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Generalized Morphic Group Rings

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Abstract: Let A be a commutative ring with identity, G be an abelian group, and consider the group ring AG. A ring A is called a generalized morphic ring (GM ring) if the annihilator of each element in A is principal. In this article, we showed that if AG is a GM ring, then so is A. The converse was proved to be false. We try to put some conditions on A or G to get the converse. Among many other results, we showed that if A is an Armendariz ring and G is a torsion free group, then AG is a GM ring if and only if A is.Moreover, if C_m denotes the multiplicative cyclic group of order m and $\mathbb{Z}_n$ the ring of integers modulo n, we justified that the ring $\mathbb{Z}_nC_m$ is a GM ring if and only if, whenever p is a prime dividing $\gcd(n,m)$, then $p^2 \nmid n$. We also proved that for an integral domain D with $char(D) = p$, the group ring DC_p is a GM ring.

Keywords: Morphic ring; Generalized morphic ring; Morphic group ring; Generalized morphic group ring.

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1 Introduction

In this article, we assume that all rings are commutative rings with identity and all groups are abelian. A ring A is said to be a **morphic ring** if for each $a \in A$ there exists $b \in A$ such that $Ann_A(a) = bA$ and $Ann_A(b) = aA$. A ring A is called a **generalized morphic ring (GM ring)** if the annihilator of each element in A is principal, see [16]. For any ring A, we denote by $Z(A)$, the set of all zero divisors in A. Moreover, the set of regular (non-zero divisors) elements and the set of all units in A are denoted by $reg(A)$ and $U(A)$, respectively. We recall that a ring A is called an **Armendariz ring** if for any two polynomials $\sum\limits_{i=0}^{n} a_i x^i$ and $\sum\limits_{i=0}^{m} b_i x^i$ over A, $(\sum\limits_{i=0}^{n} a_i x^i)(\sum\limits_{i=0}^{m} b_i x^i) = 0$ implies $a_i b_j = 0$ for all i and j. A group G is called **locally finite** if each finitely generated subgroup of G is finite. For each $n \in \mathbb{N}$, we denote the multiplicative cyclic group of order n by C_n.

For a ring A and a group G, we consider the group ring

$$AG = \left\{ \sum_{i=1}^{n} a_i g_i : a_i \in A, g_i \in G, n \in \mathbb{N} \right\}$$

Let $\varepsilon : AG \to A$ be the ring epimorphism defined by $\varepsilon\left(\sum\limits_{i=1}^{n} a_i g_i\right) = \sum\limits_{i=1}^{n} a_i$, and let

$$\Delta(G) = Ker\ \varepsilon = \left\{ \sum_{i=1}^{n} a_i g_i \in AG : \sum_{i=1}^{n} a_i = 0 \right\}$$

If G is a finite group, then we put $\hat{G} = \sum\limits_{g \in G} g$. If $g \in G$ has a finite order n, then we let $\hat{g} = 1 + g + \cdots + g^{n-1}$. For more details and terminology on group rings, see [12].

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In [5] the authors studied morphic group rings and showed that for a ring A and a group G, A is a morphic ring whenever AG is so. Then they gave some necessary conditions for the converse to be true. A continuation of this work was carried out in [8], while studying conditions under which a group ring is a principal ideal ring. Similar results were obtained for AG to be Baer (the annihilator of each non-empty subset of the ring is generated by an idempotent), quasi-Baer (the annihilator of any ideal is generated by an idempotent), or a PP-ring (every principal ideal is projective), see [14] and [15]. There are also studies that determine when AG is a clean ring, see [6].

In section 2, we studied when a group ring AG is a GM ring. We showed that if AG is a GM ring, then so is A, but not conversely. So our task is to study some situations in which the converse is true.

In section 3, we showed that if A is an Armendariz ring, and G is a **torsion free** abelian group (every non-identity element has infinite order), then AG is a GM ring if and only if A is a GM ring.

In section 4, we showed that $\mathbb{Z}_n C_m$ is a GM ring if and only if, whenever p is a prime dividing $\gcd(n,m)$, then $p^2 \nmid n$. We showed that if D is a Noetherian integral domain, $n = n.1 \in U(D)$, and $\hat{C}_n$ factors uniquely in DC_n as a product of irreducibles, then DC_n is a GM ring. Finally, we showed that if D is an integral domain, with $char(D) = p$, then DC_p is a GM ring.

2 When is AG a GM ring?

In this section, we give necessary conditions for a group ring AG to be a GM ring.

Theorem 1. *Let S be a ring and A be a subring of S such that both S and A have the same identity 1. Assume that S is a free A-module with a multiplicatively closed basis B such that $1 \in B$. Then A is a GM ring whenever S is so.*

Proof. Let $r \in A$. Then $Ann_S(r) = \alpha S$, for some $\alpha \in S$ which implies that $\alpha = \sum_{i=1}^{n} a_i g_i$, with $a_i \in A$, $g_i \in B$ for each i. Let $f : S \to A$ be defined by $f(\sum r_i g_i) = \sum r_i$. Then clearly f is a ring homomorphism, and so $0 = f(0) = f(r\alpha) = r\sum_{i=1}^{n} a_i$. Thus, $\left(\sum_{i=1}^{n} a_i\right) A \subseteq Ann_A(r)$. If $b \in Ann_A(r) \subseteq Ann_S(r) = \alpha S$, then $0 = rb$, and hence we have $b = \alpha\beta$, for some $\beta \in S$. So $b = f(b.1) = f(b) = \left(\sum_{i=1}^{n} a_i\right) f(\beta) \in \left(\sum_{i=1}^{n} a_i\right) A$. Thus, A is a GM ring.

Corollary 1. *If AG is a GM ring for a ring A and a group G, then A is a GM.*

It follows by Theorem 1, that A is a GM ring whenever the polynomial ring $A[x] = \{\sum_{i=0}^{n} a_i x^i : a_i \in A, n \in \mathbb{N}\}$ or $A[x,x^{-1}] = \{\sum_{i=-m}^{n} a_i x^i : a_i \in A, n,m \in \mathbb{N} \cup \{0\}\}$ is a GM ring, being free A-modules with bases $B_1 = \{x^n : n \in \mathbb{N} \cup \{0\}\}$ and $B_2 = \{x^n : n \in \mathbb{Z}\}$, respectively. In section 3, we will show that if A is an Armendariz ring, then the converse is also true.

In general, we note that the converse of Corollary 1 is not true. For example, the ring $\mathbb{Z}_4$ is a GM ring, while $\mathbb{Z}_4 C_2$ is not since $Ann_{\mathbb{Z}_4 C_2}(2+2g) = (2, 1-g)$ is not principal. Given that a ring A is a GM, the main objective of this paper is to determine some extra conditions on A or G that must be added to ensure that AG is also a GM ring.

Example 1.

1. It was shown in [5] that if AG is a morphic ring, then G must be a locally finite group. Now using Theorem 3.3 in [13], we see that if A is a PP-ring, then $A\mathbb{Z}$ is a PP-ring, and so it is a GM ring while $\mathbb{Z}$ is not locally finite.
2. It was shown in [14] that if AG is quasi-Baer and G is finite, then $|G|^{-1} \in A$. One can easily see that $\mathbb{Z}_2 C_2$ is a GM ring (in fact it is a morphic ring), but 2 is not even a regular element in $\mathbb{Z}_2$. Since $\mathbb{Z}_2 C_2$ is a principal ideal ring it is not a quasi-Baer ring nor a PP-ring.
3. It was shown in [14, Lemma 3.1] that if $2^{-1} \in A$, then $AC_2 \simeq A \times A$, and so A is a GM ring if and only if AC_2 is. In particular, if n is an odd integer, then $\mathbb{Z}_n C_2$ is a GM ring.
4. For any integral domain D, $Z(DC_2) = \{\alpha(1+g) : \alpha \in D\} \cup \{\beta(1-g) : \beta \in D\}$ with $Ann_{DC_2}(\alpha(1+g)) = (1-g)DC_2$ and $Ann_{DC_2}(\beta(1-g)) = (1+g)DC_2$, and so DC_2 is a GM ring. Moreover, if D is an integral domain that is not a field, with $2^{-1} \notin D$, then DC_2 is a GM ring that is not a morphic ring nor a PP-ring.

Theorem 2. *Let G be a group and A be a ring. Then AG is a GM ring if and only if AK is a GM ring for each finitely generated subgroup K of G.*

Proof. Using Theorem 1, AK is a subring of AG sharing the same identity $1.e$ and AG is a free AK-module with the multiplicatively closed basis G. Thus if AG is a GM ring, then AK is also a GM ring.

Conversely, let $u = \sum_{i=1}^{n} a_i k_i \in AG$, and let $K = \langle k_1, k_2, ..., k_n \rangle$. Then AK is a GM ring and $u \in AK$. Therefore, $Ann_{AK}(u) = cAK$ is a principal ideal for some $c \in AK$, and since $uc = 0$, we have $cAG \subseteq Ann_{AG}(u)$. Let $v \in Ann_{AG}(u)$ and $e, g_1^{'}, g_2^{'}, ...$ be the distinct representatives of the left cosets of K in G, and so $G = K \cup g_1^{'}K \cup g_2^{'}K \cup \cdots$. Thus, we have $v = \sum b_i g_i^{'}$, with $b_i \in AK$. Since $0 = uv = \sum (ub_i) g_i^{'}$, we must have $ub_i = 0$ for each i, and so $b_i = cc_i$ with $c_i \in AK$. Thus, $v = \sum b_i g_i^{'} = \sum cc_i g_i^{'} = c \sum c_i g_i^{'} \in cAG$. Hence, $Ann_{AG}(u) = cAG$ is principal, and AG is a GM ring.

Corollary 2. *Let A be a ring and let G be a locally finite group. Then AG is a GM ring if and only if AK is a GM ring for each finite subgroup K of G.*

Definition 1. *Let A be a ring and M be an A-module. For each $m \in M$ let $Ann_A(m) = \{r \in A : rm = 0\}$. An A-module M is called a GM A-module if $Ann_A(m)$ is a principal ideal for each $m \in M$.*

Theorem 3. *Let A be a ring and let G be a group. Then A is a GM ring if and only if AG is a GM A-module.*

Proof. Let $v = \sum_{i=1}^{n} a_i g_i \in AG$ and let $Ann_A(a_1, a_2, ...a_n) = cA$. It is clear that $cA \subseteq Ann_A(v)$ and if $r \in Ann_A(v)$, then $r \in Ann_A(a_1, a_2, ...a_n) = cA$. Thus, $Ann_A(v) = cA$ is principal and AG is a GM A-module. The converse is clear.

3 The Group Ring AG with G a Torsion Free Abelian Group

In this section, we discuss the GM group rings AG in the case G is a torsion free abelian group. But first, we will need the following lemma.

Lemma 1. *Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a direct system of rings such that each A_α is a GM ring. For $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, let $\varphi_\beta^\alpha : A_\alpha \longrightarrow A_\beta$ be the corresponding ring homomorphism. For $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, let $a_\alpha, b_\alpha \in A_\alpha$ and suppose that whenever $Ann_{A_\alpha}(a_\alpha) = b_\alpha A_\alpha$, we have $Ann_{A_\alpha}(\varphi_\beta^\alpha(a_\alpha)) = \varphi_\beta^\alpha(b_\alpha) A_\beta$. Then $A = \varinjlim A_\alpha$ is a GM ring.*

Proof. Using the terminology of [4, page 33] if $\alpha \leq \beta$, let $\varphi_\alpha : A_\alpha \to A_\alpha$ be a homomorphism such that $\varphi_\beta \circ \varphi_\beta^\alpha = \varphi_\beta^\alpha \circ \varphi_\alpha$. Let $a \in A$. Then there exists $\alpha \in \Lambda$ such that $a = \varphi_\alpha(a_\alpha)$. Since A_α is a GM ring, $Ann_{A_\alpha}(a_\alpha) = b_\alpha A_\alpha$. Let $b = \varphi_\alpha(b_\alpha)$. Then $ab = \varphi_\alpha(a_\alpha)\varphi_\alpha(b_\alpha) = \varphi_\alpha(a_\alpha b_\alpha) = \varphi_\alpha(0_\alpha) = 0$. So $bA \subseteq Ann_A(a)$. Let $c \in Ann_A(a)$. Then there exists $\beta \geq \alpha$ such that $c = \varphi_\beta(c_\beta)$. Now $0 = ca = \varphi_\beta(c_\beta)\varphi_\alpha(a_\alpha) = \varphi_\beta(c_\beta)\varphi_\beta(\varphi_\beta^\alpha(a_\alpha) = \varphi_\beta(c_\beta \varphi_\beta^\alpha(a_\alpha)$. Thus there exists $\gamma \geq \beta$ such that $0_\gamma = \varphi_\gamma^\beta(c_\beta \varphi_\beta^\alpha(a_\alpha)) = \varphi_\gamma^\beta(c_\beta)\varphi_\gamma^\alpha(a_\alpha)$. By assumption we must have $\varphi_\gamma^\beta(c_\beta) \in Ann_\gamma(\varphi_\gamma^\alpha(a_\alpha)) = \varphi_\gamma^\alpha(b_\alpha)A_\gamma$, that is $\varphi_\gamma^\beta(c_\beta) = \varphi_\gamma^\alpha(b_\alpha) r_\gamma$ for some $r_\gamma \in A_\gamma$. Hence $\varphi_\gamma(\varphi_\gamma^\beta(c_\beta)) = \varphi_\gamma(\varphi_\gamma^\alpha(b_\alpha))\varphi_\gamma(r_\gamma)$, and so $c = \varphi_\beta(c_\beta) = \varphi_\alpha(b_\alpha)\varphi_\gamma(r_\gamma) = b\varphi_\gamma(r_\gamma) \in bA$. Thus, $Ann_A(a) = bA$ and A is a GM ring.

Theorem 4. *Let A be an Armendariz ring and let G be a torsion free abelian group. Then A is a GM ring if and only if AG is.*

Proof. If AG is a GM ring, then A is a GM ring as shown before, so assume that A is a GM ring. Now, $AG = \varinjlim AK$ is an ascending union where K ranges over the finitely generated subgroups of G. Moreover, K is a finitely generated torsion free abelian group, and so $K \approx \mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ which implies that $AK \approx A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ which is a GM ring being a localization of the GM ring $A[x_1, \ldots, x_n]$, see [10, Theorem 2.1]. Now let $K_1 \subseteq K_2$ and let $A_1 = AK_1 = A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $A_2 = AK_2 = A[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \ldots, x_s^{\pm 1}]$. Let $a \in A_1$. Then $Ann_{A_1}(a) = bA_1$. We want to show that $Ann_{A_2}(a) = bA_2$. Since $ab = 0$ we have $bA_2 \subseteq Ann_{A_2}(a)$. Let $c \in Ann_{A_2}(a)$. Then $0 = ac = \frac{a}{1}\frac{c_1}{t_1}$ where t_1 is a monomial in $x_1, \ldots, x_s$, $c_1 \in A[x_1, \ldots, x_s]$. We can view c_1 as a polynomial in $A_1[x_{n+1}, \ldots, x_s]$. Since A is Armendariz, $\alpha_i \beta_j = 0$ for each coefficient α_i of a and coefficient β_j of c_1. Thus $\beta_j a = 0$ for each j, and so $\beta_j \in Ann_{A_1}(a) = bA_1$ and this implies that $c \in bA_2$. Hence, $Ann_{A_2}(a) = bA_2$. Using Lemma 1, we get that AG is a GM ring.

Note that $A\mathbb{Z}$ is never a morphic ring, since $\mathbb{Z}$ is not locally finite, while if A is a GM Armendariz ring, then indeed $A\mathbb{Z}$ is a GM ring.

Corollary 3. *Let A be an Armendariz ring. Then A is a GM ring if and only if $A[x,x^{-1}]$ is a GM ring if and only if $A[x]$ is a GM ring.*

Proof. A is a GM ring $\Leftrightarrow A\mathbb{Z}$ is a GM ring $\Leftrightarrow A[x,x^{-1}]$ is a GM ring.
For A is a GM ring $\Leftrightarrow A[x]$ is a GM ring, see [10].

4 The Group Ring AC_n

To work with torsion groups, we will consider the finite cyclic group C_n.
Following [3], a ring A is called an **EM ring** if for each finitely generated ideal I_1 of A there exist an element $x \in A$ and a finitely generated ideal I_2 with $I_1 = xI_2$ and $Ann_A(I_2) = \{0\}$. We recall that a ring A is said to have the **property (A)**, if whenever I is a finitely generated ideal contained in $Z(A)$, then we must have $Ann_A(I) \neq 0$.

Lemma 2. *Let A be a Noetherian ring such that $U(A) = reg(A)$. Then A is an EM ring if and only if A is a principal ideal ring.*

Proof. Suppose that A is an EM ring and I_1 is an ideal of A. Then there exist $x \in A$ and an ideal I_2 of A such that $I_1 = xI_2$ with $Ann_A(I_2) = \{0\}$. Since A has the property (A), the ideal I_2 must be regular. Hence, $I_2 = A$, and so $I_1 = xA$ is principal.

It was shown in [16, Proposition 2.8] that if A is a local ring with nilpotent Jacobson radical, then A is morphic if and only if A is GM. So if A is a finite ring, it is a product of quasi-local rings, and so a finite ring is morphic if and only if it is a GM ring. Now, we can give other equivalent conditions to the results $3.6, 3.8, 3.10$ and 3.11 in [5]. The main result is:

Theorem 5. *The following are equivalent for $n,m \in \mathbb{N} \setminus \{1\}$.*

1. $\mathbb{Z}_n C_m$ is a morphic ring.
2. $\mathbb{Z}_n C_m$ is a GM ring.
3. $\mathbb{Z}_n C_m$ is an EM ring.
4. $\mathbb{Z}_n C_m$ is a principal ideal ring.
5. If a prime integer p divides $\gcd(n,m)$, then $p^2 \nmid n$.

Proof. (1)$\Leftrightarrow$(2) Follows immediately, since $\mathbb{Z}_n C_m$ is a finite ring.
(2)$\Leftrightarrow$(3) Follows by [1, Theorem 3.26].
(3)$\Leftrightarrow$(4) Follows from Lemma 2.
(1)$\Leftrightarrow$(5) See [5, Theorem 3.10].

It was shown in [6, Lemma 3.3] that if A is a division ring, then AC_n is a morphic ring. We now give a generalization of this result. Let A be a ring, and let $a \in A$ be nonunit. Then a is called **irreducible** if $a = bc$ implies b or c is **associate** with a, that is $(a) = (b)$ or $(a) = (c)$. A commutative ring A is called **atomic** if every nonzero nonunit element of A is a finite product of irreducible elements. It is well known that if a ring A satisfies the ascending chain condition on principal ideals (in particular if A is Noetherian), then A is atomic, see [2].

Lemma 3. *Let A be a Noetherian ring and $y \in Z(A)$ such that yA is a prime principal ideal of A. If $x \in yA \setminus \{0\}$, then $x = y^n s$ for some $n \in \mathbb{N}$ and $s \in A \setminus yA$.*

Proof. See [1, Lemma 3.25].

Lemma 4. *Let A be a ring and let G be a finite group such that $|G| = n$. Then n is a unit in A if and only if $AG = \left(\hat{G}\right) \oplus \triangle(G)$ as rings.*

Proof. It is clear that for any $\alpha \in AG, \alpha\hat{G} = \varepsilon(\alpha)\hat{G}$. Thus $(n^{-1}\hat{G})^2 = n^{-1}\hat{G}$ is an idempotent, and so we have $AG = (n^{-1}\hat{G}) \oplus (1-n^{-1}\hat{G})$. But $(n^{-1}\hat{G}) = (\hat{G})$, and $1-n^{-1}\hat{G} \in \triangle(G)$. Now, if $f \in \triangle(G)$, then

$$f = fn^{-1}\hat{G} + f(1-n^{-1}\hat{G}) = \varepsilon(f)n^{-1}\hat{G} + f(1-n^{-1}\hat{G}) = f(1-n^{-1}\hat{G}) \in (1-n^{-1}\hat{G})$$

and since we have $1-n^{-1}\hat{G} \in \triangle(G)$, then $AG = \left(\hat{G}\right) \oplus \triangle(G)$ as rings. If $AG = \left(\hat{G}\right) \oplus \triangle(G)$, then $1 = \alpha\hat{G} + \beta$, where $\alpha \in AG$, and $\beta \in \triangle(G)$. Thus, $1 = \varepsilon(\alpha\hat{G} + \beta) = \varepsilon(\alpha)n + 0$.

Theorem 6. *If D is a Noetherian integral domain, $n \in U(D)$, and $\hat{C}_n$ factors uniquely in DC_n as a product of irreducibles, then DC_n is a GM ring.*

Proof. Note that DC_n is a reduced Noetherian ring, see [7, Theorem 5]. Assume that $C_n = \langle g \rangle$. Since $\Delta(C_n)$ is generated by the set $\{1-g^i : 1 \le i < n\}$ and $1-g^i = (1-g)(1+g+\cdots+g^{i-1})$, we have $\Delta(C_n) = (1-g)DC_n$ a principal prime ideal, since D is an integral domain, and so it follows by Lemma 4, that $DC_n = (\hat{g}) \oplus (1-g)$. Thus, we can write any element in DC_n as $\alpha_0\hat{g} + \alpha_1(1-g)$, where $\alpha_0 \in D$ and $\alpha_1 \in \triangle(C_n)$. Note that $1-g$ is an irreducible element in DC_n being a prime element, and assume that $\hat{g} = \gamma_1\gamma_2\cdots\gamma_m$ a finite product of irreducibles. Let $f = \alpha_0\hat{g} + \alpha_1(1-g) \in Z(DC_n) \setminus \{0\}$ and $h = \beta_0\hat{g} + \beta_1(1-g) \in Ann_{DC_n}(f) \setminus \{0\}$. Then $\alpha_0\beta_0 n\hat{g} + \alpha_1\beta_1(1-g)^2 = 0$, which implies that $\alpha_0\beta_0 n\hat{g} = -\alpha_1\beta_1(1-g)^2 \in (\hat{g}) \cap \triangle(C_n) = \{0\}$, and so $\alpha_0\beta_0 n\hat{g} = \alpha_1\beta_1(1-g)^2 = 0$. Thus we have, $\alpha_0\beta_0 = 0$, which implies that $\alpha_0 = 0$ or $\beta_0 = 0$, and since DC_n is a reduced ring we have $\alpha_1\beta_1(1-g) = 0$, and so, $\alpha_1\beta_1 \in Ann_{DC_n}(1-g) \cap \triangle(C_n) = (\hat{g}) \cap \triangle(C_n) = \{0\}$. Using Lemma 3, and that DG is an atomic ring with $1-g$ is an irreducible element, we get $\alpha_1 = a(1-g)^s, \beta_1 = b(1-g)^l$, where $a, b \in DC_n \backslash (1-g)$, and so, $0 = ab(1-g)^{s+l}$, and since DC_n is reduced, we have $ab(1-g) = 0$, and hence $ab \in (\hat{g})$. Thus $\alpha_1 = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}\alpha_2(1-g)^s$ and $\beta_1 = \frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}\beta_2(1-g)^l$. If $\alpha_0 = 0$, then $f = \alpha_1(1-g)$,

$$h = \beta_0\hat{g} + \frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}\beta_2(1-g)^l = (\beta_0\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} + \beta_2(1-g)^l)(\frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}})$$

and $Ann_{DC_n}(f) = \left(\frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}\right)$ is principal. If $\alpha_0 \neq 0$, then $f = \alpha_0\hat{g} + \alpha_1(1-g), h = \beta_1(1-g) \in \frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}(1-g)$, and $Ann_{DC_n}(f) = (\frac{\hat{g}}{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}(1-g))$ is principal. Thus, DC_n is a GM ring.

Example 2. The ring $\mathbb{Q}[x]C_n$ is a GM ring for any $n \in \mathbb{N}$, but it is not a morphic ring, since $\mathbb{Q}[x]$ is not.

We now turn to the case when n is not a unit in A. We showed in Example 1(4) that if D is an integral domain, then DC_2 is a GM ring. Now, we generalize to any prime number.

Theorem 7. *Let D be an integral domain with $char(D) = p$, an odd prime. Then DC_p is a GM ring.*

Proof. Let $C_p = \langle g \rangle$, and let $f = \sum_{i=0}^{p-1} a_i g^i \in Z(DC_p) \backslash \{0\}$. Assume $h = \sum_{i=0}^{p-1} x_i g^i \in Ann(f) \backslash \{0\}$. Then we get the system $\sum_{i+j \equiv k \pmod{p}} a_i x_j = 0$ for $k = 0, 1, \ldots, p-1$. Thus we have a circulant matrix

$$C = CIRC(a_0, a_1, \ldots, a_{p-1}) = \begin{bmatrix} a_0 & a_1 & \ldots & a_{p-2} & a_{p-1} \\ a_{p-1} & a_0 & \ldots & a_{p-3} & a_{p-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & \ldots & a_0 & a_1 \\ a_1 & a_2 & \ldots & a_{p-1} & a_0 \end{bmatrix}$$

such that $C \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Working in the field of fractions $Q(D)$ and using [9, Theorem 1.1], we get $0 = \det(C) = \sum_{i=0}^{p-1} a_i^p + pt$ for $t \in D$. Since $char(D) = p$, we get $0 = \det(C) = \sum_{i=0}^{p-1} a_i^p = \left(\sum_{i=0}^{p-1} a_i\right)^p$, and hence $\sum_{i=0}^{p-1} a_i = 0$, i.e., $f \in \Delta(C_p)$.
If $a_i = a_j$ for all i, j, then $f = a_0\hat{g}$ and $Ann(f) = (1-g)$ is principal.
If $a_i \neq a_j$ for some $i \neq j$, then C is nonrecurrent circular matrix, and so it follows by [11, Theorem 2.1] that $rank(C) = p-1$, and thus, $nullity(C) = 1$, that is $nullspace(C) = (c_0, c_1, \ldots, c_{p-1})D$ is a one dimensional $D-$module. But $h \in Ann_{DC_p}(f)$ if and only if $(x_0, x_1, \ldots, x_{p-1}) \in nullspace(C) = (c_0, c_1, \ldots, c_{p-1})D$, and so $h \in \left(\sum_{i=0}^{p-1} c_i g^i\right) DC_p$. Thus, $Ann_{DC_p}(f) = \left(\sum_{i=0}^{p-1} c_i g^i\right) DC_p$ is a principal ideal.

Declarations

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References

[1] E. Abuosba, M. Ghanem, Annihilating Content for Polynomial and Power Series Rings, J. Korean Math. Soc., **56 (5)**(2019), 1403-1418.
[2] D.D. Anderson, Silvia Valdes-leon, Factorization in Commutative Rings with Zero Divisors, Rocky Mountain Journal of Mathematics**, 26 (2)**(1996), 439-472.
[3] D.D. Anderson, E. Abuosba, M. Ghanem, Annihilating Content Polynomials and EM-rings, J. Alg. Appl., **21(5)**(2022), Article No. 2250092.
[4] M. Atiyah and I. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publication Company, USA, (1969).
[5] J. Chen, Y. Li, Y. Zhou, Morphic Group Rings, J. Pure & Applied Alg., **205** (2006), 621-639.
[6] J. Chen, W. Nicholson, Y. Zhou, Group Rings in which Every Element is Uniquely the Sum of a Unit and an Idempotent, J. Algebra, **306** (2006), 453-460.
[7] I. Connel, On Group Rings, Can. J. of Math., **15** (1963), 650-685.
[8] T. Dorsey, Morphic and Principal-ideal Group Rings, J. Algebra, **318** (2007), 393-411.
[9] M. Ghanem, A. Wyn-Jones, H. AL-Ezeh, Presimplifiable and Cyclic Group Rings, Comm. Alg., **40(8)**(2012),3036-3046.
[10] M. Ghanem, E. Abuosba, Some Extensions of Generalized Morphic Rings and EM-rings, Analele Stiintifice ale Universitatii Ovidius Constanta (Seria Matematica), **26(1)**(2018), 111-123.
[11] A. Ingleton, The Rank of Circulant Matrices, Journal of the London Mathematical Society, **1-31(4)**(1956), 445-460.
[12] C. Milies, S. Sehgal, An Introduction to Group Rings, Kluwer Academic Publishers, Netherlands, (2002).
[13] R. Schwarz, S. Glaz, Commutative Group Rings with Von Neumann Regular Total Rings of Quotients. J. Algebra, **388** (2013), 287-293.
[14] Z. Yi, Y. Zhou, Baer and Quasi-Baer Properties of Group Rings, J. Aust. Math. Soc., **83** (2007), 285-296.
[15] L. Zan, J. Chen, P.P. Properties of Group Rings, Inter. Electronic J. Algebra, **3** (2008), 117-124.
[16] H. Zhu, N. Ding, Generalized Morphic Rings and their Applications, Comm. Alg., **35** (2007), 2820-2837.