} & 0 \end{bmatrix}\right|\right) \\ &= \frac{1}{4}N^2\left(\begin{bmatrix} |AX^{\frac{1}{2}}| & 0 \\ 0 & |AX^{\frac{1}{2}}| \end{bmatrix} + \begin{bmatrix} |BY^{\frac{1}{2}}| & 0 \\ 0 & |BY^{\frac{1}{2}}| \end{bmatrix}\right) \\ &\quad (\ by\ triangle\ inequality). \\ &= \frac{1}{4}N^2\left(\begin{bmatrix} |AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}| & 0 \\ 0 & |AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}| \end{bmatrix}\right) \\ &= \frac{1}{4}N^2((|AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}|) \oplus (|AX^{\frac{1}{2}}| + |BY^{\frac{1}{2}}|)). \end{aligned}$$

Remark.

1.Letting $Y = X$ in inequality (33), we give

$$w_N(AXB^*) \leq \frac{1}{4}N^2((|AX^{\frac{1}{2}}| + |BX^{\frac{1}{2}}|) \oplus (|AX^{\frac{1}{2}}| + |BX^{\frac{1}{2}}|)). \tag{35}$$

In particular, letting $N(.) = ||.||_p$, we give

$$w_p(AXB^*) \leq 4^{\frac{1}{p}-1}|||AX^{\frac{1}{2}}| + |BX^{\frac{1}{2}}|||_p^2. \tag{36}$$

2.Letting $X = I$ in inequality (35), we give

$$w_N(AB^*) \leq \frac{1}{4}N^2((|A| + |B|) \oplus (|A| + |B|)). \tag{37}$$

In particular, letting $N(.) = ||.||_p$, we give

$$w_p(AB^*) \leq 4^{\frac{1}{p}-1}|||A| + |B|||_p^2. \tag{38}$$

3.Letting $B^* = A$ in inequality (37), we give

$$w_N(A^2) \leq \frac{1}{4}N^2((|A| + |A^*|) \oplus (|A| + |A^*|)). \tag{39}$$

In particular, letting $N(.) = ||.||_p$, we give

$$w_p(A^2) \leq 4^{\frac{1}{p}}|||A|\ ||_p^2. \tag{40}$$

4.Letting $A = B = I$ in inequality (35), we give

$$w_N(X) \leq N^2(|X^{\frac{1}{2}}| \oplus |X^{\frac{1}{2}}|). \tag{41}$$

In particular, letting $N(.) = ||.||_p$, we give

$$w_p(X) \leq 4^{\frac{1}{p}}|||X^{\frac{1}{2}}|||_p^2 = 4^{\frac{1}{p}}||X||_{\frac{p}{2}}. \tag{42}$$

Theorem 6. *If $A, B \in B(H)$, then*

$$w_p\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}^2\right) = w_p\left(\begin{bmatrix} AB^* & 0 \\ 0 & B^*A \end{bmatrix}\right) \leq 4^{\frac{1}{p}-1}\left(||\ |A^*| + |B^*|\ ||_p^p + ||\ |A| + |B|\ ||_p^p\right)^{\frac{2}{p}}. \tag{43}$$

Proof. Let $X = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ and $Y^* = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ in inequality (38), then we give

$$\begin{aligned}
w_p\left(\begin{bmatrix} AB^* & 0 \\ 0 & B^*A \end{bmatrix}\right) &\leq 4^{\frac{1}{p}-1}\left|\left|\left|\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right| + \left|\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}^*\right|\right|\right|_p^2 \\
&\leq 4^{\frac{1}{p}-1}\left|\left|\left|\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right| + \left|\begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix}\right|\right|\right|_p^2 \\
&\leq 4^{\frac{1}{p}-1}\left|\left|\begin{bmatrix} |B^*| & 0 \\ 0 & |A| \end{bmatrix} + \begin{bmatrix} |A^*| & 0 \\ 0 & |B| \end{bmatrix}\right|\right|_p^2 \\
&\leq 4^{\frac{1}{p}-1}\left(\left|\left|\begin{bmatrix} |B^*| + |A^*| & 0 \\ 0 & |A| + |B| \end{bmatrix}\right|\right|_p\right)^2 \\
&\leq 4^{\frac{1}{p}-1}\left(||\ |A^*| + |B^*|\ ||_p^p + ||\ |A| + |B|\ ||_p^p\right)^{\frac{2}{p}}.
\end{aligned}$$

We prove a generalized numerical radius inequality for product of three operators. To reach our aim, we need the following lemma.

Lemma 6. *If A, X and $Y \in B(H)$ such that A is positive semidefinite and $N(.)$ is any unitarily invariant norm, then*

$$N(YAX + X^*AY^*) \leq ||Y||\,||X||N(A \oplus A) + \frac{1}{2}N(A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \oplus A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}}). \tag{44}$$

Proof. Let $K = \begin{bmatrix} YA^{\frac{1}{2}} & X^*A^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}$ and $L^* = \begin{bmatrix} A^{\frac{1}{2}}X & 0 \\ A^{\frac{1}{2}}Y^* & 0 \end{bmatrix}$ in inequality (29), we give

$$\begin{aligned}
2N(YAX + X^*AY^*) &= 2N\left(\begin{bmatrix} YA^{\frac{1}{2}} & X^*A^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{\frac{1}{2}}X & 0 \\ A^{\frac{1}{2}}Y^* & 0 \end{bmatrix}\right) \\
&\leq N\left(\begin{bmatrix} A^{\frac{1}{2}}Y^* & 0 \\ A^{\frac{1}{2}}X & 0 \end{bmatrix}\begin{bmatrix} YA^{\frac{1}{2}} & X^*A^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{\frac{1}{2}}X & 0 \\ A^{\frac{1}{2}}Y^* & 0 \end{bmatrix}\begin{bmatrix} X^*A^{\frac{1}{2}} & YA^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right) \\
&= N\left(\begin{bmatrix} A^{\frac{1}{2}}Y^*YA^{\frac{1}{2}} + A^{\frac{1}{2}}XX^*A^{\frac{1}{2}} & A^{\frac{1}{2}}Y^*X^*A^{\frac{1}{2}} + A^{\frac{1}{2}}XYA^{\frac{1}{2}} \\ A^{\frac{1}{2}}XYA^{\frac{1}{2}} + A^{\frac{1}{2}}Y^*X^*A^{\frac{1}{2}} & A^{\frac{1}{2}}XX^*A^{\frac{1}{2}} + A^{\frac{1}{2}}Y^*YA^{\frac{1}{2}} \end{bmatrix}\right) \\
&= N\left(\begin{bmatrix} A^{\frac{1}{2}}(Y^*Y + XX^*)A^{\frac{1}{2}} & A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} & A^{\frac{1}{2}}(Y^*Y + XX^*)A^{\frac{1}{2}} \end{bmatrix}\right) \\
&\leq N\left(\begin{bmatrix} A^{\frac{1}{2}}(|Y|^2 + |X^*|^2)A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}}(|Y|^2 + |X^*|^2)A^{\frac{1}{2}} \end{bmatrix}\right) \\
&\quad + N\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} & 0 \end{bmatrix}\right) \\
&\qquad (by\ triangule\ inequality) \\
&\leq N\left(\begin{bmatrix} A^{\frac{1}{2}}(||Y||^2 + ||X||^2)A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}}(||Y||^2 + ||X||^2)A^{\frac{1}{2}} \end{bmatrix}\right) \\
&\quad + N\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} & 0 \end{bmatrix}\right) \\
&\leq (||Y||^2 + ||X||^2)N\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\right) \\
&\quad + N\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} & 0 \end{bmatrix}\right)
\end{aligned}$$

$$\leq (||Y||^2 + ||X||^2)N(A \oplus A) + N(A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}} \oplus A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}}). \tag{45}$$

Replacing X by $\sqrt{t}X$ and Y by $\frac{1}{\sqrt{t}}Y$ in inequality (45) and taking the minimum over all $t > 0$, we give inequality (44).

Theorem 7. *If A, X and $Y \in B(H)$ such that A is positive semidefinite and $N(.)$ is any unitarily invariant norm, then*

$$w_N(YAX) \leq \frac{1}{2}||Y|\,|||X||N(A \oplus A) + \frac{1}{4}N((A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}}) \oplus (A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}})). \tag{46}$$

In particular, letting $N(.) = ||.||_p$, we give

$$w_p(YAX) \leq\ 2^{\frac{1}{p}-1}||Y||\,||X||\,||A||_p + 2^{\frac{1}{p}-2}||A^{\frac{1}{2}}(Y^*X^* + XY)A^{\frac{1}{2}}||_p. \tag{47}$$

Proof.

$$\begin{aligned} w_N(YAX) &= \sup_{\theta\in\mathbb{R}} N(Re(e^{i\theta}YAX)) \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} N(e^{i\theta}YAX + e^{-i\theta}X^*AY^*) \\ &\leq \frac{1}{2}||Y||\,||X||N(A\oplus A) \\ &+ \frac{1}{4}N((A^{\frac{1}{2}}(Y^*X^*+XY)A^{\frac{1}{2}})\oplus(A^{\frac{1}{2}}(Y^*X^*+XY)A^{\frac{1}{2}})) \\ &\quad (By\ using\ inequality\ (44)). \end{aligned}$$

Remark.

1.Letting $X = Y = I$ in inequality (46), we give

$$w_N(A) \leq N(A\oplus A). \tag{48}$$

In particular, let $N(.) = ||.||$ (the usual operator norm), we give the well known inequality

$$w(A) \leq ||A||. \tag{49}$$

2.Letting $A = I$ in inequality (46), we give

$$w_N(YX) \leq \frac{1}{2}||Y||\,||X||N(I\oplus I) + \frac{1}{4}N((Y^*X^*+XY)\oplus(Y^*X^*+XY)) \tag{50}$$

In particular, let $N(.) = ||.||_p$, we give

$$w_p(YX) \leq 2^{\frac{1}{p}-1}(n^{\frac{1}{p}}||Y||\,||X|| + ||Re(XY)||_p). \tag{51}$$

3.Letting $Y = I$ and $X = XY$ in inequality (46), we give

$$w_N(AXY) \leq \frac{1}{2}||XY||N(A\oplus A) + \frac{1}{4}N(A^{\frac{1}{2}}(Y^*X^*+XY)A^{\frac{1}{2}}\oplus A^{\frac{1}{2}}(Y^*X^*+XY)A^{\frac{1}{2}}) \tag{52}$$

In particular, let $N(.) = ||.||_p$, we give

$$w_p(AXY) \leq\ 2^{\frac{1}{p}-1}||XY||\,||A||_p + 2^{\frac{1}{p}-2}||A^{\frac{1}{2}}(Y^*X^*+XY)A^{\frac{1}{2}}||_p. \tag{53}$$

4.Letting $A = I$ and $X = XY$ in inequality (46), we give

$$w_N(XY) \leq \frac{1}{2}||XY||N(I\oplus I) + \frac{1}{4}N((Y^*X^*+XY)\oplus(Y^*X^*+XY)) \tag{54}$$

In particular, let $N(.) = ||.||_p$, we give

$$w_p(XY) \leq\ 2^{\frac{1}{p}-1}n^{\frac{1}{p}}||XY|| + 2^{\frac{1}{p}-2}||XY+Y^*X^*||_p. \tag{55}$$

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References

[1] A. Abu-Omar , F. Kittaneh, A generalized of the numerical radius. Linear Algebra Appl. 569 (2015), 323 – 334.
[2] A. Aldalabih , F. Kittaneh, Hilbert-Schmidt Numerical radius inequalities for operator norm. Linear Algebra and Its Applications 581 (2019), 72 – 84.
[3] W. Audeh, M. Al-Labadi, Numerical radius inequalites for finite sums of operators. Complex Anal. oper. Theory 17 (2023).
[4] W. audeh, Singular value inequalities for operators and matrices, Ann. Func. Anal. 13 (2022).
[5] W. Audeh, Singular value inequalities for accretive-dissipative normal operators, J. Math. Ineqal. 16 (2022), 729 – 737.
[6] W. Audeh, Sinular value and norm inequlities of Davidson-Power type, J. Math. Inequal. 15 (2021), 1311 – 1320.
[7] W. Audeh, Some generalizations for singular value inequalities of compact operators, Adv. Oper. Theory 14 (2021), 1 – 10.
[8] W. Audeh, Generalizations for singular value and arithmetic-geometric mean inequalities of operators, J.Math. Anal. appl, 489 (2020), 1 – 8.
[9] W. Audeh, Generalizations for singular value inequalities of operators, Adv. Oper. Theory 5 (2020), 371 – 381.
[10] W. Audeh, Singular value inequalities and applications, Positivity, 25 (2020), 843 – 852.
[11] W. Audeh, F. kittaneh, Singular value inequalities for compact operators, Linear Algebra Appl. 437 (2012), 2516 – 2522.
[12] R. Bhatia, Matrix Analysis, Springer, NewYork, 1997.
[13] R. Bhatia, F. Kittaneh, Norm inequalities for partitioned operators and an application. Math. Ann. 287 (1990), 719 – 726.
[14] R. Bhatia, F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal.Appl. 11 (1990), 272 – 277.
[15] A. Benmakhlouf, O. Hirzallah, F. Kittaneh, On the p-numerical radii of Hilbert space operators, Linear and Multilinear Algebra, 69 (2021), 2813 – 2829.
[16] A. Frakis, F. Kittaneh, S. Soltani, Upper and Lower Bounds for the p-Numerical Radii of Operators. Results Math 79 (2024).
[17] O.Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for commutators of Hilbert space operators. Numer. Funct. Anal. Optim. 32 (2011), 739 – 749.
[18] F. Kittaneh, M. Moslehian, T. Yamazaki, Cartesian decomposition and numerical radius inequalities. Linear Algebra Appl. 471 (2015), 46 – 53.
[19] F. Kittaneh, M. Sakkijha, Inequalities for accretive-dissipative matrices, Linear and Multilinear Algebra. 67 (2018) 1563 – 5139.