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Nonlinear Noncanonical Second Order Neutral Difference Equations: Modified Oscillation Results

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Abstract: In this paper, we derive some new oscillation criteria for the second-order nonlinear noncanonical neutral difference equation. The findings are novel and acquired through the transformation of the analyzed equation into canonical form, followed by the application of the summation averaging technique. Examples are provided to demonstrate the significance of the primary results.

Keywords: Oscillation; neutral difference equation; noncanonical; second-order.

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1 Introduction

The theory of oscillation in difference equations and its practical applications have sparked substantial interest in recent decades, see for example [1,2,3] and the references cited therein. The analysis of the oscillatory behaviors of second-order functional difference equations has garnered considerable interest, notably because of their fundamental importance in tackling a diverse array of applied problems in various fields such as fluid dynamics, electromagnetism, acoustic vibrations, quantum mechanics, biological, physical and chemical phenomena, optimization, mathematics of networks, and dynamical systems, the sources [1,12] can be referred to by the reader. The numerous publications in the literature demonstrate how much interest there has been in the oscillation and asymptotic behavior of second-order difference equations with linear, sublinear, and superlinear neutral terms in recent years; see for example [4,5,6,7,8,12, 13,14,15,16,17,18,19] as well as the references cited .

In [5], the authors obtained modified conditions for the oscillation of all solutions of second-order difference equations with negative neutral term of canonical type. The authors of [11] presented a novel oscillation criteria for a second-order half-linear neutral dynamic equation. To illustrate how important the delayed function is to the oscillatory activity, a convincing example was shown. In [9], the oscillation of a second-order nonlinear dynamic equation under the noncanonical condition was considered. However, the oscillatory behavior of a second-order dynamic equation with a sublinear neutral component was discussed by the authors in [10]. They arrived at new oscillation criteria by applying the Riccati transformation and comparison concepts. In [18], the authors suggested oscillation conditions for noncanonical second-order nonlinear delay difference equations with a superlinear neutral term. The outcomes are acquired with less restrictive presumptions.In [5], the authors obtained modified conditions for the oscillation of all solutions of second-order canonical type difference equations with negative neutral term.

Inspired by the results in [9,10,11], our objective in this paper is to develop some new sufficient conditions for the oscillation of all solutions of the second-order nonlinear noncanonical neutral difference equation

$$\Delta\left(\mu(\varsigma)\Delta\left(\xi(\varsigma)+\alpha(\varsigma)\xi(\tau(\varsigma))\right)\right)+w(\varsigma)\xi^{\delta}(\sigma(\varsigma))=0, \qquad (E)$$

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where $\varsigma \geq \dot{\varsigma}_0 \geq 0$ and δ is the ratio of odd positive integers with $0 < \delta \leq 1$. Hereafter, it is assumed that:

(i) $\{\mu(\dot{\varsigma})\}$ and $\{w(\dot{\varsigma})\}$ are sequences of positive real numbers,
(ii) $\{\alpha(\dot{\varsigma})\}$ is a positive real sequence with $0 \leq \alpha(\dot{\varsigma}) \leq d < 1$,
(iii) $\{\tau(\dot{\varsigma})\}$ is sequences of integers such that $\tau(\dot{\varsigma}) \leq \dot{\varsigma} - 1$,
(iv) $\sigma(\dot{\varsigma})$ is nondecreasing sequence of integers and $\lim_{n\to\infty} \tau(\dot{\varsigma}) = \lim_{\dot{\varsigma}\to\infty} \sigma(\dot{\varsigma}) = \infty$.

Set

$$\Lambda(\dot{\varsigma}) = \sum_{s=\dot{\varsigma}}^{\infty} \frac{1}{\mu(s)},\ \dot{\varsigma} \geq \dot{\varsigma}_0,$$

and assume that

$$\Lambda(\dot{\varsigma}_0) < \infty, \tag{1}$$

that is, the equation is in noncanonical form.

Under a *solution* of equation (E), we mean a real sequence $\{\xi(\dot{\varsigma})\}$ defined and satisfying (E) for all $\dot{\varsigma} \geq \dot{\varsigma}_0$. We consider only solutions of (E) that satisfy $\sup\{|\xi(\dot{\varsigma})| : \dot{\varsigma} \geq N\} > 0$ for all $\dot{\varsigma} \geq \dot{\varsigma}_1 \geq \dot{\varsigma}_0$, and we tacitly assume that (E) possesses such solutions. A solution of (E) is said to be *oscillatory* if it is neither eventually negative nor eventually positive. Otherwise it is called *nonoscillatory*. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Unlike previous findings, the key contributions of this paper are twofold:

(i) We convert the examined equation into a canonical form without imposing any additional restrictions, therefore reducing the set of nonoscillatory solutions from two to one.
(ii) We use the summation averaging approach to derive new oscillation conditions for the examined equation. Our results go beyond those of [15,16,17,18,19] by introducing new theorems not addressed in prior literature. We formulate these theorems within a broad framework that includes specific cases.

Further the results in [5] are applicable only if the neutral coefficient is negative whereas our results are valid for positive neutral coefficients.

The paper is structured so that in Section 2, we give the primary oscillation results. Two Euler type neutral difference equations are provided as examples to support the theoretical arguments presented in Section 3. Section 4 concludes with a succinct statement.

2 Oscillation Results

For convenience, we adopt the following notations:

$$\eta(\dot{\varsigma}) = \xi(\dot{\varsigma}) + \alpha(\dot{\varsigma})\xi(\tau(\dot{\varsigma})) \text{ and } \beth(\dot{\varsigma}) = \frac{1 - p(\sigma(\dot{\varsigma}))\Lambda(\tau(\sigma(\dot{\varsigma})))}{\Lambda(\sigma(\dot{\varsigma}))}$$

for $\dot{\varsigma} \geq \dot{\varsigma}_1 \geq \dot{\varsigma}_0$. Notice that $\beth(\dot{\varsigma}) \geq 0$ for $\dot{\varsigma} \geq \dot{\varsigma}_1 \geq \dot{\varsigma}_0$. WLG, we exclusively work with positive solutions of (E) since if $\xi(\dot{\varsigma})$ is a solution of (E), then so is $-\xi(\dot{\varsigma})$.

Theorem 1. *Assume* (1) *holds. The noncanonical operator*

$$D\eta(\dot{\varsigma}) = \Delta(\mu(\dot{\varsigma})\Delta\eta(\dot{\varsigma}))$$

has the canonical representation

$$D\eta(\dot{\varsigma}) = \frac{1}{\Lambda(\dot{\varsigma}+1)}\Delta\left(\mu(\dot{\varsigma})\Lambda(\dot{\varsigma})\Lambda(\dot{\varsigma}+1)\Delta\left(\frac{\eta(\dot{\varsigma})}{\Lambda(\dot{\varsigma})}\right)\right). \tag{2}$$

Proof. A direct calculation shows that

$$\begin{aligned}\frac{1}{\Lambda(\dot{\varsigma}+1)}\Delta\left(\mu(\dot{\varsigma})\Lambda(\dot{\varsigma})\Lambda(\dot{\varsigma}+1)\Delta\left(\frac{\eta(\dot{\varsigma})}{\Lambda(\dot{\varsigma})}\right)\right) &= \frac{1}{\Lambda(\dot{\varsigma}+1)}\Delta\left(\Lambda(\dot{\varsigma})\mu(\dot{\varsigma})\Delta\eta(\dot{\varsigma}) + \eta(\dot{\varsigma})\right)\\ &= \frac{1}{\Lambda(\dot{\varsigma}+1)}\Lambda(\dot{\varsigma}+1)\Delta\left(\mu(\dot{\varsigma})\Delta\eta(\dot{\varsigma})\right)\\ &= \Delta(\mu(\dot{\varsigma})\Delta\eta(\dot{\varsigma})).\end{aligned}$$

Therefore,

$$\frac{1}{\Lambda(\dot{\varsigma}+1)}\Delta\left(\mu(\dot{\varsigma})\Lambda(\dot{\varsigma})\Lambda(\dot{\varsigma}+1)\Delta\left(\frac{\eta(\dot{\varsigma})}{\Lambda(\dot{\varsigma})}\right)\right)=D\eta(\dot{\varsigma}).$$

To see that (2) is in canonical form, note that

$$\sum_{\dot{\varsigma}=\dot{\varsigma}_0}^{\infty}\frac{1}{\mu(\dot{\varsigma})\Lambda(\dot{\varsigma})\Lambda(\dot{\varsigma}+1)}=\sum_{\dot{\varsigma}=\dot{\varsigma}_0}^{\infty}\Delta\left(\frac{1}{\Lambda(\dot{\varsigma})}\right)=\lim_{\dot{\varsigma}\to\infty}\frac{1}{\Lambda(\dot{\varsigma})}-\frac{1}{\Lambda(\dot{\varsigma}_0)}=\infty.$$

This completes the proof.

From Theorem 1, we see that equation (E) can be written in the equivalent canonical form

$$\Delta\left(b(\dot{\varsigma})\Delta\left(\frac{\eta(\dot{\varsigma})}{\Lambda(\dot{\varsigma})}\right)\right)+\Lambda(\dot{\varsigma}+1)w(\dot{\varsigma})\xi^{\delta}(\sigma(\dot{\varsigma}))=0,$$

where $b(\dot{\varsigma})=\mu(\dot{\varsigma})\Lambda(\dot{\varsigma})\Lambda(\dot{\varsigma}+1)$. Setting $z(\dot{\varsigma})=\frac{\eta(\dot{\varsigma})}{\Lambda(\dot{\varsigma})}$, we get the following result.

Theorem 2. *The noncanonical difference equation (E) own a solution $\xi(\dot{\varsigma})$ if and only if the canonical equation*

$$\Delta\left(b(\dot{\varsigma})\Delta z(\dot{\varsigma})\right)+\Lambda(\dot{\varsigma}+1)w(\dot{\varsigma})\xi^{\delta}(\sigma(\dot{\varsigma}))=0 \tag{E_1}$$

has same $\xi(\dot{\varsigma})$ as solution.

Corollary 1. *Noncanonical difference equation (E) has an eventually positive solution if and only if the canonical equation (E_1) has an eventually positive solution.*

Clearly, Corollary 1 simplifies the investigation of (E) since for (E_1) if $\{\xi(\dot{\varsigma})\}$ is an eventually positive solution, then the corresponding sequence $\{z(\dot{\varsigma})\}$ satisfies only one class, namely,

$$z(\dot{\varsigma})>0,\ \Delta z(\dot{\varsigma})>0,\ \Delta(b(\dot{\varsigma})\Delta z(\dot{\varsigma}))<0. \tag{3}$$

Before stating and proving our main results, we define

$$\digamma(\dot{\varsigma})=\Lambda(\dot{\varsigma}+1)w(\dot{\varsigma})\daleth(\sigma(\dot{\varsigma})),\ B(\dot{\varsigma})=\sum_{s=\dot{\varsigma}_1}^{\dot{\varsigma}-1}\frac{1}{b(s)}$$

and

$$\daleth(\dot{\varsigma})=\Lambda(\dot{\varsigma})\left(1-\alpha(\dot{\varsigma})\frac{\Lambda(\tau(\dot{\varsigma}))}{\Lambda(\dot{\varsigma})}\right)\ \text{ for } \dot{\varsigma}\geq\dot{\varsigma}_1\geq\dot{\varsigma}_0.$$

Note that $\daleth(\dot{\varsigma})\geq 0$ for $\dot{\varsigma}\geq\dot{\varsigma}_1\geq\dot{\varsigma}_0$. First, we state and prove three criteria for the oscillation of all solutions of (E) in the case $\{\sigma(\dot{\varsigma})\}$ is a delay argument, that is, for the case $\sigma(\dot{\varsigma})\leq\dot{\varsigma}-1$.

Theorem 3. *Let (1) hold. If*

$$\sum_{\dot{\varsigma}=\dot{\varsigma}_0}^{\infty}\digamma(\dot{\varsigma})B^{\delta}(\sigma(\dot{\varsigma}))=\infty \tag{4}$$

and

$$\limsup_{\dot{\varsigma}\to\infty}\left(\frac{1}{B(\dot{\varsigma})}\sum_{s=\dot{\varsigma}_0}^{\dot{\varsigma}-1}\digamma(s)B(s+1)B^{\delta}(\sigma(s))+B^{\delta}(\dot{\varsigma})\sum_{s=\dot{\varsigma}}^{\infty}\digamma(s)\frac{B^{\delta}(\sigma(s))}{B^{\delta}(s)}\right) > \begin{cases}1 & if\ \ \delta=1\\ 0 & if\ \ 0<\delta<1,\end{cases} \tag{5}$$

then equation (E) is oscillatory.

Proof. Let $\{\xi(\dot{\varsigma})\}$ be a nonoscillatory solution of (E), say $\xi(\dot{\varsigma})>0$, $\xi(\sigma(\dot{\varsigma}))>0$, and $\xi(\tau(\dot{\varsigma}))>0$ for $\dot{\varsigma}\geq\dot{\varsigma}_1$ for some $\dot{\varsigma}_1\geq\dot{\varsigma}_0$. Then by Corollary 1, $\{\xi(\dot{\varsigma})\}$ is also a positive solution of (E_1) and the corresponding sequence $z(\dot{\varsigma})>0$ for all $\dot{\varsigma}\geq\dot{\varsigma}_1\geq\dot{\varsigma}_0$. Thus, $z(\dot{\varsigma})$ satisfies (3). Now, from the definition of $z(\dot{\varsigma})$, we have

$$z(\dot{\varsigma})\Lambda(\dot{\varsigma})=\eta(\dot{\varsigma})=\xi(\dot{\varsigma})+\alpha(\dot{\varsigma})\xi(\tau(\dot{\varsigma}))$$

and so

$$\begin{aligned}\xi(\dot{\varsigma})&\geq\Lambda(\dot{\varsigma})z(\dot{\varsigma})-\alpha(\dot{\varsigma})\Lambda(\tau(\dot{\varsigma}))z(\tau(\dot{\varsigma}))\\&\geq\gimel(\dot{\varsigma})z(\dot{\varsigma}),\ \dot{\varsigma}\geq\dot{\varsigma}_1\geq\dot{\varsigma}_0.\end{aligned}$$

Using this in (E_1), we obtain

$$\Delta\left(b(\dot{\varsigma})\Delta z(\dot{\varsigma})\right)+\digamma(\dot{\varsigma})z^{\delta}(\sigma(\dot{\varsigma}))\leq 0,\ \dot{\varsigma}\geq\dot{\varsigma}_1. \tag{6}$$

From (3), we see that

$$z(\dot{\varsigma})=z(\dot{\varsigma}_1)+\sum_{s=\dot{\varsigma}}^{\dot{\varsigma}-1}\frac{b(s)\Delta z(s)}{b(s)}\geq B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma}),$$

or

$$z(\dot{\varsigma})-B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma})\geq 0, \tag{7}$$

and

$$\Delta\left(\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}\right)=\frac{b(\dot{\varsigma})B(\dot{\varsigma})\Delta z(\dot{\varsigma})-z(\dot{\varsigma})}{B(\dot{\varsigma})B(\dot{\varsigma}+1)b(\dot{\varsigma})}\leq 0.$$

That is

$$\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}\ \text{is decreasing} \tag{8}$$

and so

$$\lim_{\dot{\varsigma}\to\infty}\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}=M\geq 0.$$

We claim that $M=0$. If $M>0$, then there exists an integer $\dot{\varsigma}_2\geq\dot{\varsigma}_1$ such that

$$z(\dot{\varsigma})\geq\frac{M}{2}B(\dot{\varsigma}) \tag{9}$$

for $\dot{\varsigma}\geq\dot{\varsigma}_2$. Using (9) in (6) yields

$$\Delta\left(b(\dot{\varsigma})\Delta z(\dot{\varsigma})\right)+\left(\frac{M}{2}\right)^{\alpha}\digamma(\dot{\varsigma})B^{\delta}(\sigma(\dot{\varsigma}))\leq 0,\ \dot{\varsigma}\geq\dot{\varsigma}_2.$$

Summing up the last inequality from $\dot{\varsigma}_2$ to $\dot{\varsigma}$, we have

$$\left(\frac{M}{2}\right)^{\alpha}\sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}}\digamma(s)B^{\delta}(\sigma(s))\leq b(\dot{\varsigma}_2)\Delta z(\dot{\varsigma}_2)<\infty,$$

which is a contradiction, and so

$$\lim_{\dot{\varsigma}\to\infty}\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}=0. \tag{10}$$

On the other hand, we can rewrite (6) in the equivalent form

$$\Delta\left(z(\dot{\varsigma})-B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma})\right)-\digamma(\dot{\varsigma})B(\dot{\varsigma}+1)z^{\delta}(\sigma(\dot{\varsigma}))\geq 0. \tag{11}$$

Summing up (11) from $\dot{\varsigma}_2$ to $\dot{\varsigma}-1$ and using (7) gives

$$z(\dot{\varsigma})-B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma})\geq\sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1}\digamma(s)B(s+1)z^{\delta}(\sigma(s)). \tag{12}$$

Summing up (6) from $\dot{\varsigma}$ to j and letting $j \to \infty$, we get

$$B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma}) \geq B(\dot{\varsigma}) \sum_{s=\dot{\varsigma}}^{\infty} F(s) z^{\delta}(\sigma(s)). \tag{13}$$

It follows from (12) and (13) that

$$z(\dot{\varsigma}) \geq \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)z^{\delta}(\sigma(s)) + B(\dot{\varsigma}) \sum_{s=\dot{\varsigma}}^{\infty} F(s) z^{\delta}(\sigma(s)). \tag{14}$$

Taking into account the fact $z(\dot{\varsigma})$ is increasing and $\frac{z(\dot{\varsigma})}{B(n)}$ is decreasing, we have from (14) that

$$z(\dot{\varsigma}) \geq \frac{z^{\delta}(\dot{\varsigma})}{B^{\delta}(\dot{\varsigma})} \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B(\dot{\varsigma}) z^{\delta}(\dot{\varsigma}) \sum_{s=\dot{\varsigma}}^{\infty} \frac{F(s)B^{\delta}(\sigma(s))}{B^{\delta}(s)},$$

or

$$\left(\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}\right)^{1-\delta} \geq \frac{1}{B(\dot{\varsigma})} \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B^{\delta}(\dot{\varsigma}) \sum_{s=\dot{\varsigma}}^{\infty} \frac{F(s)B^{\delta}(\sigma(s))}{B^{\delta}(s)}.$$

Now taking the lim sup as $\dot{\varsigma} \to \infty$ and then using (10), we get a contradiction with (5).

Theorem 4. *Let* (1) *and* (4) *hold. If*

$$\limsup_{\dot{\varsigma}\to\infty} \left(\frac{1}{B(\sigma(\dot{\varsigma}))} \sum_{s=\dot{\varsigma}_0}^{\sigma(\dot{\varsigma})-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B^{1-\delta}(\sigma(\dot{\varsigma})) \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B^{\delta}(\sigma(s)) + B(\sigma(\dot{\varsigma})) \sum_{s=\dot{\varsigma}}^{\infty} F(s) \right) > \begin{cases} 1 & if \ \ \delta = 1 \\ 0 & if \ \ 0 < \delta < 1, \end{cases} \tag{15}$$

then equation (E) *is oscillatory.*

Proof. Let $\{\xi(\dot{\varsigma})\}$ be a positive solution of (E) such that $\xi(\dot{\varsigma}) > 0$, $\xi(\sigma(\dot{\varsigma})) > 0$, and $\xi(\tau(\dot{\varsigma})) > 0$ for $\dot{\varsigma} \geq \dot{\varsigma}_1$ for some integer $\dot{\varsigma}_1 \geq \dot{\varsigma}_0$. Then by Corollary 1, we see that $\{\xi(\dot{\varsigma})\}$ is also a positive solution of (E_1) and the corresponding sequence $z(\dot{\varsigma}) > 0$ for all $\dot{\varsigma} \geq \dot{\varsigma}_1$. Thus $z(\dot{\varsigma})$ satisfies (3). Proceeding as in the proof of Theorem 3, we again at (14), which can be written as below

$$z(\sigma(\dot{\varsigma})) \geq \sum_{s=\dot{\varsigma}_2}^{\sigma(\dot{\varsigma})-1} F(s)B(s+1)z^{\delta}(\sigma(s)) + B(\sigma(\dot{\varsigma})) \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)z^{\delta}(\sigma(s)) + B(\sigma(\dot{\varsigma})) \sum_{s=\dot{\varsigma}}^{\infty} F(s)z^{\delta}(\sigma(s)).$$

The remainder of the proof is identical to that of Theorem 3 and is thus omitted.

Theorem 5. *Let* (1) *and* (4) *hold. If*

$$\limsup_{\dot{\varsigma}\to\infty} \left(\frac{1}{B(\dot{\varsigma})} \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) \right) > \begin{cases} 1 & if \ \ \delta = 1 \\ 0 & if \ \ 0 < \delta < 1, \end{cases} \tag{16}$$

then equation (E) *is oscillatory.*

Proof. Let $\{\xi(\dot{\varsigma})\}$ be a positive solution of (E) such that $\xi(\sigma(\dot{\varsigma})) > 0$, and $\xi(\tau(\dot{\varsigma})) > 0$ for $\dot{\varsigma} \geq \dot{\varsigma}_1$ for some integer $\dot{\varsigma}_1 \geq \dot{\varsigma}_0$. Proceeding as in the proof of Theorem 3, we again get (7) and (11). Letting

$$h(\dot{\varsigma}) = z(\dot{\varsigma}) - B(\dot{\varsigma})b(\dot{\varsigma})\Delta z(\dot{\varsigma}),$$

we have $0 \leq h(\dot{\varsigma}) \leq z(\dot{\varsigma})$ since $\Delta z(\dot{\varsigma}) > 0$. It then follows from (11) that

$$\Delta h(\dot{\varsigma}) - F(\dot{\varsigma})B(\dot{\varsigma}+1)z^{\delta}(\sigma(\dot{\varsigma})) \geq 0.$$

Summing the last inequality from $\sigma(\dot{\varsigma})$ to $\dot{\varsigma}-1$, we obtain

$$h(\dot{\varsigma}) \geq \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B(s+1)z^{\delta}(\sigma(s)) \geq \frac{z^{\delta(\dot{\varsigma})}}{B^{\delta}(\dot{\varsigma})} \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s))$$
$$\geq \frac{h^{\delta(\dot{\varsigma})}}{B^{\delta}(\dot{\varsigma})} \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)),$$

so

$$\left(\frac{h(\dot{\varsigma})}{B(\dot{\varsigma})}\right)^{1-\delta} \geq \frac{1}{B(\dot{\varsigma})} \sum_{s=\sigma(\dot{\varsigma})}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)).$$

Now take the $\limsup_{\dot{\varsigma}\to\infty}$ on both sides of the last inequality. Recalling the fact that $\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})} \to 0$, which implies $\frac{h(\dot{\varsigma})}{B(\dot{\varsigma})} \to 0$, we obtain a contradiction with (16). This completes the proof.

Next, we present two oscillation results for equation (E) in the case $\{\sigma(\dot{\varsigma})\}$ is an advanced argument, that is $\sigma(\dot{\varsigma}) \geq \dot{\varsigma}+1$.

Theorem 6.*Let* (1) *and* (4) *hold. If*

$$\limsup_{\dot{\varsigma}\to\infty} \left(\frac{B^{\delta-1}(\dot{\varsigma})}{B^{\delta}(\sigma(\dot{\varsigma}))} \sum_{s=\dot{\varsigma}_0}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B^{\delta}(\dot{\varsigma}) \sum_{s=\dot{\varsigma}}^{\infty} F(s) \right) > \begin{cases} 1 & if \;\; \delta = 1 \\ 0 & if \;\; 0<\delta<1, \end{cases} \tag{17}$$

then equation (E) is oscillatory.

*Proof.*Let $\{\xi(\dot{\varsigma})\}$ be a positive solution of (E) such that $\xi(\sigma(\dot{\varsigma}))>0$, and $\xi(\tau(\dot{\varsigma}))>0$ for $\dot{\varsigma} \geq \dot{\varsigma}_1$ for some integer $\dot{\varsigma}_1 \geq \dot{\varsigma}_0$. Following the proof of Theorem 3, we reach (14). . Applying that $z(\dot{\varsigma})$ is increasing and $\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}$ is decreasing, we see that if $s \leq \dot{\varsigma}$, we have $s \leq \sigma(s) \leq \sigma(\dot{\varsigma})$ and $\sigma(\dot{\varsigma}) \geq \dot{\varsigma}+1$, so

$$\frac{z(\sigma(s))}{B(\sigma(s))} = \frac{z(\sigma(t))}{B(\sigma(t))}.$$

Now

$$\sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)z^{\delta}(\sigma(s)) \geq \left(\sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s))\right) \frac{z^{\delta}(\sigma(t))}{B^{\delta}(\sigma(t))}$$
$$\geq \frac{z^{\delta}(t)}{B^{\delta}(\sigma(t))} \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)). \tag{18}$$

Also, if $s > \dot{\varsigma}$, we have $\sigma(s) \geq \sigma(\dot{\varsigma})$, and so

$z^{\delta}(\sigma(s)) \geq z^{\delta}(\sigma(\dot{\varsigma})) \geq z^{\delta}(\dot{\varsigma})$.

Thus

$$\sum_{s=\dot{\varsigma}}^{\infty} F(s)z^{\delta}(\sigma(s)) \geq \left(\sum_{s=\dot{\varsigma}}^{\infty} F(s)\right) z^{\delta}(\dot{\varsigma}). \tag{19}$$

Using (18) and (19) in (14) gives

$$z^{1-\delta}(\dot{\varsigma}) \geq \frac{1}{B^{\delta}(\sigma(t))} \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B(\dot{\varsigma}) \sum_{\dot{\varsigma}}^{\infty} F(s)$$

or

$$\left(\frac{z(\dot{\varsigma})}{B(\dot{\varsigma})}\right)^{1-\delta} \geq \frac{B^{\delta-1}(\dot{\varsigma})}{B^{\delta}(\sigma(\dot{\varsigma}))} \sum_{s=\dot{\varsigma}_2}^{\dot{\varsigma}-1} F(s)B(s+1)B^{\delta}(\sigma(s)) + B^{\delta}(\dot{\varsigma}) \sum_{\dot{\varsigma}}^{\infty} F(s).$$

The proof reminder bears similarities to that Theorem 3 and hence is left out.

Theorem 7. *Let* (1) *and* (4) *hold. If*

$$\limsup_{\varsigma\to\infty}\left(\frac{1}{B(\sigma(\varsigma))}\sum_{s=\varsigma_0}^{\varsigma-1}F(s)B(s+1)B^{\delta}(\sigma(s))+B^{\delta-1}(\sigma(\varsigma))\sum_{s=n}^{\sigma(\varsigma)-1}F(s)B(s+1)\right.$$
$$\left.+B^{\delta}(\sigma(\varsigma))\sum_{s=\sigma(\varsigma)}^{\infty}F(s)\right)>\begin{cases}1 & if\ \ \delta=1\\ 0 & if\ \ 0<\delta<1,\end{cases} \tag{20}$$

then equation (E) *is oscillatory.*

Proof. Let $\{\xi(\varsigma)\}$ be a positive solution of (E) such that $\xi(\sigma(\varsigma))>0$, and $\xi(\tau(\varsigma))>0$ for $\varsigma\geq\varsigma_1$ for some integer $\varsigma_1\geq\varsigma_0$. Following the steps in the Theorem 3 proof, we once more reach (14), which may be expressed as

$$z(\sigma(\varsigma))\geq\sum_{s=\varsigma_2}^{\varsigma-1}F(s)B(s+1)z^{\delta}(\sigma(s))+\sum_{s=\varsigma}^{\sigma(\varsigma)-1}F(s)B(s+1)z^{\delta}(\sigma(s))$$
$$+B(\sigma(\varsigma))\sum_{s=\sigma(\varsigma)}^{\infty}F(s)z^{\delta}(\sigma(s)).$$

As in the proof of Theorem 3, using that $z(\varsigma)$ is increasing and $\frac{z(\varsigma)}{B(\varsigma)}$ is decreasing, we arrive at the intended conclusion.

3 Examples

Two examples that highlight the outcomes from the previous part are provided in this section. We consider two second-order Euler type neutral difference equations whose study has the potential to shed light on a wide range of technological and scientific problems. For example, these equations can be used to study the behavior of complex systems such as power grids, transportation networks, and communication systems.

Example 1. Consider the equation

$$\Delta\left(\varsigma(\varsigma+1)\Delta\left(\xi(\varsigma)+\frac{\varsigma+1}{4\varsigma+6}\xi(\varsigma-2)\right)\right)+\lambda\xi(\varsigma-3)=0,\ \varsigma\geq4. \tag{21}$$

Here, we have $\mu(\varsigma)=\varsigma(\varsigma+1)$, $\alpha(\varsigma)=\frac{\varsigma+1}{4\varsigma+6}$, $\tau(\varsigma)=\varsigma-2$, $\sigma(\varsigma)=\varsigma-3$, $\delta=1$, and $w(\varsigma)=\lambda>0$ is a constant. Then

$$\Lambda(\varsigma)=\frac{1}{\varsigma},\ \ b(\varsigma)=1,\ \ B(\varsigma)\approx\varsigma,\ \ \daleth(\varsigma)\approx\frac{1}{\varsigma}\left(1-\frac{\varsigma+1}{4\varsigma+6}\frac{\varsigma}{\varsigma-2}\right)\approx\frac{3}{4\varsigma}$$

and $F(\varsigma)\approx\frac{3\lambda}{4(\varsigma+1)(\varsigma-3)}$. The condition (1) is clearly satisfied. The condition (4) becomes

$$\sum_{\varsigma=4}^{\infty}\frac{3\lambda}{4(\varsigma+1)}=\infty,$$

that is, condition (4) holds. The condition (5) becomes

$$\limsup_{\varsigma\to\infty}\left(\frac{1}{\varsigma}\sum_{s=4}^{\varsigma-1}\frac{3\lambda}{4}+\varsigma\sum_{s=\varsigma}^{\infty}\frac{3\lambda}{4s(s+1)}\right)=\frac{3}{2}\lambda>1,$$

that is, condition (5) holds if $\lambda>\frac{2}{3}$. All conditions of Theorem 3 are satisfied if $\lambda>\frac{2}{3}$. Therefore, equation (21) is oscillatory if $\lambda>\frac{2}{3}$. Further by Theorem 4, the equation (21) is oscillatory if $\lambda>\frac{2}{3}$.

Example 2. Consider the equation

$$\Delta\left(\frac{\varsigma(\varsigma+1)(\varsigma+2)}{2}\Delta\left(\xi(\varsigma)+\frac{1}{8}\xi(\varsigma-2)\right)\right)+\varsigma\xi^{\frac{1}{3}}(2\varsigma)=0,\ \varsigma\geq1 \tag{22}$$

Here, we see that $\mu(\dot{\varsigma}) = \frac{\dot{\varsigma}(\dot{\varsigma}+1)(\dot{\varsigma}+2)}{2}$, $\alpha(\dot{\varsigma}) = \frac{1}{8}$, $\tau(\dot{\varsigma}) = \dot{\varsigma} - 2$, $\sigma(\dot{\varsigma}) = 2\dot{\varsigma}$, $\delta = \frac{1}{3}$, and $w(\dot{\varsigma}) = \dot{\varsigma}$. Then

$$\Lambda(\dot{\varsigma}) = \frac{1}{\dot{\varsigma}(\dot{\varsigma}+1)}, \; b(\dot{\varsigma}) = \frac{1}{2(\dot{\varsigma}+1)}, \; B(\dot{\varsigma}) \approx (\dot{\varsigma}+1)\dot{\varsigma}, \; \daleth(\dot{\varsigma}) \approx \frac{7}{8\dot{\varsigma}(\dot{\varsigma}+1)},$$

and $F(\dot{\varsigma}) \approx \left(\frac{7}{8}\right)^{\frac{1}{3}} \frac{1}{\dot{\varsigma}^{5/3}}$. The condition (1) clearly holds and the condition (4) becomes

$$\sum_{\dot{\varsigma}=1}^{\infty} \left(\frac{7}{8}\right)^{1/3} \frac{4^{1/3}}{\dot{\varsigma}} = \infty,$$

that is, condition (4) holds. Further, one can easily see that condition (17) is satisfied. Hence by Theorem 6, the equation (22) is oscillatory.

*Remark.*It is evident that none of the results from the earlier papers could explain the oscillatory behavior of equations (21) and (22), highlighting the relevance and importance of our recent findings.

Conclusion

Several oscillation criteria for the equation (E) under noncanonical conditions have been presented in this study. We demonstrate the primary theorems by first putting the examined equation into canonical form and then using the summation averaging method.The results obtained are completely novel and go beyond earlier findings. Two examples are developed to validate and highlight the relevance of theoretical discoveries, demonstrating that earlier findings cannot be applied to these situations.

The main equation is studied when $0 < \delta \leq 1$ and $0 \leq \alpha(\dot{\varsigma}) \leq d < 1$. Therefore, it would be interesting to expand the results for the situation $\alpha > 1$ where $\alpha(\dot{\varsigma})$ is either unbounded or negative.

Declarations

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