JJMS **18**, No. 4, 523-537 (2025)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/18.4.3

Adaptive Hybrid Progressive Censoring in m-Component Reliability Model and Generalized Inverse Weibull Distribution

Akram Kohansal*

Department of Statistics, Imam Khomeini International University, Qazvin, Iran.

Received: May 8, 2024 Accepted : Oct. 22, 2024

Abstract: In this paper, the authors investigate the Bayesian inference of the m-component stress-strength parameter for the generalized inverse Weibull distribution under an adaptive hybrid progressive censoring scheme. The study considers three cases. In the first step, the paper employs the MCMC method to derive a Bayesian estimate of the m-component stress-strength parameter when both the common parameters for strengths and stress variables are unknown. Secondly, assuming that the common parameters are known, two approximation methods are employed: namely, the MCMC method and Lindley's approximation. Finally, in a broader scenario where all parameters are distinct and undisclosed, the paper employs MCMC simulation to calculate a Bayesian estimate of the m-component stress-strength parameter. To evaluate and compare these methods' performance, one Monte Carlo simulation is conducted. Additionally, a real data set is used to implement theoretical methods proposed in this study.

Keywords: m-component stress-strength reliability; Lindley's approximation; MCMC method; Adaptive hybrid progressive censoring scheme.

2010 Mathematics Subject Classification. 62N05; 62F15.

1 Introduction

The primary censoring schemes include Type-I and Type-II. In Type-I, the experiment concludes at a set time, whereas in Type-II, it ends after a specified number of failures. A Type-I hybrid scheme proposed by [1] combines both Type-I and Type-II, ending the experiment at time $T^* = \min\{X_{m:n}, T\}$. Also, in a Type-II hybrid scheme, the experiment ends at time $T^* = \max\{X_{m:n}, T\}$, where $T > 0$ is a constant time, and $X_{m:n}$ denotes the m-th failure time among n items. As the previous schemes lacked the option to remove units during testing, the progressive scheme emerged. A hybrid progressive censoring scheme, as suggested by [2], merges the hybrid and progressive schemes. In this setup, if N units are employed in the experiment, the progressive samples $X_{1:n:N} \leq \ldots \leq X_{n:n:N}$ are taken with the scheme $(R_1, \ldots, R_J, \ldots, R_n)$. Moreover, a stopping time is defined as $T^* = \min\{X_{n:n:N}, T\}$, where $T > 0$. In this scenario, two possibilities emerge: In the first case, when $X_{n:n:N} < T$, the progressive censoring scheme is implemented. In the second case, if $X_{J:n:N} < T < X_{J+1:n:N}$, the experiment concludes at time T with J failures. An issue with this scheme is that the sample size is not fixed and can be quite small, which is a notable drawback of the hybrid progressive scheme. To tackle this challenge, [3] introduced the adaptive hybrid progressive censoring (AHPC) scheme. This approach can be outlined as follows: Given that N units are employed in the experiment and $X_{1:n:N} \leq \ldots \leq X_{n:n:N}$ represent the progressive samples with the scheme $(R_1, \ldots, R_J, \ldots, R_n)$. Additionally, a stopping time $T > 0$ is considered. In this scenario, two situations emerge: Firstly, if $X_{n:n:N} < T$, the progressive censoring scheme is implemented (refer to Figure 1). Secondly, if $X_{J:n:N} < T < X_{J+1:n:N}$ and we aim to observe n failure times, no item is eliminated after time T until the occurrence of the n-th failure, at which juncture all remaining items are withdrawn. Therefore, we can set

$$R_{J+1} = \cdots = R_{n-1} = 0, \; R_n = N - n - \sum_{i=1}^{J} R_i.$$

* Corresponding author e-mail: kohansal@sci.ikiu.ac.ir

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

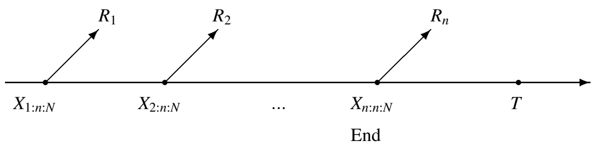


Fig. 1: Experiment ends before time T ($X_{n:n:N} \leq T$).

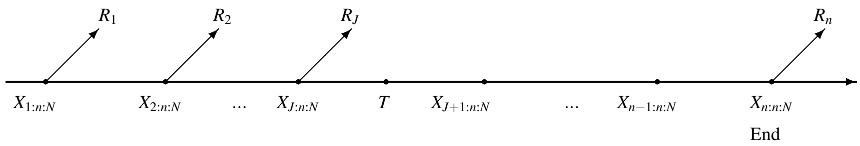


Fig. 2: Experiment ends after time T ($X_{n:n:N} \geq T$).

This case is given in Figure 2. Below, we represent an AHPC sample as $\{X_1,\ldots,X_n\}$ with the scheme of $\{N,n,T,J,R_1,\ldots R_n\}$, where $X_J < T < X_{J+1}$. This scheme allows for the derivation of various scenarios such as hybrid progressive censoring, progressive censoring, and Type-II censoring. Another advantage of this scheme is that it provides access to the n-th failure times.

The estimation of the stress-strength parameter, $R = P(X > Y)$, holds significant interest for researchers in reliability theory. In this context, the variables X and Y are linked to strength and stress, respectively. Recently, [4] developed the multi-component reliability with $\boldsymbol{k} = (k_1,k_2,\ldots,k_m)$ components as follows:

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \left(\prod_{i=1}^{m} \binom{k_i}{p_i} \right) \int_{-\infty}^{\infty} \prod_{i=1}^{m} \left((1-F_i(y))^{p_i} (F_i(y))^{k_i-p_i} \right) dF_Y(y), \tag{1}$$

where k_i components are of type i, where $i = 1,\ldots,m$, and $F_i(\cdot)$ is the cumulative distribution function (cdf) of the strengths for the i-th type components. In this context, we consider a situation where a shared stress Y with cumulative distribution function $F_Y(\cdot)$ impacts all components. If a minimum of $\boldsymbol{s} = (s_1,\ldots,s_m)$ out of the $\boldsymbol{k}$ strength components surpass this stress level, the system can be deemed reliable. [5] recently explored this model using the Kumaraswamy generalized distribution in AHPC samples. In our study, we investigate this model using the generalized inverse Weibull distribution. This model is quite general, as it can yield various cases, including:

– $\boldsymbol{k} = (1,0,\cdots,0)$ corresponds to the scenario where you are interested in calculating $P(X < Y)$.
– $\boldsymbol{k} = (k,0,\cdots,0)$ corresponds to the $R_{s,k}$ scenario.
– $\boldsymbol{k} = (k_1,k_2,0,\cdots,0)$ implies $R_{\boldsymbol{s},\boldsymbol{k}}$ in a scenario with two non-identical components.

Recently, the inference on the $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter has been meticulously examined by [6] in the context of dual components of strength variables. Furthermore, [7] have delved into reliability inference for the $R_{s,k}$ parameter derived from the Kumaraswamy-G family of distributions, utilizing progressively first failure censored samples. [8] have scrutinized $R_{s,k}$ within the framework of the unit Burr III distribution under progressive censoring. The estimation of the $R_{\boldsymbol{s},\boldsymbol{k}}$ parameter has also been addressed by [9] for the modified Weibull extension distribution, particularly in the realm of progressively censored data. [10] have investigated the inference of $R_{s,k}$ from the generalized inverted exponential lifetime distribution under the paradigm of progressive first failure censoring. Additionally, [11] have explored the inference of $R_{s,k}$ following the Topp-Leone distribution, employing progressively censored data. [12] achieved the estimation of $R_{s,k}$ when observations were subjected to Type-II censoring within the inverted exponentiated Pareto distribution. [13] have considered the inference of $R_{s,k}$ for a unit inverse Weibull distribution under Type-II censoring. Moreover, [14] have investigated $R_{\boldsymbol{s},\boldsymbol{k}}$ for progressively first failure censored samples, where the strength and stress variables adhere to the modified Kumaraswamy distribution.

The generalized inverse Weibull (GIW) distribution is capable of modeling failure rates, which are commonly observed in reliability and biological studies. Proposed by [15], this distribution may display unimodal hazard rate functions. That's interesting to hear. The GIW distribution seems to have gained significant attention and recognition in the statistical community, with over 200 citations since 2011. Its simplicity and versatility make it suitable for modeling a range of natural phenomena. The GIW distribution's probability density function (pdf), cumulative distribution function (cdf), and

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

failure rate function (frf) are defined as follows:

$$f(x) = \gamma\beta\alpha^{\beta}x^{-\beta-1}e^{-\gamma(\frac{\alpha}{x})^{\beta}}, \quad x>0, \tag{2}$$

$$F(x) = e^{-\gamma(\frac{\alpha}{x})^{\beta}}, \qquad x>0, \tag{3}$$

$$h(x) = \frac{\gamma\beta\alpha^{\beta}x^{-\beta-1}e^{-\gamma(\frac{\alpha}{x})^{\beta}}}{1-e^{-\gamma(\frac{\alpha}{x})^{\beta}}}, \quad x>0,$$

where $\alpha,\beta,\gamma>0$. The inverse Weibull distribution is a special case of the GIW when $\gamma=1$. Figure 3 illustrates some potential shapes of the pdf and the frf of the GIW distribution.

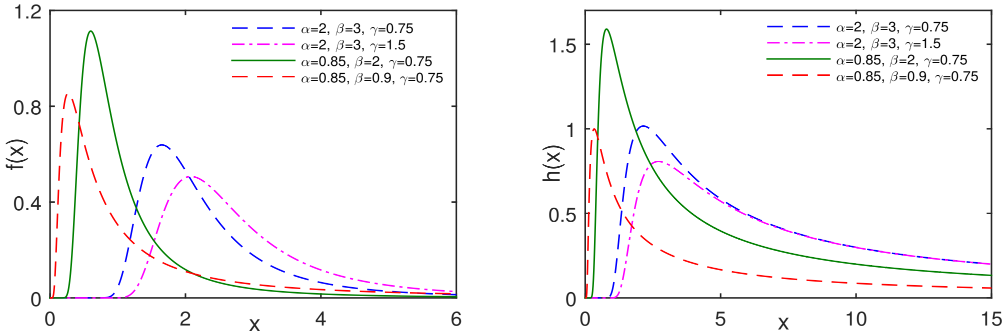


Fig. 3: Shape of frf (right) and pdf (left) functions of GIW distribution.

The other sections of this article are arranged in the following manner. In Section 2, we employ the Markov Chain Monte Carlo (MCMC) method for conducting Bayesian inference on $R_{s,k}$ concerning the undisclosed common α and β parameters. Additionally, we establish highest posterior density (HPD) credible intervals for $R_{s,k}$. In Section 3, we derive Bayesian inference for $R_{s,k}$ with the given shared α and β parameters. We use the MCMC and Lindley's approximation techniques as two approximation methods and establish HPD credible intervals for $R_{s,k}$. In Section 4, we accomplish Bayesian inference for $R_{s,k}$ under general conditions. Section 5 covers simulation and data analysis, while Section 6 outlines the study's conclusions.

2 Inference on $R_{s,k}$ with unknown common α and β

Consider independent random variables $X_1 \sim GIW(\alpha,\beta,\gamma_1),\ldots,X_m \sim GIW(\alpha,\beta,\gamma_m)$, and $Y \sim GIW(\alpha,\beta,\gamma)$. Under this condition, the m-component stress-strength parameter, $R_{s,k}$, from (2) and (3), can be obtained as

$$R_{s,k} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^{\infty}\gamma\beta\alpha^{\beta}y^{-\beta-1}e^{-\gamma(\frac{\alpha}{y})^{\beta}}e^{-(\frac{\alpha}{y})^{\beta}\left(\sum_{l=1}^{k}\gamma_l(k_l-p_l)\right)}$$
$$\times\prod_{l=1}^{m}\left(1-e^{-\gamma_l(\frac{\alpha}{y})^{\beta}}\right)^{p_l}dy \quad \text{Put: } t=e^{-(\frac{\alpha}{y})^{\beta}}$$

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

$$
\begin{aligned}
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \gamma \int_0^1 t^{\sum_{l=1}^{m} \gamma_l (k_l - p_l) + \gamma - 1} \prod_{l=1}^{m} (1 - t^{\gamma_l})^{p_l} dt \\
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{p_1} \cdots \sum_{q_m=0}^{p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \binom{p_1}{q_1} \cdots \binom{p_m}{q_m} \\
&\times (-1)^{\sum_{l=1}^{m} q_l} \gamma \int_0^1 t^{\sum_{l=1}^{m} \gamma_l (k_l - p_l + q_l) + \gamma - 1} dt \\
&= \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{p_1} \cdots \sum_{q_m=0}^{p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \\
&\times \binom{p_1}{q_1} \cdots \binom{p_m}{q_m} \frac{(-1)^{\sum_{l=1}^{m} q_l} \gamma}{\sum_{l=1}^{m} \gamma_l (k_l - p_l + q_l) + \gamma}. \qquad (4)
\end{aligned}
$$

To obtain the Bayes estimation, the likelihood function can be constructed by the following samples:

$$
Y = \overset{\text{Observed stress variables}}{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}} \quad \text{and} \quad X_l = \overset{\text{Observed strength variables}}{\begin{pmatrix} X_{11}^{(l)} & \dots & X_{1k_l}^{(l)} \\ \vdots & \ddots & \vdots \\ X_{n1}^{(l)} & \dots & X_{nk_l}^{(l)} \end{pmatrix}}, \quad l = 1, \dots, m,
$$

where $\{Y_1, \dots, Y_n\}$ is an AHPC sample from $GIW(\alpha, \beta, \gamma)$ with the $\{N, n, T, J, S_1, \dots, S_n\}$ censoring scheme. Also, $\{X_{i1}^{(l)}, \dots, X_{ik_l}^{(l)}\}$, $i = 1, \dots, n$, $l = 1, \dots, m$ are l AHPC samples from $GIW(\alpha, \beta, \gamma_l)$ with schemes $\{K_l, k_l, T_i^{(l)}, J_i^{(l)}, R_{i1}^{(l)}, \dots, R_{ik_l}^{(l)}\}$. Now, we obtain the likelihood function of the parameters as

$$
\begin{aligned}
L(\alpha, \beta, \gamma, \gamma_1, \dots, \gamma_m | \text{data}) \propto \prod_{i=1}^{n} \Big(\prod_{l=1}^{m} \Big(\prod_{j_l=1}^{k_l} f_l(x_{ij_l}^{(l)}) \prod_{j_l=1}^{J_i^{(l)}} \big(1 - F_l(x_{ij_l}^{(l)})\big)^{R_{ij_l}^{(l)}} \big(1 - F_l(x_{ik_l}^{(l)})\big)^{R_{ik_l}^{(l)}} \Big) \Big) \\
\times f_Y(y_i) \prod_{i=1}^{J} \big(1 - F_Y(y_i)\big)^{S_i} \big(1 - F_Y(y_n)\big)^{S_n}.
\end{aligned}
$$

Within this section, we investigate Bayesian inference for $R_{\boldsymbol{s},\boldsymbol{k}}$, under squared error loss functions, assuming independence among $\alpha, \beta, \gamma_1, \dots, \gamma_m$. We use squared error loss functions in our analysis due to their simplicity and interpretability, as they measure the average of the squares of errors, making it easy to understand model performance. This loss function possesses desirable mathematical properties, such as differentiability, which facilitates optimization methods like MCMC for Bayesian inference. Additionally, squaring the errors emphasizes larger discrepancies, which is particularly important in reliability modeling where significant errors can have serious implications. The squared error loss also aligns well with the assumption of normally distributed errors, leading to natural derivations of estimators that minimize this loss. Overall, its robustness, interpretability, and widespread acceptance in statistical applications make it a suitable choice for estimating parameters in the m-component stress-strength model. The graph below depicts the joint posterior density function derived from the censored samples:

$$
\pi(\alpha, \beta, \gamma, \gamma_1, \dots, \gamma_m | \text{data}) \propto L(\text{data} | \alpha, \beta, \gamma, \gamma_1, \dots, \gamma_m) \times \pi(\alpha) \pi(\beta) \big(\prod_{l=1}^{m} \pi(\gamma_l) \big) \pi(\gamma) \qquad (5)
$$

where

$$
\begin{aligned}
&\pi(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \alpha, a, b > 0, \pi(\beta) \propto \beta^{c-1} e^{-d\beta}, \beta, c, d > 0, \\
&\pi(\gamma) \propto \gamma^{e-1} e^{-f\gamma}, \gamma, e, f > 0, \pi(\gamma_l) \propto \gamma_l^{e_l - 1} e^{-f_l \gamma_l}, \gamma_l, e_l, f_l > 0, l = 1, \dots, m.
\end{aligned}
$$

Function (5) appears to be highlighting the difficulty in finding a closed-form solution for the Bayes estimate, which necessitates turning to the MCMC method for approximation. This method is frequently employed when analytical

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

solutions are not attainable. Subsequently, by utilizing equation (5), the posterior pdfs of $\alpha,\beta,\gamma,\gamma_1,\dots,\gamma_m$ can be derived as follows:

$$\pi(\alpha|\beta,\gamma,\gamma_1,\dots,\gamma_m,\text{data}) \propto \alpha^{n\beta(\sum_{l=1}^{m}k_l+1)+a-1}e^{-b\alpha}$$
$$\times e^{-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{J_i^{(l)}}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}\big)^{R_{ij_l}^{(l)}}\prod_{i=1}^{n}\prod_{l=1}^{m}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ik_l}^{(l)}})^\beta}\big)^{R_{ik_l}^{(l)}}$$
$$\times\prod_{i=1}^{n}y_i^{-\beta-1}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^\beta}\prod_{i=1}^{J}\big(1-e^{-\gamma(\frac{\alpha}{y_i})^\beta}\big)^{S_i}\big(1-e^{-\gamma(\frac{\alpha}{y_n})^\beta}\big)^{S_n},$$

$$\pi(\beta|\alpha,\gamma,\gamma_1,\dots,\gamma_m,\text{data}) \propto \beta^{n(\sum_{l=1}^{m}k_l+1)+c-1}\alpha^{n\beta}e^{-d\beta}\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{k_l}(x_{ij_l}^{(l)})^{-\beta-1}$$
$$\times e^{-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}\prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{J_i^{(l)}}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}\big)^{R_{ij_l}^{(l)}}\prod_{i=1}^{n}\prod_{l=1}^{m}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ik_l}^{(l)}})^\beta}\big)^{R_{ik_l}^{(l)}}$$
$$\times\prod_{i=1}^{n}y_i^{-\beta-1}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^\beta}\prod_{i=1}^{J}\big(1-e^{-\gamma(\frac{\alpha}{y_i})^\beta}\big)^{S_i}\big(1-e^{-\gamma(\frac{\alpha}{y_n})^\beta}\big)^{S_n},$$

$$\pi(\gamma|\alpha,\beta,\text{data}) \propto \gamma^{n+e-1}e^{-f\gamma}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^\beta}\prod_{i=1}^{J}\big(1-e^{-\gamma(\frac{\alpha}{y_i})^\beta}\big)^{S_i}\big(1-e^{-\gamma(\frac{\alpha}{y_n})^\beta}\big)^{S_n},$$

$$\pi(\gamma_l|\alpha,\beta,\text{data}) \propto \gamma_l^{nk_l+e_l-1}e^{-f_l\gamma_l}e^{-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}$$
$$\times\prod_{i=1}^{n}\prod_{j_l=1}^{J_i^{(l)}}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^\beta}\big)^{R_{ij_l}^{(l)}}\prod_{i=1}^{n}\big(1-e^{-\gamma_l(\frac{\alpha}{x_{ik_l}^{(l)}})^\beta}\big)^{R_{ik_l}^{(l)}},\ l=1,\dots,m,$$

It is evident that sampling from the probability density functions of $\alpha,\beta,\gamma,\gamma_1,\dots,\gamma_m$ requires the utilization of the Metropolis-Hastings (MH) method since their probability density functions are unknown. Therefore, the procedure for implementing the Gibbs sampling algorithm is as follows:

1.Begin with initial values $(\alpha_{(0)},\beta_{(0)},\gamma_{(0)},\gamma_{1(0)},\dots,\gamma_{m(0)})$.

2.Set $t=1$.

3.With $N(\alpha_{(t-1)},1)$ as proposal distribution and $\pi(\alpha|\beta_{(t-1)},\gamma_{(t-1)},\gamma_{1(t-1)},\dots,\gamma_{m(t-1)},\text{data})$ as target distribution, produce $\alpha(t)$ with MH algorithm as follows:

(i)Generate $w_{(t)}$ from $W(\cdot|\alpha_{(t-1)},1)=N(\alpha_{(t-1)},1)$ and u from $U(0,1)$ distribution.

(ii)If $u<\min\{1,\delta\}$ then set $\alpha_{(t)}=w_{(t)}$, where

$$\delta=\frac{\pi(w_{(t)}|\text{data})W(\alpha_{(t)}|w_{(t)},1)}{\pi(\alpha_{(t-1)}|\text{data})W(w_{(t)}|\alpha_{(t-1)},1)},$$

else return step (i).

4.With $N(\beta_{(t-1)},1)$ as proposal distribution and $\pi(\beta|\alpha_{(t-1)},\gamma_{(t-1)},\gamma_{1(t-1)},\dots,\gamma_{m(t-1)},\text{data})$ as target distribution, produce $\beta_{(t)}$ with MH algorithm.

5.With $N(\gamma_{(t-1)},1)$ as proposal distribution and $\pi(\gamma|\alpha_{(t-1)},\beta_{(t-1)},\text{data})$ as target distribution, produce $\gamma_{(t)}$ with MH algorithm.

6.With $N(\gamma_{1(t-1)},1)$ as proposal distribution and $\pi(\gamma_1|\alpha_{(t-1)},\beta_{(t-1)},\text{data})$ as target distribution, produce $\gamma_{1(t)}$ with MH algorithm.

$\vdots$

m+5.With $N(\gamma_{m(t-1)},1)$ as proposal distribution and $\pi(\gamma_m|\alpha_{(t-1)},\beta_{(t-1)},\text{data})$ as target distribution, produce $\gamma_{m(t)}$ with MH algorithm.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

m+6.Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{p_1} \cdots \sum_{q_m=0}^{p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{p_1}{q_1} \cdots \binom{p_m}{q_m} \frac{(-1)^{\sum_{l=1}^{m} q_l} \gamma_{(t)}}{\sum_{l=1}^{m} \gamma_{l(t)}(k_l - p_l + q_l) + \gamma_{(t)}}.$$

m+7.Set $t = t+1$.
m+8.Repeat steps 3 - m+7, T times.

Hence, under the squared error loss functions, the Bayesian estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ is:

$$\widehat{R}^{MB}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T} \sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{6}$$

Additionally, the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ can be determined using the method proposed by [16]. To establish the HPD interval, first arrange the values of $R_{(1)\boldsymbol{s},\boldsymbol{k}}, \ldots, R_{(T)\boldsymbol{s},\boldsymbol{k}}$ in ascending order as $R_{((1)\boldsymbol{s},\boldsymbol{k})} < \cdots < R_{((T)\boldsymbol{s},\boldsymbol{k})}$. Next, create all the confidence intervals for $R_{\boldsymbol{s},\boldsymbol{k}}$ with a confidence level of $100(1-\eta)\%$ as follows:

$$\Big(R_{\left((1)\boldsymbol{s},\boldsymbol{k}\right)}, R_{\left(([T(1-\eta)])\boldsymbol{s},\boldsymbol{k}\right)}\Big), \ldots, \Big(R_{\left(([T\eta])\boldsymbol{s},\boldsymbol{k}\right)}, R_{\left(([T])\boldsymbol{s},\boldsymbol{k}\right)}\Big).$$

Here, the symbol $[T]$ represents the largest integer less than or equal to T. The HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$ is the interval with the shortest length.

3 Inference on $R_{\mathbf{s},\mathbf{k}}$ with known common α and β

In this part, we will determine the Bayesian estimate and its associate credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$ using the squared error loss function. We assume that $\gamma, \gamma_1, \ldots, \gamma_m$ have independent gamma distributions as prior distributions, similar to Section 2. The posterior pdfs of these parameters are obtained as follows:

$$\pi(\gamma|\alpha,\beta,\text{data}) \propto \gamma^{n+e-1} e^{-f\gamma} e^{-\sum_{i=1}^{n} \gamma(\frac{\alpha}{y_i})^{\beta}} \prod_{i=1}^{J} \left(1 - e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\right)^{S_i} \left(1 - e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\right)^{S_n},$$

$$\pi(\gamma_l|\alpha,\beta,\text{data}) \propto \gamma_l^{nk_l+e_l-1} e^{-f_l\gamma_l} e^{-\sum_{i=1}^{n}\sum_{l=1}^{m}\sum_{j_l=1}^{k_l} \gamma_l(\frac{\alpha}{x^{(l)}_{ij_l}})^{\beta}} \times \prod_{i=1}^{n}\prod_{l=1}^{m}\prod_{j_l=1}^{J_i^{(l)}} \left(1 - e^{-\gamma_l(\frac{\alpha}{x^{(l)}_{ij_l}})^{\beta}}\right)^{R^{(l)}_{ij_l}} \prod_{i=1}^{n}\prod_{l=1}^{m} \left(1 - e^{-\gamma_l(\frac{\alpha}{x^{(l)}_{ik_l}})^{\beta}}\right)^{R^{(l)}_{ik_l}},\ l = 1,\ldots,m,$$

Therefore, the procedure for implementing the Gibbs sampling algorithm is as follows:

1.Begin with initial values $(\gamma_{(0)}, \gamma_{1(0)}, \ldots, \gamma_{m(0)})$.
2.Set $t = 1$.
3.With $N(\gamma_{(t-1)}, 1)$ as proposal distribution and $\pi(\gamma|\alpha,\beta,\text{data})$ as target distribution, produce $\gamma_{(t)}$ with MH algorithm.
4.With $N(\gamma_{1(t-1)}, 1)$ as proposal distribution and $\pi(\gamma_1|\alpha,\beta,\text{data})$ as target distribution, produce $\gamma_{1(t)}$ with MH algorithm.
⋮
m+3.With $N(\gamma_{m(t-1)}, 1)$ as proposal distribution and $\pi(\gamma_m|\alpha,\beta,\text{data})$ as target distribution, produce $\gamma_{m(t)}$ with MH algorithm.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

m+4.Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1} \cdots \sum_{p_m=s_m}^{k_m} \sum_{q_1=0}^{p_1} \cdots \sum_{q_m=0}^{p_m} \binom{k_1}{p_1} \cdots \binom{k_m}{p_m} \times \binom{p_1}{q_1} \cdots \binom{p_m}{q_m} \frac{(-1)^{\sum_{l=1}^{m} q_l} \gamma_{(t)}}{\sum_{l=1}^{m} \gamma_{l(t)}(k_l - p_l + q_l) + \gamma_{(t)}}.$$

m+5.Set $t = t+1$.
m+6.Repeat steps 3 - m+5, T times.

Therefore, the Bayesian estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ under the squared error loss function is:

$$\widehat{R}^{MB}_{\boldsymbol{s},\boldsymbol{k}} = \frac{1}{T} \sum_{t=1}^{T} R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{7}$$

Furthermore, to establish the $100(1-\eta)\%$ HPD credible interval of $R_{\boldsymbol{s},\boldsymbol{k}}$, you can use the method described in [16], as explained in Section 2.

3.1 Lindley's approximation

In the influential work, [17] introduced a highly significant numerical method for approximating the Bayes estimation of a parameter. It is well-known that the Bayes estimation of $U(\Theta)$ under the squared error loss can be obtained through the following derivation:

$$\mathbb{E}(u(\theta)|\text{data}) = \frac{\int u(\theta) e^{Q(\theta)} d\theta}{\int e^{Q(\theta)} d\theta}, \tag{8}$$

where $Q(\theta) = \rho(\theta) + \ell(\theta)$, $\rho(\theta)$ and $\ell(\theta)$ are logarithm of the prior density of θ and log-likelihood function, respectively. [17] provided an approximation for equation (8) in the following manner:

$$\mathbb{E}(u(\theta)|\text{data}) = u + \frac{1}{2}\sum_i \sum_j (u_{i,j} + 2u_i \rho_j)\sigma_{i,j} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_p \ell_{i,j,k}\sigma_{i,j}\sigma_{k,p}u_p \bigg|_{\theta=\widehat{\theta}}, \tag{9}$$

where $\theta = (\theta_1, \ldots, \theta_m)$, $i, j, k, p = 1, \ldots, m$, $\widehat{\theta}$ is the MLE of θ, $u = u(\theta)$, $u_i = \partial u/\partial \theta_i$, $u_{i,j} = \partial^2 u/(\partial \theta_i \partial \theta_j)$, $\ell_{i,j,k} = \partial^3 \ell/(\partial \theta_i \partial \theta_j \partial \theta_k)$, $\rho_j = \partial \rho/\partial \theta_j$, and $\sigma_{i,j} = (i,j)$-th element in the inverse of matrix $[-\ell_{i,j}]$ all evaluated at the MLE of parameters. By re-writing (9), for $m+1$ parameters, we obtain

$$\widehat{u}^{Lin} = u + (\sum_{i=1}^{m+1} u_i d_i + d_{m+2} + d_{m+3}) + \frac{1}{2}\sum_{i=1}^{m+1} A_i (\sum_{j=1}^{m+1} u_j \sigma_{i,j}), \tag{10}$$

where

$$d_i = \sum_{j=1}^{m+1} \rho_j \sigma_{i,j},\ i = 1, \cdots, m+1,\ \ d_{m+2} = \sum_{\substack{i=1 \\ i<j}}^{m+1} \sum_{j=1}^{m+1} u_{i,j}\sigma_{i,j},\ \ d_{m+3} = \frac{1}{2}\sum_{i=1}^{m+1} u_{i,i}\sigma_{i,i},$$

$$A_i = \sum_{\substack{j=1 \\ j \le k}}^{m+1} \sum_{k=1}^{m+1} \ell_{j,k,i} \times \begin{cases} \sigma_{j,k} & j = k, \\ 2\sigma_{j,k} & j < k, \end{cases} \quad i = 1, \cdots, m+1.$$

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

For $(\theta_1,\cdots,\theta_m,\theta_{m+1}) \equiv (\gamma_1,\cdots,\gamma_m,\gamma)$ and $u \equiv u(\gamma_1,\cdots,\gamma_m,\gamma) = R_{\boldsymbol{s},\boldsymbol{k}}$, we have

$$\rho_l = \frac{a_l-1}{\gamma_l} - b_l,\ l=1,\cdots,m,\ \ \rho_{m+1} = \frac{a-1}{\gamma} - b,$$

$$\ell_{l,l} = -\frac{nk_l}{\gamma_l^2} - \sum_{i=1}^{n}\sum_{j_l=1}^{J_i^{(l)}} R_{ij_l}^{(l)}\left(\frac{\alpha}{x_{ij_l}^{(l)}}\right)^{2\beta} \frac{e^{-\gamma(\frac{\alpha}{x_{ij_l}^{(l)}})^{\beta}}}{\left(1-e^{-\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^{\beta}}\right)^2} - \sum_{i=1}^{n} R_{ik_l}^{(l)}\left(\frac{\alpha}{x_{ik_l}^{(l)}}\right)^{2\beta} \frac{e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}}{\left(1-e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}\right)^2},\ l=1,\cdots,m,$$

$$\ell_{m+1,m+1} = -\frac{n}{\gamma^2} - \sum_{i=1}^{J} S_i\left(\frac{\alpha}{y_i}\right)^{2\beta} \frac{e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}}{\left(1-e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\right)^2} - S_n\left(\frac{\alpha}{y_n}\right)^{2\beta} \frac{e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}}{\left(1-e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\right)^2},$$

$$\ell_{l,k} = 0,\ l=1,\cdots,m+1, l\neq k.$$

Using $\ell_{i,j}, i,j=1,\cdots,m+1$, we can obtain $\sigma_{i,j}, i,j=1,\cdots,m+1$ and

$$\ell_{l,l,l} = \frac{2nk_l}{\gamma_l^3} + \sum_{i=1}^{n}\sum_{j_l=1}^{J_i^{(l)}} R_{ij_l}^{(l)}\left(\frac{\alpha}{x_{ij_l}^{(l)}}\right)^{3\beta} \frac{e^{-\gamma(\frac{\alpha}{x_{ij_l}^{(l)}})^{\beta}}\left(1+e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}\right)}{\left(1-e^{-\gamma_l(\frac{\alpha}{x_{ij_l}^{(l)}})^{\beta}}\right)^3} + \sum_{i=1}^{n} R_{ik_l}^{(l)}\left(\frac{\alpha}{x_{ik_l}^{(l)}}\right)^{3\beta} \frac{e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}\left(1+e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}\right)}{\left(1-e^{-\gamma(\frac{\alpha}{x_{ik_l}^{(l)}})^{\beta}}\right)^3},\ l=1,\cdots,m,$$

$$\ell_{m+1,m+1,m+1} = \frac{2n}{\gamma^3} + \sum_{i=1}^{J} S_i\left(\frac{\alpha}{y_i}\right)^{3\beta} \frac{e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\left(1+e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\right)}{\left(1-e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\right)^2} + S_n\left(\frac{\alpha}{y_n}\right)^{3\beta} \frac{e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\left(1+e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\right)}{\left(1-e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\right)^3},$$

and other $\ell_{i,j,k} = 0$. Furthermore,

$$u_l = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{p_1}\cdots\sum_{q_m=0}^{p_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\binom{p_1}{q_1}\cdots\binom{p_m}{q_m}(-1)^{\sum_{l=1}^{m} q_l} \times A,$$

and

$$u_{l,k} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\sum_{q_1=0}^{p_1}\cdots\sum_{q_m=0}^{p_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\binom{p_1}{q_1}\cdots\binom{p_m}{q_m}(-1)^{\sum_{l=1}^{m} q_l} \times B,$$

where

$$A = \begin{cases} \dfrac{\gamma(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma\right)^2} & l=1,\cdots,m, \\ \dfrac{\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma\right)^2} & l=m+1, \end{cases}$$

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

and

$$B = \begin{cases} \frac{2\gamma(k_l-p_l+q_l)(k_k-p_k+q_k)}{\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma\right)^3} & l,k=1,\cdots,m, \\ -\frac{2\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)}{\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma\right)^3} & l=m+1, \\ -\frac{(k_l-p_l+q_l)\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)-\gamma\right)}{\left(\sum_{l=1}^{m}\gamma_l(k_l-p_l+q_l)+\gamma\right)^3} & l=1,\cdots,m,\ k=m+1. \end{cases}$$

After acquiring the aforementioned values, $\widehat{R}^{Lin}_{\boldsymbol{s},\boldsymbol{k}}$, which represents Lindley's estimation of $R_{\boldsymbol{s},\boldsymbol{k}}$, can be calculated from equation (10). It is important to compute all parameters at $(\widehat{\gamma},\widehat{\gamma}_1,\cdots,\widehat{\gamma}_m)$, which are the MLEs for $(\gamma,\gamma_1,\cdots,\gamma_m)$ and are obtained by employing one numerical method like Newton-Raphson algorithm.

4 Inference on $R_{\mathbf{s},\mathbf{k}}$ in general case

Consider independent random variables $X_1 \sim GIW(\alpha_1,\beta_1,\gamma_1),\ldots,X_m \sim GIW(\alpha_m,\beta_m,\gamma_m)$, and $Y \sim GIW(\alpha,\beta,\gamma)$. Under this condition, m-component stress-strength parameter, $R_{\boldsymbol{s},\boldsymbol{k}}$, from (2) and (3), can be obtained as

$$R_{\boldsymbol{s},\boldsymbol{k}} = \sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^\infty \gamma\beta\alpha^\beta y^{-\beta-1}e^{-\gamma(\frac{\alpha}{y})^\beta} \times\prod_{l=1}^{m}e^{-\gamma_l(k_l-p_l)(\frac{\alpha_l}{y})^{\beta_l}}\prod_{l=1}^{m}\left(1-e^{-\gamma_l(\frac{\alpha_l}{y})^{\beta_l}}\right)^{p_l}dy.$$

In this portion, we examine the Bayesian inference for $R_{\boldsymbol{s},\boldsymbol{k}}$ using squared error loss functions, given that $\alpha,\alpha_1,\ldots,\alpha_m,\beta,\beta_1,\ldots,\beta_m,\gamma,\gamma_1,\ldots,\gamma_m$ are independent random variables as follows:

$$\pi(\alpha)\propto\alpha^{a-1}e^{-b\alpha},\alpha,a,b>0,\pi(\alpha_l)\propto\alpha_l^{a_l-1}e^{-b_l\alpha_l},\alpha_l,a_l,b_l>0,l=1,\ldots,m,$$
$$\pi(\beta)\propto\beta^{c-1}e^{-d\beta},\beta,c,d>0,\pi(\beta_l)\propto\beta_l^{c_l-1}e^{-d_l\beta_l},\beta_l,c_l,d_l>0,l=1,\ldots,m,$$
$$\pi(\gamma)\propto\gamma^{e-1}e^{-f\gamma},\gamma,e,f>0,\pi(\gamma_l)\propto\gamma_l^{e_l-1}e^{-f_l\gamma_l},\gamma_l,e_l,f_l>0,l=1,\ldots,m.$$

Similar to Section 2, it is not possible to evaluate the Bayes estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ in a closed form. Therefore, we use the MCMC method to approximate it. By using the joint posterior density function, we can derive the posterior pdfs of parameters by

$$\pi(\alpha|\beta,\gamma\text{data})\propto\alpha^{n\beta+a-1}e^{-b\alpha}\prod_{i=1}^{n}y_i^{-\beta-1}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^\beta}\prod_{i=1}^{J}\left(1-e^{-\gamma(\frac{\alpha}{y_i})^\beta}\right)^{S_i}\left(1-e^{-\gamma(\frac{\alpha}{y_n})^\beta}\right)^{S_n},$$

$$\pi(\alpha_l|\beta_l,\gamma_l,\text{data})\propto\alpha_l^{n\beta_lk_l+a_l-1}e^{-b_l\alpha_l}e^{-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha_l}{x^{(l)}_{ij_l}})^{\beta_l}}$$
$$\times\prod_{i=1}^{n}\prod_{j_l=1}^{J_i^{(l)}}\left(1-e^{-\gamma_l(\frac{\alpha_l}{x^{(l)}_{ij_l}})^{\beta_l}}\right)^{R^{(l)}_{ij_l}}\prod_{i=1}^{n}\left(1-e^{-\gamma_l(\frac{\alpha_l}{x^{(l)}_{ik_l}})^{\beta_l}}\right)^{R^{(l)}_{ik_l}},\ l=1,\ldots,m,$$

$$\pi(\beta|\alpha,\gamma,\text{data})\propto\beta^{n+c-1}\alpha^{n\beta}e^{-d\beta}\prod_{i=1}^{n}y_i^{-\beta-1}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^\beta}\prod_{i=1}^{J}\left(1-e^{-\gamma(\frac{\alpha}{y_i})^\beta}\right)^{S_i}\left(1-e^{-\gamma(\frac{\alpha}{y_n})^\beta}\right)^{S_n},$$

$$\pi(\beta_l|\alpha_l,\gamma_l,\text{data})\propto\beta^{nk_l+c_l-1}\alpha_l^{n\beta_l}e^{-d_l\beta_l}\prod_{i=1}^{n}\prod_{j_l=1}^{k_l}(x^{(l)}_{ij_l})^{-\beta_l-1}$$
$$\times e^{-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha_l}{x^{(l)}_{ij_l}})^{\beta}_l}\prod_{i=1}^{n}\prod_{j_l=1}^{J_i^{(l)}}\left(1-e^{-\gamma_l(\frac{\alpha_l}{x^{(l)}_{ij_l}})^{\beta_l}}\right)^{R^{(l)}_{ij_l}}\prod_{i=1}^{n}\left(1-e^{-\gamma_l(\frac{\alpha_l}{x^{(l)}_{ik_l}})^{\beta_l}}\right)^{R^{(l)}_{ik_l}},\ l=1,\ldots,m,$$

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

$$\pi(\gamma|\alpha,\beta,\text{data}) \propto \gamma^{n+e-1}e^{-f\gamma}e^{-\sum_{i=1}^{n}\gamma(\frac{\alpha}{y_i})^{\beta}}\prod_{i=1}^{J}\left(1-e^{-\gamma(\frac{\alpha}{y_i})^{\beta}}\right)^{S_i}\left(1-e^{-\gamma(\frac{\alpha}{y_n})^{\beta}}\right)^{S_n},$$

$$\pi(\gamma_l|\alpha,\beta,\text{data}) \propto \gamma_l^{nk_l+e_l-1}e^{-f_l\gamma_l}e^{-\sum_{i=1}^{n}\sum_{j_l=1}^{k_l}\gamma_l(\frac{\alpha}{x^{(l)}_{ij_l}})^{\beta}}$$
$$\times\prod_{i=1}^{n}\prod_{j_l=1}^{J_i^{(l)}}\left(1-e^{-\gamma_l(\frac{\alpha}{x^{(l)}_{ij_l}})^{\beta}}\right)^{R^{(l)}_{ij_l}}\prod_{i=1}^{n}\left(1-e^{-\gamma_l(\frac{\alpha}{x^{(l)}_{ik_l}})^{\beta}}\right)^{R^{(l)}_{ik_l}},\ l=1,\ldots,m,$$

It is evident that sampling from probability density functions of $\alpha,\alpha_1,\ldots,\alpha_m,\beta,\beta_1,\ldots,\beta_m,\gamma,\gamma_1,\ldots,\gamma_m$ requires the utilization of the Metropolis-Hastings method since their probability density functions are unknown. Therefore, the procedure for implementing the Gibbs sampling algorithm is as follows:

1.Begin with initial values $(\alpha_{(0)},\alpha_{1(0)},\ldots,\alpha_{m(0)},\beta_{(0)},\beta_{1(0)},\ldots,\beta_{m(0)},\gamma_{(0)},\gamma_{1(0)},\ldots,\gamma_{m(0)})$.
2.Set $t=1$.
3.With $N(\alpha_{(t-1)},1)$ as proposal distribution and $\pi(\alpha|\beta_{(t-1)},\gamma_{(t-1)},\text{data})$ as target distribution, produce $\alpha_{(t)}$ with MH algorithm.
4.With $N(\alpha_{1(t-1)},1)$ as proposal distribution and $\pi(\alpha_1|\beta_{1(t-1)},\gamma_{1(t-1)},\text{data})$ as target distribution, produce $\alpha_{1(t)}$ with MH algorithm.
⋮
m+3.With $N(\alpha_{m(t-1)},1)$ as proposal distribution and $\pi(\alpha_m|\beta_{m(t-1)},\gamma_{m(t-1)},\text{data})$ as target distribution, produce $\alpha_{m(t)}$ with MH algorithm.
m+4.With $N(\beta_{(t-1)},1)$ as proposal distribution and $\pi(\beta|\alpha_{(t-1)},\gamma_{(t-1)},\text{data})$ as target distribution, produce $\beta_{(t)}$ with MH algorithm.
m+5With $N(\beta_{1(t-1)},1)$ as proposal distribution and $\pi(\beta_1|\alpha_{1(t-1)},\gamma_{1(t-1)},\text{data})$ as target distribution, produce $\beta_{1(t)}$ with MH algorithm.
⋮
2m+4With $N(\beta_{m(t-1)},1)$ as proposal distribution and $\pi(\beta_m|\alpha_{m(t-1)},\gamma_{m(t-1)},\text{data})$ as target distribution, produce $\beta_{m(t)}$ with MH algorithm.
2m+5.With $N(\gamma_{(t-1)},1)$ as proposal distribution and $\pi(\gamma|\alpha_{(t-1)},\beta_{(t-1)},\text{data})$ as target distribution, produce $\gamma_{(t)}$ with MH algorithm.
2m+6.With $N(\gamma_{1(t-1)},1)$ as proposal distribution and $\pi(\gamma_1|\alpha_{1(t-1)},\beta_{1(t-1)},\text{data})$ as target distribution, produce $\gamma_{1(t)}$ with MH algorithm.
⋮
3m+5.With $N(\gamma_{m(t-1)},1)$ as proposal distribution and $\pi(\gamma_m|\alpha_{m(t-1)},\beta_{m(t-1)},\text{data})$ as target distribution, produce $\gamma_{m(t)}$ with MH algorithm.
3m+6.Evaluate

$$R_{(t)\boldsymbol{s},\boldsymbol{k}}=\sum_{p_1=s_1}^{k_1}\cdots\sum_{p_m=s_m}^{k_m}\binom{k_1}{p_1}\cdots\binom{k_m}{p_m}\int_0^{\infty}\gamma_{(t)}\beta_{(t)}\alpha_{(t)}^{\beta_{(t)}}y^{-\beta_{(t)}-1}e^{-\gamma_{(t)}(\frac{\alpha_{(t)}}{y})^{\beta_{(t)}}}$$
$$\times\prod_{l=1}^{m}e^{-\gamma_{l(t)}(k_l-p_l)(\frac{\alpha_{l(t)}}{y})^{\beta_{l(t)}}}\prod_{l=1}^{m}\left(1-e^{-\gamma_{l(t)}(\frac{\alpha_{l(t)}}{y})^{\beta_{l(t)}}}\right)^{p_l}dy.$$

3m+7.Set $t=t+1$.
3m+8.Repeat steps 3 - 3m+7, T times.

Therefore, the Bayesian estimate of $R_{\boldsymbol{s},\boldsymbol{k}}$ under the squared error loss function is:

$$\widehat{R}^{MB}_{\boldsymbol{s},\boldsymbol{k}}=\frac{1}{T}\sum_{t=1}^{T}R_{(t)\boldsymbol{s},\boldsymbol{k}}. \tag{11}$$

Furthermore, the $100(1-\eta)\%$ HPD credible interval for $R_{\boldsymbol{s},\boldsymbol{k}}$ can be computed using the method described by [16], as elaborated in Section 2.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

5 Simulation study and data analysis

5.1 Numerical experiment and discussion

This section utilizes Monte Carlo simulation to compare various estimators. We assess point estimates using mean square errors (MSEs) and evaluate interval estimates through average confidence lengths (AL) and coverage percentages (CP). The simulation studies involve different censoring schemes, parameter values, and hyper-parameters. The findings stem from 2000 repetitions, where the Gibbs sampling algorithm iterates for $T = 3000$ cycles. HPD credible intervals are calculated at a significance level of 0.95. We consider a simulated system with two strength components. The results are based on various censoring schemes (C.S) outlined in Table 1.

(k_l, K_l)		C.S.	(n,N)		C.S.
(5,10)	R_1	(0,0,0,0,5)	(5,10)	S_1	(0,0,0,0,5)
	R_2	(5,0,0,0,0)		S_2	(5,0,0,0,0)
	R_3	(1,1,1,1,1)		S_3	(1,1,1,1,1)
(10,20)	R_4	$(0^{*9},10)$	(10,20)	S_4	$(0^{*9},10)$
	R_5	$(10,0^{*9})$		S_5	$(10,0^{*9})$
	R_6	(1^{*10})		S_6	(1^{*10})

Table 1: Different censoring schemes.

We will now consider three cases. In the first case, we assume that the common parameters α and β are unknown, and we use $(\alpha,\beta,\gamma,\gamma_1,\gamma_2,\gamma_3) = (1,3,2,2.5,3.5,1.5)$ to obtain the simulation results. Additionally, we employ two priors: Prior 1 with $a,c,e,e_l = 0,\ b,d,f,f_l = 0$ for $l = 1,\ldots,3$, and Prior 2 with $a,c,e,e_l = 0.4,\ b,d,f,f_l = 0.35$ for $l = 1,\ldots,3$. These priors are used to compare the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ obtained from equation (6). The simulation results can be found in Table 2. Secondly, assuming that the common parameters α and β are known, we use $(\alpha,\beta,\gamma,\gamma_1,\gamma_2,\gamma_3) = (2,1,2,1,3,4)$ to obtain the simulation results. Additionally, we employ two priors, namely Prior 3: $e,e_l = 0,\ f,f_l = 0$ for $l = 1,\ldots,3$, and Prior 4: $e,e_l = 0.45,\ f,f_l = 0.7$ for $l = 1,\ldots,3$, to compare the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$. In this case, we use equations (7) and (10) to obtain the simulation results, which are provided in Table 3. In the third case, we use $(\alpha,\alpha_1,\alpha_2,\alpha_3,\beta,\beta_1,\beta_2,\beta_3,\gamma,\gamma_1,\gamma_2,\gamma_3) = (1,5,3,1.5,2,2.5,3,42,4,3,0.5)$ to obtain the simulation results. Additionally, we employ two priors, namely Prior 5: $a,a_l,c,c_l,e,e_l = 0,\ b,b_l,d,d_l,f,f_l = 0$ for $l = 1,\ldots,3$, and Prior 6: $a,a_l,c,c_l,e,e_l = 0.8,\ b,b_l,d,d_l,f,f_l = 0.65$ for $l = 1,\ldots,3$. These priors are used to compare the Bayes estimates of $R_{\boldsymbol{s},\boldsymbol{k}}$ obtained from equation (11). The simulation results can be found in Table 4.

Based on Tables 2-4, it is evident that the informative priors (priors 2, 4, and 6) exhibit the best performance based on the MSE values. Furthermore, in the second scenario, the Bayes estimates derived from the MCMC method demonstrate superior performance compared to those obtained through Lindley's approximation. Additionally, the HPD intervals derived from informative priors (specifically priors 2, 4, and 6) exhibit the most favorable performance in terms of the AL and CP values among the intervals analyzed.

Moreover, the following general trends can be observed from Tables 2-4:

– As the sample size n increases, while keeping $\boldsymbol{s}$ and $\boldsymbol{k}$ constant, the MSEs and ALs decrease, indicating improved accuracy and precision in the estimates. Conversely, the CPs increase, suggesting a higher likelihood of capturing the true parameter value within the confidence interval.
– As the value of $\boldsymbol{k}$ increases, while keeping $\boldsymbol{s}$ and n constant, the MSEs and ALs decrease, indicating improved accuracy in the estimates. However, the CPs increase, suggesting a higher likelihood of capturing the true parameter value within the confidence interval.

These trends may be attributed to as the value of n increases, the frequency of failures also increases, leading to the accumulation of more information. Consequently, this improvement in the amount of information contributes to the enhanced performance of the estimates.

5.2 Real data analysis

Within this section, we examine an authentic dataset to demonstrate the current topic. The dataset comprises strength measurements in GPa for individual carbon fibers subjected to tension tests at gauge lengths of 50 mm, 10 mm, and 1 mm. The original dataset reference is provided in [18]. In a recent study, [19] have explored this data type using a stress-strength model dependent on the two-parameter Rayleigh distribution. Consider a system with two distinct gauge lengths of individual fibers. Here, the strength of a single fiber at 1 mm and 10 mm gauge lengths is denoted by X_1 and

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	MCMC					
		Prior 1			Prior 2		
		MSE	AL	CP	MSE	AL	CP
(5,5,5,5,2,2,2)	(R_1,R_1,R_1,S_1)	0.0512	0.5545	0.942	0.0405	0.5122	0.945
	(R_2,R_2,R_2,S_2)	0.0510	0.5536	0.943	0.0400	0.5130	0.944
	(R_3,R_3,R_3,S_3)	0.0513	0.5524	0.940	0.0409	0.5167	0.945
(5,5,5,10,2,2,2)	(R_1,R_1,R_1,S_4)	0.0405	0.5034	0.945	0.0385	0.4625	0.948
	(R_2,R_2,R_2,S_5)	0.0400	0.5085	0.945	0.0379	0.4633	0.949
	(R_3,R_3,R_3,S_6)	0.0401	0.5033	0.944	0.0374	0.4675	0.948
(10,10,10,5,2,2,2)	(R_4,R_4,R_4,S_1)	0.0384	0.4562	0.948	0.0304	0.4025	0.950
	(R_5,R_5,R_5,S_2)	0.0395	0.4574	0.947	0.0300	0.4031	0.950
	(R_6,R_6,R_6,S_3)	0.0378	0.4503	0.949	0.0309	0.4011	0.949
(10,10,10,10,2,2,2)	(R_4,R_4,R_4,S_4)	0.0301	0.4025	0.950	0.0285	0.3452	0.952
	(R_5,R_5,R_5,S_5)	0.0305	0.4053	0.951	0.0295	0.3411	0.953
	(R_6,R_6,R_6,S_6)	0.0308	0.4065	0.950	0.0265	0.3485	0.952
(5,5,5,5,4,4,4)	(R_1,R_1,R_1,S_1)	0.0523	0.5532	0.943	0.0408	0.5103	0.945
	(R_2,R_2,R_2,S_2)	0.0524	0.5514	0.943	0.0406	0.5174	0.944
	(R_3,R_3,R_3,S_3)	0.0531	0.5590	0.942	0.0400	0.5190	0.944
(5,5,5,10,4,4,4)	(R_1,R_1,R_1,S_4)	0.0405	0.5024	0.946	0.0368	0.4627	0.948
	(R_2,R_2,R_2,S_5)	0.0406	0.5066	0.945	0.0370	0.4638	0.948
	(R_3,R_3,R_3,S_6)	0.0408	0.5094	0.945	0.0380	0.4602	0.948
(10,10,10,5,4,4,4)	(R_4,R_4,R_4,S_1)	0.0345	0.4536	0.949	0.0307	0.4068	0.951
	(R_5,R_5,R_5,S_2)	0.0367	0.4522	0.948	0.0306	0.4033	0.950
	(R_6,R_6,R_6,S_3)	0.0358	0.4574	0.948	0.0303	0.4005	0.950
(10,10,10,10,4,4,4)	(R_4,R_4,R_4,S_4)	0.0309	0.4030	0.951	0.0276	0.3495	0.952
	(R_5,R_5,R_5,S_5)	0.0308	0.4058	0.950	0.0270	0.3466	0.953
	(R_6,R_6,R_6,S_6)	0.0310	0.4066	0.950	0.0264	0.3475	0.952

Table 2: Simulation results when common parameters α and β are unknown.

$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	MCMC						Lindley	
		Prior 3			Prior 4			Prior 3	Prior 4
		MSE	AL	CP	MSE	AL	CP	MSE	MSE
(5,5,5,5,2,2,2)	(R_1,R_1,R_1,S_1)	0.0605	0.5965	0.941	0.0576	0.5525	0.945	0.0652	0.0615
	(R_2,R_2,R_2,S_2)	0.0603	0.5952	0.940	0.0578	0.5530	0.944	0.0659	0.0619
	(R_3,R_3,R_3,S_3)	0.0609	0.5931	0.941	0.0573	0.5514	0.945	0.0653	0.0613
(5,5,5,10,2,2,2)	(R_1,R_1,R_1,S_4)	0.0546	0.5133	0.943	0.0468	0.4963	0.949	0.0618	0.0568
	(R_2,R_2,R_2,S_5)	0.0548	0.5120	0.944	0.0463	0.4922	0.948	0.0619	0.0563
	(R_3,R_3,R_3,S_6)	0.0549	0.5164	0.943	0.0462	0.4930	0.949	0.0620	0.0560
(10,10,10,5,2,2,2)	(R_4,R_4,R_4,S_1)	0.0486	0.4865	0.948	0.0415	0.3715	0.950	0.0586	0.0535
	(R_5,R_5,R_5,S_2)	0.0482	0.4885	0.947	0.0416	0.3766	0.951	0.0581	0.0530
	(R_6,R_6,R_6,S_3)	0.0480	0.4890	0.948	0.0413	0.3741	0.950	0.0580	0.0533
(10,10,10,10,2,2,2)	(R_4,R_4,R_4,S_4)	0.0412	0.4025	0.950	0.0371	0.3325	0.952	0.0512	0.0475
	(R_5,R_5,R_5,S_5)	0.0416	0.4011	0.951	0.0379	0.3341	0.951	0.0519	0.0470
	(R_6,R_6,R_6,S_6)	0.0419	0.4069	0.950	0.0376	0.3300	0.952	0.0513	0.0474
(5,5,5,5,4,4,4)	(R_1,R_1,R_1,S_1)	0.0609	0.5931	0.940	0.0576	0.5522	0.944	0.0664	0.0612
	(R_2,R_2,R_2,S_2)	0.0606	0.5947	0.941	0.0570	0.5547	0.945	0.0663	0.0615
	(R_3,R_3,R_3,S_3)	0.0607	0.5930	0.940	0.0571	0.5596	0.945	0.0659	0.0618
(5,5,5,10,4,4,4)	(R_1,R_1,R_1,S_4)	0.0543	0.5166	0.943	0.0475	0.4966	0.948	0.0619	0.0566
	(R_2,R_2,R_2,S_5)	0.0546	0.5174	0.943	0.0472	0.4922	0.949	0.0610	0.0563
	(R_3,R_3,R_3,S_6)	0.0549	0.5199	0.942	0.0470	0.4915	0.948	0.0613	0.0561
(10,10,10,5,4,4,4)	(R_4,R_4,R_4,S_1)	0.0489	0.4820	0.948	0.0410	0.3755	0.950	0.0584	0.0540
	(R_5,R_5,R_5,S_2)	0.0486	0.4830	0.947	0.0419	0.3733	0.950	0.0583	0.0533
	(R_6,R_6,R_6,S_3)	0.0483	0.4823	0.948	0.0418	0.3766	0.951	0.0580	0.0539
(10,10,10,10,4,4,4)	(R_4,R_4,R_4,S_4)	0.0410	0.4066	0.950	0.0381	0.3352	0.952	0.0510	0.0480
	(R_5,R_5,R_5,S_5)	0.0411	0.4022	0.950	0.0375	0.3322	0.952	0.0519	0.0475
	(R_6,R_6,R_6,S_6)	0.0415	0.4081	0.951	0.0389	0.3344	0.952	0.0511	0.0477

Table 3: Simulation results when common parameters α and β are known.

X_2, respectively. At the same time, the system's stress is determined by the single fiber at a 50 mm gauge length, labeled as Y. Hence, we can view the observations of X_1, X_2, and Y as follows:

$$\begin{bmatrix} 3.126 & 3.245 & 3.328 & 3.355 & 3.383 \\ 3.572 & 3.581 & 3.681 & 3.726 & 3.727 \\ 3.728 & 3.783 & 3.785 & 3.786 & 3.896 \\ 3.912 & 3.964 & 4.05 & 4.063 & 4.082 \\ 4.111 & 4.118 & 4.141 & 4.216 & 4.251 \\ 4.262 & 4.326 & 4.402 & 4.457 & 4.466 \\ 4.519 & 4.542 & 4.555 & 4.614 & 4.632 \\ 4.634 & 4.636 & 4.678 & 4.698 & 4.738 \\ 4.832 & 4.924 & 5.043 & 5.099 & 5.134 \\ 5.359 & 5.473 & 5.571 & 5.684 & 5.721 \end{bmatrix}, \begin{bmatrix} 2.35 & 2.361 & 2.396 & 2.397 & 2.445 \\ 2.454 & 2.454 & 2.474 & 2.518 & 2.522 \\ 2.525 & 2.532 & 2.575 & 2.614 & 2.616 \\ 2.618 & 2.624 & 2.659 & 2.675 & 2.738 \\ 2.74 & 2.856 & 2.917 & 2.928 & 2.937 \\ 2.937 & 2.977 & 2.996 & 3.03 & 3.125 \\ 3.139 & 3.145 & 3.22 & 3.223 & 3.235 \\ 3.243 & 3.264 & 3.272 & 3.294 & 3.332 \\ 3.346 & 3.377 & 3.408 & 3.435 & 3.493 \\ 3.501 & 3.537 & 3.554 & 3.562 & 3.628 \end{bmatrix}, \begin{bmatrix} 1.613 \\ 1.812 \\ 1.864 \\ 2.051 \\ 2.162 \\ 2.211 \\ 2.308 \\ 2.39 \\ 2.471 \\ 2.593 \end{bmatrix}.$$

By standardizing the data on a scale of 0 to 1, we have transformed the values so that they all fall within this range. This allows us to compare and analyze the data more easily. It also eliminates any differences in units or scales that may exist between different variables, making calculations more straightforward and consistent. It is important to note that this normalization does not affect the statistical inference. After fitting the GIW distribution to each of the three datasets individually, the results are as follows:

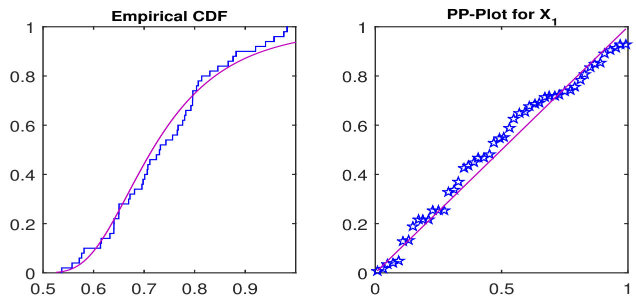
–For X_1: $(\alpha,\beta,\gamma) = (0.9963, 7.0870, 0.0668)$ with a p-value of 0.7840.
–For X_2: $(\alpha,\beta,\gamma) = (0.9997, 8.3220, 0.0748)$ with a p-value of 0.2329.
–For Y: $(\alpha,\beta,\gamma) = (1.5420, 6.4043, 0.0088)$ with a p-value of 0.9115.

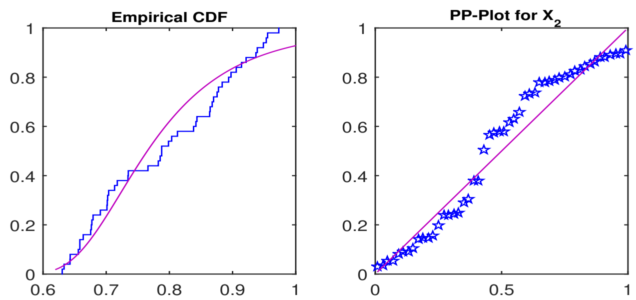
© 2025 YU

$(k_1,k_2,k_3,n,s_1,s_2,s_3)$	C.S	MCMC					
		Prior 5			Prior 6		
		MSE	AL	CP	MSE	AL	CP
(5,5,5,5,2,2,2)	(R_1,R_1,R_1,S_1)	0.0756	0.7025	0.942	0.0712	0.6340	0.945
	(R_2,R_2,R_2,S_2)	0.0768	0.7065	0.941	0.0716	0.6333	0.946
	(R_3,R_3,R_3,S_3)	0.0759	0.7024	0.941	0.0719	0.6324	0.945
(5,5,5,10,2,2,2)	(R_1,R_1,R_1,S_4)	0.0723	0.6235	0.944	0.0677	0.5274	0.949
	(R_2,R_2,R_2,S_5)	0.0721	0.6241	0.943	0.0673	0.5233	0.948
	(R_3,R_3,R_3,S_6)	0.0720	0.6281	0.944	0.0678	0.5214	0.949
(10,10,10,5,2,2,2)	(R_4,R_4,R_4,S_1)	0.0632	0.5964	0.949	0.0603	0.4949	0.950
	(R_5,R_5,R_5,S_2)	0.0637	0.5932	0.947	0.0601	0.4966	0.950
	(R_6,R_6,R_6,S_3)	0.0633	0.5901	0.949	0.0600	0.4927	0.951
(10,10,10,10,2,2,2)	(R_4,R_4,R_4,S_4)	0.0583	0.5234	0.951	0.0503	0.4055	0.952
	(R_5,R_5,R_5,S_5)	0.0583	0.5201	0.950	0.0501	0.4066	0.952
	(R_6,R_6,R_6,S_6)	0.0581	0.5291	0.950	0.0500	0.4028	0.951
(5,5,5,5,4,4,4)	(R_1,R_1,R_1,S_1)	0.0762	0.7051	0.942	0.0703	0.6322	0.945
	(R_2,R_2,R_2,S_2)	0.0751	0.7063	0.940	0.0710	0.6334	0.945
	(R_3,R_3,R_3,S_3)	0.0763	0.7008	0.940	0.0701	0.6345	0.946
(5,5,5,10,4,4,4)	(R_1,R_1,R_1,S_4)	0.0729	0.6244	0.944	0.0674	0.5274	0.949
	(R_2,R_2,R_2,S_5)	0.0721	0.6255	0.943	0.0679	0.5211	0.948
	(R_3,R_3,R_3,S_6)	0.0719	0.6208	0.943	0.0673	0.5261	0.949
(10,10,10,5,4,4,4)	(R_4,R_4,R_4,S_1)	0.0628	0.5961	0.947	0.0606	0.4961	0.950
	(R_5,R_5,R_5,S_2)	0.0639	0.5941	0.949	0.0600	0.4925	0.951
	(R_6,R_6,R_6,S_3)	0.0627	0.5934	0.947	0.0601	0.4937	0.951
(10,10,10,10,4,4,4)	(R_4,R_4,R_4,S_4)	0.0573	05214	0.950	0.0501	0.4037	0.952
	(R_5,R_5,R_5,S_5)	0.0570	0.5245	0.950	0.0510	0.4068	0.952
	(R_6,R_6,R_6,S_6)	0.0583	0.5248	0.951	0.0509	0.4066	0.951

Table 4: Simulation results in general case.

Considering the p-values, we infer that the GIW distribution fits well for the X_1, X_2, and Y datasets. The estimated parameters indicate for describing these sets of data, the general scenario is suitable. In Figure 4, we display the empirical distribution functions and PP-plots for these three datasets.





For the complete data set, using $\boldsymbol{s}=(2,2)$ and $\boldsymbol{k}=(5,5)$ with non-informative priors, we calculate $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ as 0.4419 and associated 95% HPD interval is $(0.2245, 0.7753)$. MCMC diagnostics in the case of real data is provided. For this aim, the trace plot, autocorrelation plot of MCMC chains, Gelman and Rubin's diagnostic plot which shows the pattern of Gelman and Rubin's shrink factor as the number of iterations increases, and the density plot of the posterior distribution of $R_{\boldsymbol{s},\boldsymbol{k}}$, for complete data in case $\boldsymbol{s}=(2,2)$ and non-informative priors, is presented in Figure 5. From this figure, the trace plot shows the convergence of the MCMC algorithm. Also, the auto-correlation plot shows the chain achieves stationarity, the shrink

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

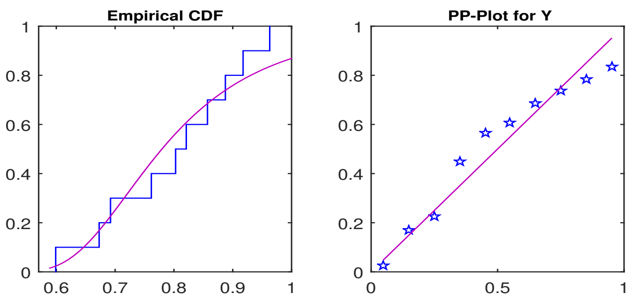


Fig. 4: The left side shows the empirical distribution function and the right side shows the PP-plot for X_1 in the first row, for X_2 in the middle row, and for Y in the third row.

factor values are smaller than the general acceptable top value of 1.1 and the density plots are symmetric and unimodal. Hence, we conclude that the MCMC chain is converged. Moreover, we monitor the overall convergence of MCMC chains in all cases of the real data, and these are omitted for the sake of brevity. Next, we will create two distinct censoring AHP

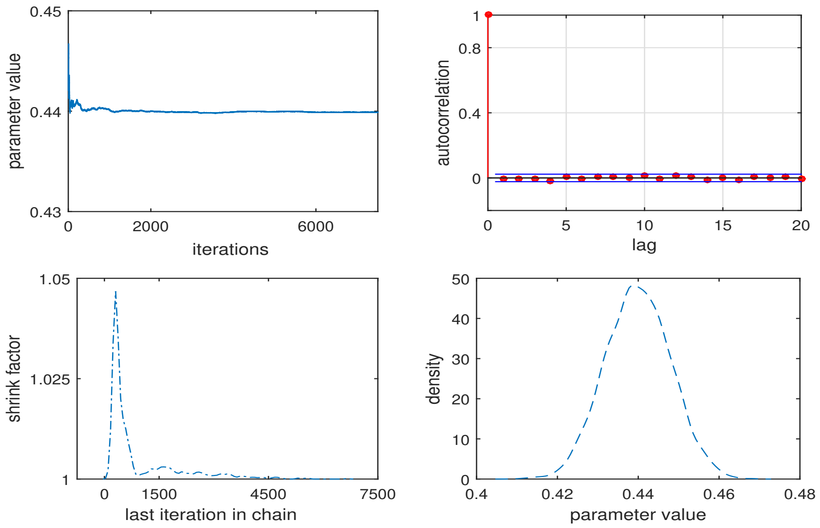


Fig. 5: MCMC diagnostic plots of $R_{\mathbf{s},\mathbf{k}}$, in data set I.

schemes outlined as follows:

$$\begin{aligned}
\text{Scheme 1: } & R^{(1)} = R^{(2)} = [1,0,0,0], \;\; S = [1,1,0,0,0,0,1], \;\; (\boldsymbol{k} = (4,4), \boldsymbol{s} = (2,2)), \\
& T = T_1^{(1)} = T_2^{(l)} = 0.75, \; l = 1,2. \\
\text{Scheme 2: } & R^{(1)} = R^{(2)} = [1,0,1], \quad S = [2,1,0,1,1], \quad (\boldsymbol{k} = (3,3), \boldsymbol{s} = (2,2)), \\
& T = T_1^{(1)} = T_2^{(l)} = 0.35, \; l = 1,2.
\end{aligned}$$

In Scheme 1, employing non-informative priors, we compute $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ to be 0.4966, with a 95% HPD interval of $(0.2373, 0.8166)$. In Scheme 2, also with non-informative priors, we derive $\widehat{R}^{MC}_{\boldsymbol{s},\boldsymbol{k}}$ as 0.5103, with a corresponding 95% HPD interval of $(0.2912, 0.9055)$. Upon comparing the estimates and intervals for both schemes, Scheme 1 demonstrates superior performance, as anticipated.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

6 Conclusion

This paper focuses on statistically inferring the GIW distribution in combination with the AHPC scheme, for a stress-strength system consisting of m non-identical components with varying strengths. Specifically, the point and interval Bayesian estimates are considered in three cases: in scenarios where the common parameters are not known, are known, or fall under the general case. Given that $R_{s,k}$ and the AHPC scheme can be adapted to various scenarios, the problem addressed in this paper is quite general.

Through the Monte Carlo simulation study, we conducted a comparison of different estimates. The simulation results indicate that informative priors consistently yield superior results compared to non-informative priors in different estimates. Furthermore, Bayesian estimates obtained through the MCMC method surpass those obtained using Lindley's approximation. Moreover, as the number of failures increases, resulting in enhanced accuracy of the estimates, because more relevant information is accumulated.

Declarations

Competing interests: The author declares that there are no competing interests.
Authors' contributions: The author is solely responsible for all aspects of the work.
Funding: This work received no funding.
Availability of data and materials: The data generated and analyzed during the current study are included in the paper.
Acknowledgments: The author wishes to thank the editor and the reviewers for their valuable feedback and constructive suggestions, which greatly improved the quality of this work.

References

[1] B. Epstein, Truncated life tests in the exponential case, *Ann. Math. Stat.*, **25**(1954), 555–564.
[2] D. Kundu, A. Joarder, Analysis of type-II progressively hybrid censored data, *Comput. Stat. Data. Anal.*, **50**(2006), 2509–2528.
[3] H.K.T. Ng, D. Kundu, P.S. Chan, Statistical analysis of exponential lifetimes under an adaptive Type-II progressively censoring scheme, *Nav. Res. Log.*, **56**(2009), 687–698.
[4] T.R. Rasethuntsa, M. Nadar, Stress-strength reliability of a non-identical-component strengths system based on upper record values from the family of Kumaraswamy generalized distributions, *Statistics.*, **52**(2018), 684–716.
[5] A. Kohansal, C.J. Pérez-González, A.J. Fernández, Family of Kumaraswamy generalized distribution in estimation of multi-component reliability under the adaptive hybrid progressive censoring schemes, *Qual. Technol. Quant. M.*, (2024), DOI: 10.1080/16843703.2024.2315830.
[6] A. Kohansal, A.J. Fernández, C.J. Pérez-González, Multi-component stress-strength parameter estimation of a non-identical-component strengths system under the adaptive hybrid progressive censoring samples, *Statistics.*, **55**(2021), 925–962.
[7] S. Saini, R. Garg, Reliability inference for multicomponent stress-strength model from Kumaraswamy-G family of distributions based on progressively first failure censored samples, *Comput Stat*, **37**(2022), 1795–1837.
[8] D.P. Singh, M.K. Jha, Y. Tripathi, L. Wang, Reliability estimation in a multicomponent stress-strength model for unit Burr III distribution under progressive censoring, *Qual. Technol. Quant. M.*, **19**(2022), 605–632.
[9] A. Kohansal, C.J. Pérez-González, A.J. Fernández, Multi-component reliability inference in modified Weibull extension distribution and progressive censoring scheme, *B. Malays. Math. Sci. So.*, **46**(2023), 61.
[10] A. Kumari, S. Kumar, K. Kumar, Inference for reliability in a multicomponent stress-strength model from generalized inverted exponential lifetime distribution under progressive first failure censoring, *J. Stat. Comput. Sim.*, **93**(2023), 863–887.
[11] S. Saini, S. Tomer, R. Garg, Inference of multicomponent stress-strength reliability following Topp-Leone distribution using progressively censored data, *J. Appl. Stat.*, **50**(2023), 1538–1567.
[12] R. Kumari, Y.M. Tripathi, R.K. Sinha, L. Wang, Reliability estimation for the inverted exponentiated Pareto distribution, *Qual. Technol. Quant. M.*, **20**(2023), 485–510
[13] K. Singh, A.K. Mahto, Y. Tripathi, L. Wang, Inference for reliability in a multicomponent stress-strength model for a unit inverse Weibull distribution under type-II censoring, *Qual. Technol. Quant. M.*, **21**(2024), 147–176.
[14] A. Kohansal, C.J. Pérez-González, A.J. Fernández, Inference on the stress-strength reliability of multi-component systems based on progressive first failure censored samples, *P. I. Mech. Eng. O-J Ris.*, (2023) DOI: 10.1177/1748006X231188075.
[15] F.R.S. Gusmão, E.M.M. Ortega, G.M. Cordeiro, The generalized inverse Weibull distribution, *Stat. Pap.*, **52**(2011), 591–619.
[16] Chen MH, Shao, QM, Monte Carlo estimation of Bayesian Credible and HPD intervals, *J. Comput. Graph. Stat.*, **8**(1999), 69–92.
[17] D.V. Lindley, Approximate Bayesian methods, *Trabajos. de. Estadistica.*, **3**(1980), 281–288.
[18] M.G. Badar, A.M. Priest, Statistical aspects of fiber and bundle strength in hybrid composites. In: Hayashi T, Kawata K, Umekawa S, editors. Progress in Science and Engineering Composites, Tokyo: ICCM-IV; (1982), 1129–1136.
[19] A. Kohansal, S. Rezakhah, Inference of $R = P(Y < X)$ for two-parameter Rayleigh distribution based on progressively, *Statistics.*, **53**(2019), 81–100.

© 2025 YU
Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.