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Fuzzy Equivalence Relation And Fundamental Relation On Hyperalgebras

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Abstract: We introduce and study fuzzy hypercongruence relation on hyperalgebras as an extension of fuzzy congruence relation of algebras and present some important properties and isomorphism theorems. Then by applying the product concept for two fuzzy hypercongruence relations, it is determined that their product also constitutes a fuzzy hypercongruence relation. Finally, fundamental relation on the product of fuzzy subhyperalgebras are investigated.

Keywords: Hyperalgebra; Fuzzy equivalence relation; Fuzzy hypercongruence relation; Fuzzy factor hypercongruence; fundamental relation.

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1 Introduction

The concept of fuzzy sets [38], as a generalization of crisp sets, extends to fuzzy relations, which can be viewed as a generalization of ordinary relations between two elements. Zadeh [38] defined a fuzzy relation between A and B as a fuzzy set of $A*B$. Since then, various articles have been released in this field among which we can mention to fuzzy group (Rosenfeld [28]), fuzzy graph (Mordeson et al. [20]), fuzzy topological graphs (Atef et al. [7]), fuzzy vector space (Ranjbar et al. [27]), fuzzy geometric spaces (Ameri et al. [1], [2]), fuzzy probability theory (Bugajski [9]), lattice of fuzzy subalgebras (Murali [22]) etc. Equivalence relations are well-known in crisp sets for instance, Murali ([22], [23]) studied the fuzzy equivalence relations and congruence relations in fuzzy set theory. Following this, Samhan ([30], [31])]), applying the concept of fuzzy quotient algebra and fuzzy factor congruence, presented fuzzy congruence on a subgroup and a fuzzy congruence relation on a universal algebra.

Algebraic hyperstructures present a natural expansion of algebraic structures. As we know, the composition of two elements in hyperalgebra is a nonempty set, while the composition of two elements in algebra was an element. Marty [18] defined hyperstructures in 8th congress of Scandinavian Mathematicians. Various studies in this field led to great progress in the field of mathematics and computer science, among which are applications of hyperstructure theory (Corsini and Leoreanu [11]), join n-spaces and lattices (Leoreanu and Davvaz [17]), (Zahedi and Ameri [39]), congruence relation in hyperalgebra (Schweigert [32]), *el*-hyperstructures (Novak [21]), direct limit and inverse limit (Leoreanu [16]), weak hyperstructures (Vougiouklis et al. [37]), and homomorphism and subalgebra of multialgebra (Pickett [26]). Most works in this notion were given in book by Corsini and Leoreanu [11] that Vougiouklis et al. [37] followed it. As we know, there have been several studies investigating the connection between hyperstructures and fuzzy sets. These studies can be categorized into three types. The first group include the articles that concern the crisp hyperoperation defined on fuzzy sets that was an idea from Corsini [10], expanded upon Leoreanu [16], Corsini and Leoreanu [12] and some other researchers. The second group include the articles that study fuzzy hyperalgebras that are the extension of fuzzy algebras. This idea was initiated by Ameri and Zahedi [6] Zahedi et al. [40], and Davvaz and Corsini [15]. The last group is an idea from Corsini and Tofan [13] that is also about fuzzy hyperalgebra but in a completely different way which can be expressed as follows: Similar to crisp hyperoperations assigning a crisp set to all pair of elements, a fuzzy hyperoperation assigns

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a fuzzy set to each pair of elements, as continued by Sen et al. [33]. The current paper falls in the second group, aiming to generalize Samhan's article [31] to hyperalgebras by introducing the concept of fuzzy hypercongruence relation and quotient hyperalgebra. Additionally, the notion of direct product for any fuzzy relations, obtaining relevant results will be introduced and explored.

2 Preliminaries

In this segment, we introduce several properties related to hyperalgebras and fuzzy sets.
Suppose that S is a fixed nonempty set and $P^*(S)$ is the family of all nonempty subsets of S. For a positive integer n we denote by S^n the set of n-tuple over S and an n-ary hyperoperation β^n on S is a map $\beta^n : S\times\cdots\times S \to P^*(S)$ which a nonzero subset $\beta^n(\xi_1,\ldots,\xi_n)$ with any n-tuple $(\xi_1,\ldots,\xi_n)$ of elements of S. We refer to say that n as the arity of β.

To define a hyperalgebra, the classical algebraic concept (see [8]) can be generalized and expressed as follows: A hyperalgebra $\mathbf{S} = \langle S,(\beta_i, i\in I)\rangle$ is the set S along with a collection $(\beta_i : i \in I)$ of n_i-ary hyperoperations on S, where $(n_i : i \in I)$ indicates the type of this hyperalgebra.

A hypergroupoid $\langle S,*\rangle$ is a hyperalgebra (multialgebra) of type 2, that is a set s together with a single binary hyperoperation $*$. Also a hypergroupoid $\langle S,*\rangle$, which is satisfying associative, that is $x*(y*z) = (x*y)*z$ for all $x,y,z \in S$, is called semihypergroup. A hypergroup is a semihypergroup satisfying the productive law: for all $x\in S$ we have $x*S = S*x = S$. An element $e \in S$ of a hypergroup $\langle S,*\rangle$ is called an identity of S if for all $x \in S$, $x\in(e*x)\cap(x*e)$.

*Example 1.*Let $S = \{a,b,c\}$. We can define a hyperoperation β^2 on S, as follows:

$$\beta^2 : S\times S \longrightarrow P^*(S)$$

$$(x,y) \longmapsto x\beta^2 y = \{x,y\}$$

for all $(x,y) \in S^2$. So $\mathbf{S} = \langle S,\beta^2\rangle$ is an hyperalgebra.

Let π be a subset of S. We say π is closed under the n-ary hyperoperation β If $(\xi_1,\ldots,\xi_n) \in \pi^n$ give results $\beta(\xi_1,\ldots,\xi_n) \subseteq \pi$, where π^n represent the set of all n-tuple on π. In the case of $\mathbf{S} = \langle S,(\beta_i, i\in I)\rangle$ is a hyperalgebra, π is called a subhyperalgebra of S, if π is closed under each the n-ary hyperoperation β_i, for all $i \in I$.

In this paper we use $\vee$ and $\wedge$ to refer supremum and infimum in $[0,1]$, respectively.

To elaborate the point, we first state the important definitions in fuzzy relations (see Murali [23] and Samhan [31] for more details).

Let $S_1,S_2,...,S_n$ be nonempty sets. A fuzzy n-ary relation θ on $S_1,S_2,...,S_n$ is defined as a fuzzy set of the product set $S_1\times S_2\times...\times S_n$. Especially, a fuzzy binary relation on S and G is a fuzzy set θ on $S\times G$. In this article, sets S and G are considered to be the same sets.

A fuzzy relation (FR) always refers to a fuzzy binary relation $\theta : S\times S \to [0,1]$ unless specified otherwise.

Definition 1.(see Ameri et al. [5]) Let θ_1 and θ_2 be two FR on S. The composition of θ_1 and θ_2 is shown by $\theta_1\circ\theta_2$ and defined by $(\theta_1\circ\theta_2)(\xi,\tau) = \bigvee_{\zeta\in S}(\theta_1(\xi,\zeta)\wedge\theta_2(\zeta,\tau))$.

*Example 2.*Let θ_1 and θ_2 be two fuzzy equivalence relations on S. Define

θ_1	1	2
1	0.2	0.3
2	0.3	0.4

and

θ_2	1	2
1	0.1	0.2
2	0.2	0.4

Simple calculations show that

$\theta_1\circ\theta_2$	1	2
1	0.2	0.3
2	0.2	0.4

For example $(\theta_1\circ\theta_2)(2,1) = \bigvee_{s\in S}(\theta_1(2,s)\wedge\theta_2(s,1)) = (\theta_1(2,1)\wedge\theta_2(1,1))\bigvee(\theta_1(2,2)\wedge\theta_2(2,1)) = (0.3\wedge 0.1)\bigvee(0.4\wedge 0.2) = 0.2$.

A fuzzy relation θ on S is termed a fuzzy equivalence relation when the following conditions are met:

(i)θ is reflexive, $\theta(\xi,\xi)=1, \quad for\ \xi\in S.$
(ii)θ is symmetric, $\theta(\xi,\tau)=\theta(\tau,\xi), \quad for\ \xi,\tau\in S.$
(iii)θ is transitive, $\theta\circ\theta\leq\theta.$

We denote fuzzy equivalence relation by FER.

Proposition 1.(see Murali [23]) If θ is a FER on $\mathbf{S}$, then $\theta\circ\theta=\theta$.

Let $\mathbf{S}=\langle S,\beta^{\mathbf{S}}\rangle$, $\mathbf{S}'=\langle S',\beta^{\mathbf{S}'}\rangle$ be two hyperalgebras. A function $g:\mathbf{S}\to\mathbf{S}'$ is called a homomorphism from $\mathbf{S}$ to $\mathbf{S}'$ if for all $\xi_1,\ldots,\xi_n\in S$, it follows that

$$g(\beta^{\mathbf{S}}(\xi_1,\ldots,\xi_n))=\beta^{\mathbf{S}'}(g(\xi_1),\ldots,g(\xi_n)).$$

An equivalence relation θ on $S\times S$ of $\mathbf{S}=\langle S,F\rangle$ is called a congruence of $\mathbf{S}$ if $\xi_i\theta\tau_i$ for $1\leq i\leq n$, then $\beta^{\mathbf{S}}(\xi_1,\ldots,\xi_n)\theta\beta^{\mathbf{S}}(\tau_1,\ldots,\tau_n)$ for all $\xi_i,\tau_i\in S$.

Definition 2. Let $\lambda:S\times S\to[0,1]$ be a fuzzy set. For $p\in[0,1]$, a $p-$level (strang $p-$level) subset of λ on $S\times S$ are defined as follows:

$$\lambda_p=\{(x_1,x_2)\in S^2:\ \lambda(x_1,x_2)\geq p\}\quad(\bar{\lambda}_p=\{(x_1,x_2)\in S^2:\ \lambda(x_1,x_2)>p\}).$$

A new attitude to the fuzzy algebra was presented by Samhan [31]. Now we follow it in order to extend this notion into FR on hyperalgebras.

3 Product of fuzzy hypercongruences

Definition 3. Suppose that $\mathbf{S}=\langle S,F\rangle$ is a hyperalgebra and θ be a FER on $\mathbf{S}$. We say θ is fuzzy hypercongruence relation on $\mathbf{S}$ if for any elements $\xi_i,\tau_i\in S$ and for all n-ary $\beta\in F$, it follows that

$$\theta(\xi,\tau)\geq\bigwedge_{i=1}^{n}\theta(\xi_i,\tau_i),$$

where $\xi\in\beta(\xi_1,\ldots,\xi_n)$, $\tau\in\beta(\tau_1,\ldots,\tau_n)$.

The set of all fuzzy hypercongruences on S will be indicated by $FHC(\mathbf{S})$.
We define Δ_S and ∇_S the least and the greatest elements of $FHC(\mathbf{S})$, where

$$\Delta_S(\xi,\tau)=\begin{cases}1 & \xi=\tau\\ 0 & \xi\neq\tau\end{cases}\quad and\quad \nabla_S(\xi,\tau)=1,$$

for all $\xi,\tau\in S$.

Definition 4. Let $\mathbf{S}$ and $\mathbf{S}'$ be two hyperalgebras and $g:\mathbf{S}\to\mathbf{S}'$ be a homomorphism and θ,π be two FR on $\mathbf{S}$ and $\mathbf{S}'$, respectively. The inverse image of π, shown by $g^{-1}(\pi)$, is the FR on $\mathbf{S}$ defined by

$$g^{-1}(\pi)(\xi,\tau)=\pi\big(g(\xi),g(\tau)\big)\quad\forall\xi,\tau\in S,$$

and the image of θ, is indicated by $g(\theta)$, it is the FR on $\mathbf{S}'$ defined by

$$g(\theta)(\xi',\tau')=\bigvee_{\xi,\tau}\{\theta(\xi,\tau):\xi\in g^{-1}(\xi'),\tau\in g^{-1}(\tau')\}.$$

If $g^{-1}(\xi')$ or $g^{-1}(\tau')$ is empty, then $g(\theta)(\xi',\tau')$ is defined as 0.
Also the kernel of g, written $kerg$, is defined by

$$kerg=\{(a,b)\in S^2:\ g(a)=g(b)\}.$$

In this section based on fuzzy hypercongruence relations, we define fuzzy quotient hyperalgebra and factor congruence. Finally, we prove some homomorphism theorems and correspondence theorem.

Let $\mathbf{S}$ be a hyperalgebra and $\theta,\pi\in FHC(\mathbf{S})$ and $\xi\in S$. We indicate the fuzzy hypercongruence class determined by ξ and θ by ξ/θ, it's the fuzzy set of S defined by

$$(\xi/\theta)(\tau)=\theta(\xi,\tau)\quad\forall\tau\in S.$$

Proposition 2. If $\theta \in FHC(\mathbf{S})$ and $\xi, \tau \in S$, then

$$\xi/\theta = \tau/\theta \Leftrightarrow \theta(\xi, \tau) = 1.$$

*Proof.*Let $\xi/\theta = \tau/\theta$ then for all $\zeta \in S$, it follows that $(\xi/\theta)(\zeta) = (\tau/\theta)(\zeta)$ so $\theta(\xi, \zeta) = \theta(\tau, \zeta)$, in a special situation for $\tau = \zeta$, it follows that $\theta(\xi, \tau) = \theta(\tau, \tau) = 1$.

Conversly, let $\theta(\xi, \tau) = 1$. Since θ is fuzzy hypercongruence relation, by Proposition 1, it follows that

$$\theta(\xi, \zeta) = \theta \circ \theta(\xi, \zeta) \geq \theta(\xi, \tau) \wedge \theta(\tau, \zeta) = \theta(\tau, \zeta),$$

similarity, $\theta(\tau, \zeta) \geq \theta(\xi, \zeta)$.

Then it follows that $\theta(\tau, \zeta) = \theta(\xi, \zeta)$ i.e.

$$(\xi/\theta)(\zeta) = (\tau/\theta)(\zeta) \qquad for \;\; all \;\; \zeta \in S.$$

Definition 5. Let $\mathbf{S}$ be a hyperalgebra and $\theta \in FHC(\mathbf{S})$. Define $S/\theta = \{\xi/\theta : \xi \in S\}$ be the set of all congruence classes S by θ. Suppose $\beta \in F$ be an n-ary function, the hyperoperation on S/θ induceed by β on S is defined by

$$\beta^{\mathbf{S}/\theta}(\xi_1/\theta, \ldots, \xi_n/\theta) = \beta^{\mathbf{S}}(\xi_1, \ldots, \xi_n)/\theta.$$

Definition 6. Suppose that M and N are two sets. We say $M\theta N$ if and only if $\xi\theta\tau$ for all $\xi \in M$, $\tau \in N$.

Proposition 3. The hyperoperation $\beta^{\mathbf{S}/\theta}$ is well-defined.

*Proof.*For all $\xi_i, \tau_i \in S$, let $\xi_i/\theta = \tau_i/\theta$. Then by Proposition 2, it follows that $\theta(\xi_i, \tau_i) = 1$ for $1 \leq i \leq n$. Thus, $\theta(\xi, \tau) \geq \theta(\xi_i, \tau_i) = 1$ for $\xi \in \beta(\xi_1, \ldots, \xi_n), \tau \in \beta(\tau_1, \ldots, \tau_n)$. Thus $\theta(\xi, \tau) = 1$, by Proposition 2, we have

$$\beta(\xi_1, \ldots, \xi_n)/\theta = \beta(\tau_1, \ldots, \tau_n)/\theta$$

i.e. $\beta^{S/\theta}$ is well-defined.

The hyperalgebra $\mathbf{S}/\theta = \langle S/\theta, F\rangle$ is termed the fuzzy quotient hyperalgebra of S resulted by θ. Let $\mathbf{S}$ be a hyperalgebra and $\theta \in FHC(\mathbf{S})$. The natural map $\nu_\theta : \mathbf{S} \to \mathbf{S}/\theta$ is defined by $\nu_\theta(\xi) = \xi/\theta$. Clearly ν_θ is onto homomorphism.

Lemma 1. Suppose that $\mathbf{S}, \mathbf{S}'$ are two hyperalgebras and $g : \mathbf{S} \to \mathbf{S}'$ be homomorphism and $\pi \in FHC(\mathbf{S}')$. Then $g^{-1}(\pi) \in FHC(\mathbf{S})$.

*Proof.*Suppose that $\xi \in \beta(\xi_1, \ldots, \xi_m), \tau \in \beta(\tau_1, \ldots, \tau_m)$. Then

$$g^{-1}(\pi)(\xi, \tau) = \pi(g(\xi), g(\tau)) \geq \bigwedge_{j=1}^{m} \pi(g(\xi_j), g(\tau_j)) = \bigwedge_{j=1}^{m} g^{-1}(\pi)(\xi_j, \tau_j).$$

Thus $g^{-1}(\pi) \in FHC(\mathbf{S})$.

The subsequent theorem bears resemblance to the homomorphism theorem in universal algebra theory.

Theorem 1. Suppose that $\mathbf{S}, \mathbf{S}'$ are two hyperalgebras and $g : \mathbf{S} \to \mathbf{S}'$ be a homomorphism and $\pi \in FHC(\mathbf{S}')$. Then

$$\mathbf{S}/g^{-1}(\pi) \cong \mathbf{S}'/\pi.$$

*Proof.*By Lemma 1 we have $g^{-1}(\pi) \in FHC(\mathbf{S})$. Define the map $f : \mathbf{S}/g^{-1}(\pi) \to \mathbf{S}'/\pi$ by $f\big(\xi/g^{-1}(\pi)\big) = g(\xi)/\pi$ for all $\xi \in S$. By Proposition 2 and Definition 4, we have

$$\begin{aligned}
\xi/g^{-1}(\pi) = \tau/g^{-1}(\pi) \Longleftrightarrow &\quad g^{-1}(\pi)(\xi, \tau) = 1 \\
\Longleftrightarrow &\quad \pi\big(g(\xi), g(\tau)\big) = 1 \\
\Longleftrightarrow &\quad g(\xi)/\pi = g(\tau)/\pi \\
\Longleftrightarrow &\quad f\big(\xi/g^{-1}(\pi)\big) = f\big(\tau/g^{-1}(\pi)\big).
\end{aligned}$$

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Then f is well-defined and one-to-one. On the other hand f is onto and also it follows that

$$\begin{aligned} f\big(\beta^{\mathbf{S}}(\xi_1,\ldots,\xi_n)/g^{-1}(\pi)\big) &= g\big(\beta^{\mathbf{S}}(\xi_1,\ldots,\xi_n)\big)/\pi \\ &= \beta^{\mathbf{S}'}\big(g(\xi_1),\ldots,g(\xi_n)\big)/\pi \\ &= \beta^{\mathbf{S}'/\pi}\big(g(\xi_1)/\pi,\ldots,g(\xi_n)/\pi\big) \\ &= \beta^{\mathbf{S}'/\pi}\big(f(\xi_1/g^{-1}(\pi)),\ldots,f(\xi_n/g^{-1}(\pi))\big). \end{aligned}$$

Thus f is a homomorphism. So the proof is complete.

Corollary 1. Suppose that $\mathbf{S},\mathbf{S}'$ are two hyperalgebras and $g:\mathbf{S}\to\mathbf{S}'$ be an onto homomorphism and $\pi\in FHC(\mathbf{S}')$. Then

$$\mathbf{S}/1_{kerg}\cong\mathbf{S}'/\Delta_S\cong\mathbf{S}',$$

where 1_{kerg} is the characteristic function of kernel g.

The ensuing definition serves as the fuzzy analog to the factor congruence definition in hyperalgebras.

Definition 7. Let $\mathbf{S}$ be a hyperalgebra. We say θ and π are a pair of fuzzy factor hypercongruences of S if $\theta\cap\pi=\Delta_{\mathbf{S}}$ and $\theta\circ\pi=\nabla_{\mathbf{S}}$.

Theorem 2. Suppose that θ and π is a pair of fuzzy factor hypercongruences on a hyperalgebra $\mathbf{S}$ and $|\mathrm{Im}\theta|$ and $|\mathrm{Im}\pi|$ are finite. Then

$$\mathbf{S}\cong\mathbf{S}/\pi\times\mathbf{S}/\theta.$$

*Proof.*Define the map $\underset{g(\xi)=\langle\xi/\pi,\xi/\theta\rangle}{g:\mathbf{S}\to\mathbf{S}/\pi\times\mathbf{S}/\theta}$ for all $\xi\in S$.

g is well-defined and one-to-one. Let $g(\xi)=g(\tau)$ for all $\xi,\tau\in S$, then $\xi/\pi=\tau/\pi$ and $\xi/\theta=\tau/\theta$, so by Proposition 2 it follows that $\pi(\xi,\tau)=1$ and $\theta(\xi,\tau)=1$ hence $\Delta_{\mathbf{S}}(\xi,\tau)=1$ so $\xi=\tau$.

We can easily prove g is onto: suppose $\langle\xi/\pi,\tau/\theta\rangle\in\mathbf{S}/\pi\times\mathbf{S}/\theta$. Since $\theta\circ\pi=\nabla_{\mathbf{S}}$, it follows that

$$1=\nabla_{\mathbf{S}}(\xi,\tau)=\theta\circ\pi(\xi,\tau)\geq\bigvee_{\zeta\in S}\big(\theta(\xi,\zeta)\wedge\pi(\zeta,\tau)\big).$$

On the other hand, since $|\mathrm{Im}\theta|$ and $|\mathrm{Im}\pi|$ are finite then there is a $\zeta\in S$ s.t. $\theta(\xi,\zeta)\wedge\pi(\zeta,\tau)=1$ so $\theta(\xi,\zeta)=\pi(\zeta,\tau)=1$. Thus $\xi/\theta=\zeta/\theta$ and $\zeta/\pi=\tau/\pi$ so

$$g(t)=\langle\zeta/\pi,\zeta/\theta\rangle=\langle\xi/\pi,\tau/\pi\rangle.$$

Later, we define direct product for two FR then we prove that direct product of two fuzzy hypercongruences belongs to fuzzy hypercongruence relation too. Finally, we extend this theorem for all fundamental relations.

Definition 8. Let $\mathbf{S}$ and $\mathbf{S}'$ be two hyperalgebras and θ, π are FR on S and S', respectively. The product of θ and π is shown as $\theta\times\pi$ and defined by

$$(\theta\times\pi)\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)=\bigwedge\{\theta(\xi_1,\xi_2),\pi(\tau_1,\tau_2)\}$$

for all $\xi_1,\xi_2\in S$ and $\tau_1,\tau_2\in S'$.

Theorem 3. Suppose that θ is a fuzzy hypercongruence relation on $\mathbf{S}$. Then all nonempty p-level θ_p is hypercongruence on $\mathbf{S}$.

*Proof.*We prove that θ_p is p-level relation. Let $\beta\in F$ and $x_1\in\beta(\xi_1,\ldots,\xi_n)$, $x_2\in\beta(\tau_1,\ldots,\tau_n)$ for all $x_1,x_2\in S$ and $(\xi_i,\tau_i)\in S^n$. Suppose $\xi_i\theta_p\tau_i$ for $i=1,\ldots,n$ then $\theta(\xi_i,\tau_i)\geq p$ for $0\leq p\leq 1$. Since θ is a fuzzy hypercongruence relation it follows that $\theta(x_1,x_2)\geq\bigwedge_{i=1}^{n}\theta(\xi_i,\tau_i)\geq p$. So $x_1\theta_p x_2$ for all $x_1\in\beta(\xi_1,\ldots,\xi_n)$, $x_2\in\beta(\tau_1,\ldots,\tau_n)$.

Theorem 4.(Davvaz and Cristea [14]). Let $\mathbf{S}$ and $\mathbf{S}'$ be two hyperalgebras and θ, π be FR on $\mathbf{S},\mathbf{S}'$, respectively. Then

$$(\theta\times\pi)_P=\theta_P\times\pi_P.$$

Theorem 5. Suppose that $\mathbf{S}$ and $\mathbf{S}'$ are two hyperalgebras and $\theta \in FHC(\mathbf{S})$ and $\pi \in FHC(\mathbf{S}')$. Then

$$(\theta \times \pi) \in FHC(\mathbf{S} \times \mathbf{S}').$$

*Proof.*Suppose that $(m,\xi) \in (m_1,\xi_1) * (m_2,\xi_2)$ and $(n,\tau) \in (n_1,\tau_1) * (n_2,\tau_2)$. Then it follows that

$$\begin{aligned}
&(\theta \times \pi)\big((m,\xi),(n,\tau)\big) = \bigwedge \{\theta(m,n),\pi(\xi,\tau)\} \\
&\geq \bigwedge \Big\{ \bigwedge \{ (\theta(m_1,n_1),\pi(m_2,n_2) \}, \bigwedge \{ (\theta(\xi_1,\tau_1),\pi(\xi_2,\tau_2) \} \Big\} \\
&\geq \bigwedge \Big\{ \bigwedge \{ (\theta(m_1,n_1),\pi(\xi_1,\tau_1) \}, \bigwedge \{ (\theta(m_2,n_2),\pi(\xi_2,\tau_2) \} \Big\} \\
&= \bigwedge \{ (\theta \times \pi)\big((m_1,\xi_1),(n_1,\tau_1)\big), (\theta \times \pi)\big((m_2,\xi_2),(n_2,\tau_2)\big) \}.
\end{aligned}$$

Thus $(\theta \times \pi) \in FHC(\mathbf{S} \times \mathbf{S}')$.

Definition 9. Let $\mathbf{S}$ be a hyperalgebra and $\theta,\pi \in FHC(\mathbf{S})$, $\theta \leq \pi$. Then for all $\xi,\tau \in S$ the FR π/θ on $\mathbf{S}/\theta$ is defined by

$$(\pi/\theta)(\xi/\theta,\tau/\theta) = \pi(\xi,\tau).$$

Proposition 4. The FR π/θ is well-defined.

*Proof.*For all $\xi,\xi_1,\tau,\tau_1 \in S$, let $\xi/\theta = \xi_1/\theta$ and $\tau/\theta = \tau_1/\theta$. Then by Proposition 2, it follows that $\theta(\xi,\xi_1) = 1$ and $\theta(\tau,\tau_1) = 1$. Thus it follows that

$$\begin{aligned}
(\pi/\theta)(\xi/\theta,\tau/\theta) = \pi(\xi,\tau) &\geq \pi \circ \pi(\xi,\tau) \\
&\geq \bigwedge \{\pi(\xi,\xi_1),\pi(\xi_1,\tau)\} \\
&\geq \bigwedge \{\pi(\xi,\xi_1),\pi \circ \pi(\xi_1,\tau)\} \\
&\geq \bigwedge \{\pi(\xi,\xi_1),\pi(\xi_1,\tau_1),\pi(\tau_1,\tau)\} \\
&= \pi(\xi_1,\tau_1) \\
&= (\pi/\theta)(\xi_1/\theta,\tau_1/\theta).
\end{aligned}$$

Similarly, we can demonstrate that $(\pi/\theta)(\xi_1/\theta,\tau_1/\theta) \geq (\pi/\theta)(\xi/\theta,\tau/\theta)$. Thus $(\pi/\theta)(\xi_1/\theta,\tau_1/\theta) = (\pi/\theta)(\xi/\theta,\tau/\theta)$. i.e. π/θ is well-defined.

Now we proceed to state and prove the second isomorphism theorem in FHC(H).

Theorem 6. Suppose that $\mathbf{S}$ is a hyperalgebra and $\theta,\pi \in FHC(\mathbf{S})$, $\theta \leq \pi$. Then

$$(\mathbf{S}/\theta)/(\pi/\theta) \cong \mathbf{S}/\pi.$$

*Proof.*By Definition 9 we have $\pi/\theta \in FHC(\mathbf{S}/\theta)$. Define the map $f_1 : (\mathbf{S}/\theta)/(\pi/\theta) \to \mathbf{S}/\pi$ by $f_1\big((\xi/\theta)/(\pi/\theta)\big) = \xi/\pi$ for all $\xi \in S$. By Proposition 2 and Definition 5, it follows that

$$\begin{aligned}
(\xi/\theta)/(\pi/\theta) = (\tau/\theta)/(\pi/\theta) &\Longleftrightarrow \quad (\pi/\theta)(\xi/\theta,\tau/\theta) = 1 \\
&\Longleftrightarrow \quad \pi\big(\xi,\tau\big) = 1 \\
&\Longleftrightarrow \quad \xi/\pi = \tau/\pi \\
&\Longleftrightarrow \quad f_1\big((\xi/\theta)/(\pi/\theta)\big) = f_1\big((\tau/\theta)/(\pi/\theta)\big).
\end{aligned}$$

Then f_1 is well-defined and 1-1. On the other hand f_1 is onto and also it follows that

$$\begin{aligned}
f_1\big((\beta^{\mathbf{S}/\theta}(\xi_1/\theta,\ldots,\xi_n/\theta))/(\pi/\theta)\big) &= f_1\big((\beta^{\mathbf{S}}(\xi_1,\ldots,\xi_n)/\theta)/(\pi/\theta)\big) \\
&= \beta^{\mathbf{S}}\big(\xi_1,\ldots,\xi_n\big)/\pi \\
&= \beta^{\mathbf{S}/\pi}\big(\xi_1/\pi,\ldots,\xi_n/\pi\big) \\
&= \beta^{\mathbf{S}/\pi}\big(f_1\big((\xi_1/\theta)/(\pi/\theta)\big),\ldots,(f_1\big((\xi_n/\theta)/(\pi/\theta)\big)\big).
\end{aligned}$$

Thus f_1 is a homomorphism. So the proof is complete.

4 Fundamental relations and fuzzy subhyperalgebras

Let $\mathbf{S} = (S, \beta)$ be hypergroup (resp. hypergroupoid, semihypergroup) and n be a natural number, for every x_1 and x_2 in S; we can write $x_1 \beta_n x_2$ if and only if $\exists\ \xi_1, \ldots, \xi_n \in S$, s.t. $\{x_1, x_2\} \subseteq \prod_{i=1}^{n} \xi_i$. Define $\beta = \bigcup_{n \geq 1} \beta_n$ and suppose that $\beta_1 = \{(x_1, x_1) | x_1 \in S\}$. We can easily show that the relation defined above is a reflexive and symmetric. Use the β^* symbol the transitive closure of β. Hence, the relation β^* is the smallest equivalence relation on S, s.t. the quotient S/β^* is a group (resp. groupoid, semigroup). Note that β^* is called the fundamental relation on S and S/β^* is called the fundamental group (resp. groupoid, semigroup). As the fundamental relation has a crucial role in the theory of algebraic hyperstructures, it has been extended to other classes of algebraic hyperstructures (refer to [9], [14], [34], [35] and [36]).

In this regard, Pelea [24] and Ameri [4] and [25] presented and studied the fundamental relation on multialgebras and fuzzy hyperalgebras using the definition of term functions respectively. In the following, we extend the fundamental relation to the product of fuzzy subhyperalgebras and explore its fundamental properties.

Definition 10.[4] A fuzzy n-ary hyperoperation f^n on S is a map $f^n : S \times \cdots \times S \to F^*(S)$ which associates a nonzero fuzzy subset $f^n(a_1, \ldots, a_n)$ with any n-tuple $(a_1, \ldots, a_n)$ of elements of S. The couple (S, f^n) is called a fuzzy n-ary hypergroupoid. A fuzzy nullary hyperoperation on S is just an element of $F^*(S)$, i.e. a nonzero fuzzy subset of S.

Example 3.[4] Let $\mathbf{S} = \langle S, (\beta_i, i \in I) \rangle$ be a hyperalgebra and μ be a nonzero fuzzy subset of S. Define the following fuzzy n-ary hyperoperations on S, for every $i \in I$ and for all $(a_1, \ldots, a_{n_i}) \in S^{n_i}$:

$$\beta_i^0(a_1, \ldots, a_{n_i}) = \begin{cases} \mu(a_1) \wedge \ldots \wedge \mu(a_{n_i}) & t \in \beta_i(a_1, \ldots, a_{n_i}) \\ 0 & otherwise, \end{cases}$$

and let $\beta_i^1(a_1, \ldots, a_{n_i}) = \chi_{a_1, \wedge \ldots \wedge, a_{n_i}}$. Clearly, $\mathbf{S} = \langle S, (\beta_i^0, i \in I) \rangle$ and $\mathbf{S} = \langle S, (\beta_i^1, i \in I) \rangle$ are fuzzy hyperalgebras.

Theorem 7.[4] *Let $\mathbf{S} = \langle S, (\beta_i, i \in I) \rangle$ be a hyperalgebra, then for every $i \in I$ and every $(a_1, \ldots, a_{n_i}) \in S^{n_i}$ we have $\beta_i(\chi_{a_1}, \ldots, \chi_{a_{n_i}}) = \beta_i(a_1, \ldots, a_{n_i})$.*

Definition 11. Suppose that $\mathbf{S}$ is a hyperalgebra and λ be a fuzzy set of $\mathbf{S}$. The fuzzy set λ_{β^*} on $\mathbf{S}/\beta^*$ is defined as follow:

$$\begin{array}{l} \lambda_{\beta^*} : \mathbf{S}/\beta^* \to [0,1] \\ \lambda_{\beta^*}(x/\beta^*) = \bigvee_{\xi \in (x/\beta^*)} \{\lambda(\xi)\} \end{array}$$

We remind that a subhyperalgebra of hyperalgebra $\mathbf{S} = \langle S, F \rangle$ is a fuzzy set $\lambda : S \to [0,1]$ s.t. for all $\xi_1, \ldots, \xi_{n_i} \in S$, we have:

$$\lambda(\xi_1) \wedge \ldots \wedge \lambda(\xi_n) \leq \bigwedge \{\lambda(x) | x \in \beta_i(\xi_1, \ldots, \xi_{n_i})\},$$

Notice that if the arity of β_i is zero, the left side member is infimum of $\emptyset$, so for any $x \in \beta_i$ we consider $\lambda(x) = 1$.

Lemma 2.[19] Let M and N be any arbitrary index sets. Then it follows that

$$\bigvee_{\substack{j \in M \\ k \in N}} \left\{ \bigwedge (\xi_j, \tau_k) \right\} = \bigwedge \left\{ \bigvee_{j \in M} \{\xi_j\}, \bigvee_{k \in N} \{\tau_k\} \right\}.$$

Theorem 8.*Suppose that* $\mathbf{S}$ *is a hyperalgebra and* λ *be a fuzzy subhyperalgebra of* $\mathbf{S}$*. Then* λ_{β^*} *is a fuzzy subhyperalgebra of* $\mathbf{S}/\beta^*$.

*Proof.*Let $\beta^*(\xi_1), \ldots, \beta^*(\xi_{n_i}) \in \mathbf{S}/\beta^*$. Then, it follows that:

$$\begin{aligned} & \lambda_{\beta^*}(\beta^*(\xi_1)) \wedge \ldots \wedge \lambda_{\beta^*}(\beta^*(\xi_{n_i})) \\ & = \Big(\bigvee_{x_1 \in \beta^*(\xi_1)} \lambda(x_1) \Big) \wedge \ldots \wedge \Big(\bigvee_{x_{n_i} \in \beta^*(\xi_{n_i})} \lambda(x_{n_i}) \Big) \\ & = \bigvee_{\substack{x_1 \in \beta^*(\xi_1) \\ \ldots \\ x_{n_i} \in \beta^*(\xi_{n_i})}} \{\lambda(x_1) \wedge \ldots \wedge \lambda(x_{n_i})\} \end{aligned}$$

Since λ is a fuzzy subhyperalgebra of **S**, thus

$$\leq \bigvee_{\substack{x_1 \in \beta^*(\xi_1) \\ \dots \\ x_{n_i} \in \beta^*(\xi_{n_i})}} \left\{ \bigwedge \lambda(x) | \ x \in \beta_i(x_1, \dots, x_{n_i}) \right\}$$

By lemma 2 it follows that

$$\begin{aligned}
&\lambda_{\beta^*}(\beta^*(\xi_1)) \wedge \dots \wedge \lambda_{\beta^*}(\beta^*(\xi_{n_i})) \\
&\leq \bigwedge_{x \in \beta_i(\xi_1, \dots, \xi_{n_i})} \left\{ \bigvee_{x_1 \in \beta^*(\xi_1)} \lambda(x_1), \ \dots, \bigvee_{x_{n_i} \in \beta^*(\xi_{n_i})} \lambda(x_{n_i}) \right\} \\
&= \bigwedge_{x \in \beta_i(\xi_1, \dots, \xi_{n_i})} \{\lambda_{\beta^*}(\beta^*(\xi_1)), \ \dots, \ \lambda_{\beta^*}(\beta^*(\xi_{n_i}))\} \\
&= \bigwedge \{\lambda_{\beta^*}(x) | \ x \in \beta_i(x_1, \dots, x_{n_i}), \ x_1 \in \beta^*(\xi_1), \dots, \ x_{n_i} \in \beta^*(\xi_{n_i})\} \\
&= \bigwedge \{\lambda_{\beta^*}(x) | \ x \in \beta_i(\beta^*(\xi_1), \dots, \beta^*(\xi_{n_i}))\}
\end{aligned}$$

Theorem 9. *Suppose that* **S**, **S**$'$ *are two hyperalgebras and* β_1^*, β_2^* *and* β^* *are fundamental equivalence relations on* S, S' *and* $\mathbf{S} \times \mathbf{S}'$, *respectively. Let* λ *and* ν *are fuzzy subhyperalgebras of* **S**, **S**$'$, *respectively. Then*

$$(\lambda \times \nu)_{\beta^*} = \lambda_{\beta_1^*} * \nu_{\beta_2^*}.$$

Proof. Let $x_1, y_1 \in S$, $x_2, y_2 \in S'$. Then,

$$\begin{aligned}
&(\lambda \times \nu)_{\beta^*}\big((x_1, y_1), (x_2, y_2)\big)/\beta^* \\
&= \bigvee_{\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) \in \big((x_1,y_1),(x_2,y_2)\big)/\beta^*} \{(\lambda \times \nu)\big((\xi_1, \tau_1), (\xi_2, \tau_2)\big)\} \\
&= \bigvee_{\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) \in \big((x_1,y_1),(x_2,y_2)\big)/\beta^*} \left\{ \bigwedge \{\lambda(\xi_1, \xi_2), \nu(\tau_1, \tau_2)\} \right\} \\
&= \bigvee_{\substack{(\xi_1, \xi_2) \in (x_1, y_1)/\beta_1^* \\ (\tau_1, \tau_2) \in (x_2, y_2)/\beta_2^*}} \left\{ \bigwedge (\lambda(\xi_1, \xi_2), \nu(\tau_1, \tau_2)) \right\} .
\end{aligned}$$

By Lemma 2 it follows that

$$\begin{aligned}
&= \bigwedge_{\substack{(\xi_1, \xi_2) \in (x_1, y_1)/\beta_1^* \\ (\tau_1, \tau_2) \in (x_2, y_2)/\beta_2^*}} \left(\bigvee \{\lambda(\xi_1, \xi_2)\}, \bigvee \{\nu(\tau_1, \tau_2)\} \right) \\
&= \bigwedge \Big(\lambda_{\beta_1^*}\big((x_1, y_1)/\beta_1^*\big), \nu_{\beta_2^*}\big((x_2, y_2)/\beta_2^*\big)\Big) \\
&= (\lambda_{\beta_1^*} \times \nu_{\beta_2^*})\Big((x_1, y_1)/\beta_1^*, (x_2, y_2)/\beta_2^*\Big).
\end{aligned}$$

Since $(\xi, \tau) \in \beta^*(x, y)$ if and only if $\xi \beta_1^* x$ and $\tau \beta_2^* y$, then the proof of theorem is completed.

Theorem 10. *Suppose that* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are two hyperalgebras and* β_1^*, β_2^* *and* β^* *are fundamental equivalence relations on* $\mathbf{S}_1$, $\mathbf{S}_2$ *and* $\mathbf{S}_1 \times \mathbf{S}_2$, *respectively. Let* λ *be a fuzzy subhyperalgebra of* $\mathbf{S}_1 \times \mathbf{S}_2$, *define*

$$\lambda_1(m) = \bigvee_{\xi \in S_2} \{\lambda(m, \xi)\} \quad \textit{and} \quad \lambda_2(n) = \bigvee_{\tau \in S_1} \{\lambda(\tau, n)\}.$$

Then $(\lambda_1)_{\beta_1^*}$ *and* $(\lambda_2)_{\beta_2^*}$ *are fuzzy subalgebras of* $\mathbf{S}_1/\beta_1^*$ *and* $\mathbf{S}_2/\beta_2^*$ *and*

$$\lambda_{\beta^*} \subseteq (\lambda_1)_{\beta_1^*} \times (\lambda_2)_{\beta_2^*}.$$

*Proof.*By theorem 8 we know $(\lambda_i)_{\beta_i^*}$ is a fuzzy subhyperalgebra of $\mathbf{S}_i/\beta_i^*$ for $i = 1,2$. By definition 8 and 10 we have:

$$\begin{aligned}&\big((\lambda_1)_{\beta_1^*} \times (\lambda_2)_{\beta_2^*}\big)(\beta_1^*(m), \beta_2^*(n))\\ &= \bigwedge \Big[(\lambda_1)_{\beta_1^*}(\beta_1^*(m), (\lambda_2)_{\beta_2^*}(\beta^*(n))\Big]\\ &= \bigwedge \left[\bigvee_{\xi\in\beta^*(m)} \{\lambda_1(\xi)\}, \bigvee_{\tau\in\beta^*(n)} \{\lambda_2(\tau)\}\right].\end{aligned}$$

By lemma 2, we have

$$= \bigvee_{\substack{\xi \in \beta_1^*(m)\\ \tau \in \beta_2^*(n)}} \Big[\bigwedge(\lambda_1(\xi), \lambda_2(\tau))\Big]$$

By assumption it follows that

$$\geq \bigvee_{\substack{\xi \in \beta_1^*(m)\\ \tau \in \beta_2^*(n)}} \Big[\bigwedge(\lambda(\xi,\tau), \lambda(\xi,\tau))\Big]$$

Thus by definition 11, we have

$$\geq \bigvee_{(\xi,\tau)\in\beta^*(m,n)} \{\lambda(\xi,\tau)\} = \lambda_{\beta^*}\big(\beta^*(m,n)\big).$$

So

$$\big((\lambda_1)_{\beta_1^*} \times (\lambda_2)_{\beta_2^*}\big)(\beta_1^*(m), \beta_2^*(n)) \geq \lambda_{\beta^*}\big(\beta^*(m,n)\big).$$

Since $(\xi,\tau) \in \beta^*(m,n)$ iff $\xi\beta_1^* m$ and $\tau\beta_2^* n$, then the proof of theorem is completed.

5 Conclusion

This paper showed that a fuzzy quotient hyperalgebra can be defined by fuzzy hypercongruence relations, as defined earlier for universal algebra so based in this, we proved the important homomorphism theorems. On the other hand, we defined the product for both arbitrary fuzzy hypercongruence relations and used the smallest equivalence relation, called the fundamental relation, to examine its effect on the product of fuzzy hypercongruence relations.

Declarations

Competing interests: The authors declare that they have no competing interests.
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