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Effect of Cyclotron Frequency on Vibrational Partition Function and Enthalpy for Hellmann Potential

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Abstract: This study presents a comparative analysis of the eigenvalues, vibrational partition function, and vibrational enthalpy of the Hellmann potential under different quantum states, cyclotron frequencies, and temperatures. The study reveals that the eigenvalue at higher quantum states exhibit less negativity values, with the present method consistently yielding lower eigenvalues than the NU and AP methods. The effect of cyclotron frequency on eigenvalues indicates an upward energy shift with increasing frequency, while spacing between successive energy levels increases at higher quantum numbers. The vibrational partition function exhibits temperature-dependent behaviour, increasing steadily in some cases while showing fluctuations and saturation effects at specific values of cyclotron frequency. Similarly, the vibrational enthalpy trends suggest stabilization at higher temperatures, with variations in magnitude and rate of change depending on cyclotron frequency. These findings highlight the complex interplay between quantum state, cyclotron frequency, and thermal effects on the Hellmann potential system.

Keywords: Bound state; Wave equation; Eigensolutions; Thermodynamic properties; Potential model

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

The determination of bound state or eigen solutions of the wave equation for different physical potential model of interest emerged long ago even before the 19th century. The focus of the authors in the early stage of the development of the determination of the eigen solution is to obtain the energy eigenvalue equation and the corresponding wave functions [1] - [5]. It was clear that the numerical package gave the most accurate results and is used as a standard for comparison. As time goes by, the approximate determination of these solutions for some few existing potential models become subject of interest for both the two forms of the wave equations. In the contest of the relativistic wave equation, the Dirac equation and the Klein-Gordon equation respectively, the bound state has been studied in details for different potentials. The Dirac equation in its own part has been studied under spin symmetry and pseudospin symmetry where energy degeneracies were produced and means of splitting them were also determined [6] - [11]. As knowledge advances, the quest to apply solutions of the wave equation as energy equation and wave function to different areas aroused. The application of quantum mechanics in statistical mechanics prevailed. Thus, the use of the energy equation to calculate the thermodynamic properties such as the entropy, heat capacity, mean and free mean energy via partition function became the subject of interest in the recent times. For instance, Dong and Cruz-Irisson [12] calculated the energy equation and wave function for modified Rosen-Morse potential. They also calculated the thermal properties of the modified Rosen-Morse potential. The thermal properties were studied in details as a function of the temperature parameter and the maximum quantum state. In their results, the heat capacity as a function of temperature parameter tends to be constant after the maximum turning point. They also showed that the entropy as a function of the maximum quantum state decreases monotonically after the maximum turning point. Ikot et al. [13], obtained eigensolutions of the shifted Tietz-Wei potential type and calculated the thermodynamic properties. Their results showed that some thermal properties

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against the temperature parameter decreased while others increased as the temperature parameter increases. In the work of Khordad and Sedehi for double ring-shape potential [14], it was shown that the different thermodynamic properties have different variation with the temperature. Okon et al. [15] in one of their reported works, obtained both the energy equation and the wave function of the Mobius square plus Kratzer potential and then, calculated the thermodynamic properties. It was shown that the various thermal properties exhibit different variation. Inyang et al. [16], studied Eckart-Hellman potential via the conventional Nikiforov-Uvarov method. After obtaining the energy equation and wave function of the interacted potential, they calculated the various thermal properties. It was shown that the thermal properties as a function of the maximum quantum state, exhibit different features as the temperature parameter increases. Omugbe et al. [17], calculated the various thermodynamic properties of mixed hyperbolic Pöschl–Teller potential and have a similar result. Other studies on the thermodynamic properties for different potential models can be found in reference therein [18] -[21]. All the reports given above only discussed thermal properties such as partition function, mean energy, heat capacity, free energy and entropy. The enthalpy for many potential models is yet to be studied. Motivated by the interest in thermodynamic properties, the present study wants to examine the thermodynamic properties like the pure vibrational partition function and vibrational enthalpy for an interaction of the Coulomb potential and Yukawa potential popularly called Hellmann potential in the presence of a cyclotron frequency. Cyclotron frequency is the frequency at which a charged particle moves under the influence of a magnetic field. It plays a crucial role in various applications, such as particle accelerators, mass spectrometry, and plasma physics. Cyclotron frequency depends only on the charge-to-mass ratio and the magnetic field strength. The cyclotron frequency contains a magnetic field which significantly influences different systems as can be seen in the fluid dynamics of electrically conducting nanofluids [22] -[24]. The interacting potential in this case, is written as

$$V(r) = -\frac{a + be^{-\alpha r}}{r} \tag{1}$$

where a and b are the potential strengths. The solution of the radial Schrödinger equation will be obtained using the traditional method of supersymmetry quantum mechanics.

2 Bound State Solutions

To obtain the bound state solution of any quantum system for a given physical potential model $V(r)$ of interest, the radial Schrödinger equation with a non-relativistic energy $E_{n,l}$ and the radial wave function $R_{n,l}(r)$ is given by the formula

$$-\frac{h^2}{2\mu}\frac{d^2R_{n,l}(r)}{dr^2} + \left[V(r) - E_{n,l} + \frac{h^2 l(l+1)}{2\mu r^2}\right] R_{n,l}(r) = 0, \tag{2}$$

where μ is the reduced mass, h is the reduced Planck's constant, and are quantum and angular momentum number respectively. The inverse squared term in (2) can be addressed by a suitable approximation scheme given by the formula [25]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha r})^2}. \tag{3}$$

The above approximation is valid when $\alpha 1$. Substituting (1) and (3) together with the cyclotron frequency into the radial equation in (2), then, a second-order differential equation of the form

$$\frac{d^2R_{n,l}(r)}{dr^2} = \left[\frac{l(l+1)\alpha^2 e^{-\alpha r}}{(1-e^{-\alpha r})^2} + \frac{\left(\frac{2m\alpha(a-b-0.5\omega_c m)}{h^2} + l(l+1)\alpha^2\right)e^{-\alpha r}}{1-e^{-\alpha r}} + E_T\right] R_{n,l}(r), \tag{4}$$

is obtained where

$$E_T = \frac{m^2\alpha\omega_c - 2m(a\alpha + E_{n,l})}{h^2} + \frac{m^2\omega_c^2}{4h} + l(l+1)\alpha^2. \tag{5}$$

where ω_c is a cyclotron frequency and m is mass. As stated in the introductory part, the supersymmetry approach will be used to obtained the solution of (4). Hence, the ground state wave function [26]-[28] for the system can be written in the form

$$R_{0,l}(r) = exp\left(-\int W(r)dr\right). \tag{6}$$

The term $W(r)$ in equation (6) is known as the superpotential in supersymmetry quantum mechanics. Superpotential is a key function used to construct the partner Hamiltonians in the SUSY framework. It plays a fundamental role in defining the potential energy functions for a pair of quantum systems such as the partner potentials related by supersymmetry. It allows the Hamiltonian to be factorized in terms of ladder operators. In supersymmetric approach, is a central function providing a systematic way to construct solvable quantum systems, analyze spectral properties, and explore algebraic solutions. On substituting the ground state wave function in Eq. (6) into the differential equation given in (4), a Riccati equation which pave way for easy solution of the system is obtain as follows

$$W^2(r) - \frac{dW(r)}{dr} = \frac{l(l+1)\alpha^2 e^{-\alpha r}}{(1-e^{-\alpha r})^2} + \frac{\left(\frac{2m\alpha(a-b-0.5\omega_c m)}{h^2} + l(l+1)\alpha^2\right) e^{-\alpha r}}{1-e^{-\alpha r}} + E_T, \tag{7}$$

where the superpotential for this system is propose to be of the form

$$W(r) = A + \frac{A}{1-e^{-\alpha r}}. \tag{8}$$

Equation (8) is proposed based on the complete structure of the interacting potential in relation to the Riccati equation given in equation (7). This proposition is the solution to equation (7) and thus, required mathematical understanding of the property of the two sides of equation (7). The values of the parameters and can be obtained by the help of (7) and (8) together with some mathematical manipulations. Substituting the superpotential given in (8) into the Riccati equation given in (7), and compare the properties of both sides of (7), the two constants in (7) can easily be obtain as follows

$$A^2 = E_T. \tag{9}$$

By comparison, $2AB = 0$, $\alpha B = l(l+1)\alpha^2 \rightarrow B = l(l+1)\alpha$. This shows that either $A = 0$ or $B = 0$. If $B = 0$, then, $l(l+1)\alpha = 0, \rightarrow \alpha l^2 = -\alpha l$. therefore, $l^2 = -l$ and $l = -1$. With this simplifications,

$$B = -\alpha(l+1). \tag{10}$$

Similarly, the comparison generates $2AB = 0$, $-\frac{2m\alpha(a-b-0.5\omega_c)}{h^2} + l(l+1)\alpha^2 = 0$, as well as $B^2 = 0$. These three equations are equal since their right-hand sides are the same (zero). Using this equality,

$$A = \frac{\frac{2m\alpha(a-b-0.5\omega_c)}{h^2} - l(l+1)\alpha^2 - B^2}{2B} \tag{11}$$

To proceed for the deduction of the energy equation for the system, the partner Hamiltonian for the problem can be written as

$$H_- = \hat{A}^\dagger \hat{A} = -\frac{d}{dr} + V_-(r)$$
$$H_+ = \hat{A}\hat{A}^\dagger = -\frac{d}{dr} + V_+(r)$$
$$\hat{A} = \frac{d}{dr} - W(r)$$
$$\hat{A}^\dagger = -\frac{d}{dr} - W(r)$$

which by taking into consideration the superpotential of (8) with its parameters leads to the family potentials [29], [30]given by which by taking into consideration the superpotential of Eq. (8) with its parameters leads to the family potentials [29], [30]given by

$$V_+(r) = W^2(r) + \frac{dW(r)}{dr} = A^2 - \frac{B(B+2A)e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{B(B+\alpha)e^{-\alpha r}}{(1-e^{-\alpha r})^2}, \tag{12}$$

$$V_-(r) = W^2(r) - \frac{dW(r)}{dr} = A^2 - \frac{B(B+2A)e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{B(B+\alpha)e^{-\alpha r}}{(1-e^{-\alpha r})^2}, \tag{13}$$

The family potentials above are related by a formula which consider the remainder term $R(a_1)$, that do not depend of the variable r in the family potentials. Thus

$$V_+(a_0, r) = V_-(a_1, r) + R(a_1). \tag{14}$$

The term a_0 is an old set of parameters where the term a_1 a new set of parameters is deduced via mapping of the form $a_1 = f(a_0) - \alpha$, where $a_0 = B$. Rearranging (14) and consider the parameter of the system under study, the remainder term can be written in a recurrence relation and then generalized to a whole problem

$$R(a_1) = \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_0^2}{2a_0}\right] - \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_1^2}{2a_1}\right], \tag{15}$$

$$R(a_2) = \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_1^2}{2a_1}\right] - \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_2^2}{2a_2}\right], \tag{16}$$

$$R(a_2) = \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_2^2}{2a_2}\right] - \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_3^2}{2a_3}\right], \tag{17}$$

$$R(a_2) = \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_{n-1}^2}{2a_{n-1}}\right] - \left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1)\alpha^2 - a_n^2}{2a_n}\right], \tag{18}$$

The energy level of the system can be obtained via

$$E_{n,l} = \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + R(a_3) + \ldots + R(a_n). \tag{19}$$

On substituting (9), (10) and (11), the complete energy level for the Coulomb-inverse potential is obtained as

$$E_{n,l} = E_{T_0} - \frac{\alpha^2 h^2}{2m}\left[\frac{\frac{2m(a-b-0.5m\omega_c)}{ah^2} - l(l+1) - (n+l+1)^2}{2(n+l+1)}\right]^2, \tag{20}$$

$$E_T = \frac{m\omega}{2} + \frac{mh\omega^2}{8} + \frac{l(l+1)\alpha^2 h^2}{2m} - \alpha a \tag{21}$$

3 Thermodynamic Properties

The pure vibrational state $(l=0)$ energy is required to obtain the vibrational partition function. Thus, (20) can be reduced to

$$E_n = \frac{m\omega}{2} + \frac{mh\omega_c^2}{8} - \alpha a - \frac{\alpha^2 h^2}{8m}\left[\frac{\frac{2m(a-b-0.5m\omega_c)}{\alpha h^2} - (n+1)^2}{n+1}\right]^2 \tag{22}$$

For compatibility, equation (22) can be written as

$$E_n = \Lambda - \Lambda_0^2\left(\frac{\Lambda_1 - (n+1)^2}{n+1}\right)^2, \tag{23}$$

where

$$\Lambda = \frac{m\omega}{2} + \frac{mh\omega_c^2}{8} - \alpha a; \quad \Lambda_0^2 = \frac{\alpha^2 h^2}{8m}; \quad \Lambda_1 = \frac{2m(a-b-0.5m\omega_c)}{\alpha h^2} \tag{24}$$

The vibrational partition function for any system or model with energy E_n is written as

$$Z = \sum_{n=0}^{n_{max}} e^{-\beta E_n}. \tag{25}$$

where $\beta = (K_B T)^{-1}$, k_B is the Boltzmann constant, T is the temperature and n_{max} upper bound vibrational quantum number obtained from the first derivative of equation (23) with respect to the quantum number. For (23), $n_{max} = -1 - \sqrt{\Lambda_1}$. Plugging (23) into (25) gives

$$Z = e^{-\beta\Lambda}\sum_{n=0}^{n_{max}} e^{-\beta\Lambda_0^2\left(\frac{\Lambda_1-(n+1)^2}{n+1}\right)^2}. \tag{26}$$

The partition function in this study will be evaluated using the modified Poisson summation formula (MPS) (Chem. Phys. 439 (2007) 209). Thus,

$$Z = e^{-\beta\Lambda} \sum_{n=0}^{n_{max}} f(n), f(n) = e^{\beta\Lambda_0^2\left(\frac{\Lambda_1-(n+1)^2}{n+1}\right)^2}. \tag{27}$$

In the present evaluation, we consider the lowest order approximation. The lowest-order approximate expression to the summation in equation (27) is written as

$$\sum_{n=0}^{n_{max}} f(n) = \frac{1}{2}\left\{f(0) - f(n_{max}+1)\right\} + \int_0^{n_{max}+1} f(x)dx. \tag{28}$$

Substituting equation (27) into equation (28) leads to

$$e^{-\beta\Lambda}\sum_{n=0}^{n_{max}} f(n) = e^{-\beta\Lambda}\left[\frac{1}{2}\left[e^{\beta\Lambda_0^2(\Lambda_1-1)^2} - e^{\beta\Lambda_0^2\left(\frac{2\sqrt{\Lambda_1}-1}{1-\sqrt{\Lambda_1}}\right)^2}\right] + \int_{\Lambda_0(\Lambda_1-1)\sqrt{\beta}}^{\Lambda_0\sqrt{\beta}\left(\frac{2\sqrt{\Lambda_1}-1}{1-\sqrt{\Lambda_1}}\right)} e^{\Lambda\frac{2}{2}} d\Lambda_2\right], \tag{29}$$

$$Z = \frac{e^{\Lambda\frac{2}{3}-\beta\Lambda} - e^{\Lambda\frac{2}{4}-\beta\Lambda}}{2} + \frac{\sqrt{\pi}e^{-\beta\Lambda}\left\{erf(\Lambda_4) - erf(\Lambda_3) + \Lambda_5\left(erf(\Lambda_6) - erf(\Lambda_7)\right)\right\}}{4\Lambda_0\sqrt{\beta}} \tag{30}$$

where we have defined the following for simplicity

$$\Lambda_3 = \Lambda_0(\Lambda_1 - 1)\sqrt{\beta}; \quad \Lambda_4 = \Lambda_0\sqrt{\beta}\left(\frac{2\sqrt{\Lambda_1 - 1}}{1 - \Lambda_1}\right); \quad \Lambda_5 = e^{4\beta\Lambda_1\Lambda_0^2 - 8\beta\Lambda_0^2};$$

$$\Lambda_6 = \sqrt{\Lambda_4^2 + 8\beta\Lambda_0^2 - 4\beta\Lambda_0^2}; \quad \Lambda_7 = \sqrt{\Lambda_3^2 + 8\beta\Lambda_0^2 - 4\beta\Lambda_0^2}.$$

The pure vibrational enthalpy H is given as

$$H = k_\beta T^2\left(\frac{\delta lnZ}{\delta T}\right). \tag{31}$$

Table 1: Comparative of the eigenvalues of Hellmann potential for various states with $b = -1$, $h = 2m = 1$, and $a = 2$

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State	α	Present	NU [31]	AP [31]
1s	0.001	-2.2505002	-2.249000	-2.248981
	0.005	-2.2525062	-2.245000	-2.244993
	0.010	-2.2550250	-2.240050	-2.240030
2s	0.001	-0.5611301	-0.561502	-0.561502
	0.005	-0.5557516	-0.557550	-0.557549
	0.010	-0.5492563	-0.552697	-0.552697
2p	0.001	-0.5633736	-0.561502	-0.561502
	0.005	-0.5668391	-0.557541	-0.557541
	0.010	-0.5711063	-0.552664	-0.552664
3s	0.001	-0.2478507	-0.249004	-0.249004
	0.005	-0.2396007	-0.245111	-0.245110
	0.010	-0.2300694	-0.240435	-0.240435
3p	0.001	-0.2501680	-0.249004	-0.249004
	0.005	-0.2508674	-0.245103	-0.245102
	0.010	-0.2518028	-0.240404	-0.240404
3d	0.001	-0.2508287	-0.249003	-0.249003
	0.005	-0.2540507	-0.245086	-0.245086
	0.010	-0.2578694	-0.240341	-0.240341

Table 2: Eigenvalues of Hellmann potential for various quantum state with $a = 2$, $b = -1$, $2m = h = l = 1$ for three values of the cyclotron frequency

n	$\omega_c = 0$	$\omega_c = 5$	$\omega_c = 10$
0	-0.6914062	0.8710938	1.6523438
1	-0.3767361	0.7343750	1.4982639
2	-0.5947266	0.2646484	0.9287109
3	-1.3256250	-0.6256250	-0.0506250
4	-2.5629340	-1.9726562	-1.4691840
5	-4.3038903	-3.7936862	-3.3472577
6	-6.5471191	-6.0979004	-5.6975098
7	-9.2918596	-8.8906250	-8.5279707
8	-12.537656	-12.175156	-11.843906
9	-16.284220	-15.953642	-15.648890
10	-20.531359	-20.227539	-19.945421

Table 3: The vibrational partition function of the Hellmann potential with three values of cyclotron frequency for various temperature with and $\alpha = 0.5$, $m = \mu = h = 1$, $b = 0.4$ and $a = 0.5$.

n	$\omega_c = 0$	$\omega_c = 5$	$\omega_c = 10$
100	3.579606009	2.529112034	5.098839035
200	5.103055049	6.079211837	11.69826800
300	5.113286743	5.367744681	10.64184713
400	5.264149589	5.118773218	10.25090003
500	5.431306786	4.998123388	10.05146707
600	5.593722263	4.930572239	9.932927720
700	5.746998861	4.130567491	9.855877634
800	5.890711323	4.864103020	9.802804276
900	6.025497663	4.847759652	9.764760095
1000	6.152237428	4.837471083	9.736709200

Table 4: The vibrational enthalpy of the Hellmann potential with three values of cyclotron frequency for various temperature with $\alpha = 0.5$, $m = \mu = h = 1$, $b = 0.4$ and $a = 0.5$.

n	$\omega_c = 0$	$\omega_c = 5$	$\omega_c = 10$
100	-0.2471378399	4.137749632	14.75882035
200	-0.2480620758	4.131577636	14.75348188
300	-0.2481094954	4.131375706	14.75345953
400	-0.2481881776	4.131073261	14.75333195
500	-0.2482601544	4.130801222	14.75320260
600	-0.2483223419	4.130567491	14.75308558
700	-0.2483759541	4.130366432	14.75298180
800	-0.2484225778	4.130191761	14.75288974
900	-0.2484635566	4.130038322	14.75280763
1000	-0.2484999419	4.129902130	14.75273389

4 Discussion

Table 1 presents a comparative analysis of the eigenvalues of the Hellmann potential for various quantum states $(1s, 2s, 2p, 3s, 3p, 3d)$ under three different values of α. The eigenvalues are compared across three different methods: Present, NU [31], and AP [31]. For all the methods, as α increases (from 0.001 to 0.010), the eigen values tend to decrease for $1s, 2p, 3p$ and $3d$ where $n = 0$, indicating a stronger potential effect on energy levels. The eigenvalues for higher quantum states (e.g. 3d, 3p) are generally less negative than those for lower states (e.g. 1s), following expected quantum mechanics principles. The values in the present method are slightly more negative than the other two methods for most states. The differences between the three methods are minor, but the present method consistently yields the lowest eigenvalues. Table 2 presents the eigenvalues of the Hellmann potential for various quantum states with three different values of the cyclotron frequency. The eigenvalues decrease as the quantum state increases. In the absence of the cyclotron frequency, the trend begins from the first excited state while the trend starts from the ground state in the presence of the cyclotron frequency, indicating that higher quantum states have lower energy levels. This trend holds across all three cyclotron frequencies. The three rows correspond to different values of the cyclotron frequency. For a given quantum state, the eigenvalue increases as the cyclotron frequency increases. This suggests that a higher cyclotron frequency leads to a less negative eigenvalue, meaning the system's energy levels shift upwards. The difference between successive eigenvalues increases as the quantum state index increases. This suggests that the spacing between energy levels is not uniform but increases at higher quantum numbers. All eigenvalues are negative, showing that the potential is binding in nature. The most negative value is observed at the highest quantum state for the lowest cyclotron frequency. Table 3 presents the vibrational partition function of the Hellmann potential at three different cyclotron frequencies across various temperatures (from $100K$ to $1000K$). The values of the vibrational partition function generally increase as temperature rises in the absence of the cyclotron frequency. This is expected because higher temperatures provide more energy, allowing a system to access higher vibrational states. The first dataset shows a steady increase in the partition function from 3.58 at $100K$ to 6.15 at $1000K$ which suggests a gradual and continuous increase, indicating a stable response to temperature changes. The second dataset $\omega_c = 5$ fluctuates more, peaking at $200K(6.08)$ before decreasing and stabilizing around 4.83–5.09 at higher temperatures suggesting potential resonance effects or non-linear dependence on temperature. The third dataset $\omega_c = 5$ starts at 5.09 and increases sharply to 11.69 at $200K$ before gradually decreasing over the rest of the temperature range, possibly due to saturation effects in the vibrational states. From the analysis, it is clear that the Hellmann potential's vibrational partition function behavior is influenced by temperature and cyclotron frequency. At specific cyclotron frequencies, the vibrational states might become more accessible at certain temperatures, leading to peaks or fluctuations. The differences in trends suggest that different cyclotron frequencies impact the vibrational energy distribution in distinct ways. Table 4 presents the vibrational enthalpy of the Hellmann potential at different temperatures (T) for three values of cyclotron frequency. The values are consistently negative and show a decreasing trend in magnitude as temperature increases in the first column. The rate of change decreases with increasing temperature, suggesting that the enthalpy stabilizes at higher temperatures. The difference between consecutive values becomes smaller, implying a saturation effect. In the second column, the values are positive but gradually decreasing with temperature. The rate of decrease is small, indicating a slow decline. The change in values suggests a weak dependence of vibrational enthalpy on temperature. In the third column, the values exhibit a very slight decline with temperature. The rate of change is even smaller than in the second column, showing a more stable trend. The overall decrease indicates a minor dependence on temperature, but the impact is not significant.

5 Conclusion

The examination of the Hellmann potential across different quantum states and cyclotron frequencies reveals significant trends in eigenvalues, vibrational partition functions, and vibrational enthalpy. The comparative analysis indicates that variations in quantum state, cyclotron frequency, and temperature brought about corresponding shifts in the characteristics of energy of the system. The vibrational partition function exhibits temperature-dependent fluctuations, with varying cyclotron frequencies influencing energy distribution. Moreover, vibrational enthalpy trends suggest stabilization effects at elevated temperatures, displaying distinct patterns across different frequency values. Overall, the findings underscore the complex interplay between the Hellmann potential and quantum state properties, providing insights into the impact of external factors such as cyclotron frequency on energy distributions and thermodynamic properties. The result can be applied in plasma confinement for fusion reactor as it helps to optimize conditions for maintaining stable nuclear fusion reactions. The result is also useful in designing quantum dot and other nanoscale devices where external magnetic fields influence which aids the development of quantum computing and advanced semiconductors with controlled electronic properties. Knowledge of vibrational partition functions reveals information about reaction kinetics modeling, helping predict reaction behavior in drug design and chemical engineering. This can optimize reaction conditions for pharmaceutical synthesis and catalysis. As short coming, the study focuses on a combination of Coulomb and Yukawa potentials. While useful, real-world interactions in quantum systems may involve more complex potentials that are not fully captured by this model. See also [32] and [33].

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