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Modified Conformable Self Adjoint Equation and Sturm Liouville Problems with Applications

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Abstract: Within this research, we introduce a new modified conformable operator and study its properties in detail. Furthermore, we investigate the self-adjoint modified conformable equation and its connection to modified conformable initial value problems. Additionally, we analyze the modified conformable Sturm–Liouville problem, determining its eigenvalues and corresponding eigenfunctions, while establishing results on orthogonality and dependence. Finally, illustrative examples are provided to demonstrate the applicability of our findings.

Keywords: Modified conformable derivative; Modified conformable integral; Modified conformable Sturm Liouville problem; Modified conformable self adjoint equation; Orthogonality; Dependence.

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1 Introduction

Fractional derivative concepts date back to the early development of calculus. In 1695, L'Hôpital raised the question about the meaning of $\frac{d^n f}{dt^n}$ when $n = \frac{1}{2}$, initiating centuries of mathematical exploration. Since then, various definitions of fractional derivatives have been proposed, including the Riemann–Liouville fractional derivative [1], the Atangana–Baleanu fractional derivative [2], the Caputo–Fabrizio fractional derivative [3], and the Caputo fractional derivative [1]. The reader may refer to [4,5,6,7,8,9,10,11,12,13] for a broader overview of the definitions, theoretical foundations, and applications of fractional calculus.

Most existing definitions attempt to generalize differentiation, but only a few preserve fundamental properties from classical calculus. While linearity is commonly retained, many fractional derivatives fail to satisfy essential rules such as the product rule and chain rule. To address these limitations, the authors in [14] introduced a new local and well-structured derivative known as the *conformable fractional derivative*.

Definition 1([14]). *Assume that $f : [0,\infty) \to \mathbb{R}$ is a given function. The conformable fractional derivative of f of order $\alpha \in (0,1)$ at a point $x > 0$ is defined by*

$$T_\alpha(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

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If the limit exists, we denote it by $f^{\alpha}(x)$. That is,

$$f^{\alpha}(x) = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

Moreover, if f is α-differentiable on an interval $(0,a)$ and the following limit exists,

$$f^{(\alpha)}(0) = \lim_{x\to 0^+} f^{\alpha}(x),$$

then $f^{(\alpha)}(0)$ is defined accordingly.

Essentially, the aforementioned definition adheres to the basic concept of a limit and retains most of the key properties of the classical derivative. Several authors [15,16,17,18,19] have further refined this derivative and generalized many of its useful and elegant properties. Since the geometric interpretation of the conformable derivative via the concept of fractional cords was introduced by Khalil et al. in [20], the derivative has gained significant attention and has been employed in solving various problems. For additional information and applications of the conformable fractional derivative, we refer the reader to [21,22,23,24,25,26]. Sturm–Liouville problems play a central role in applied mathematics due to their frequent appearance in the study of separable linear partial differential equations. More details on this topic can be found in [27,28,29,30,31].

Recent work by Al-Refai et al. [32] has explored fundamental properties of conformable Sturm–Liouville eigenvalue problems. Additionally, Anderson has studied self-adjoint differential equations involving the proportional-derivative (conformable) operator, focusing on both second-order and even-order boundary value problems [33]. Building upon these contributions, this study introduces a modified conformable operator—also referred to as the proportional derivative—and extends the analysis of self-adjoint structures within the context of Sturm–Liouville problems. In this work, the term "self-adjoint" is used in the classical sense, referring to the structural form of the modified conformable Sturm–Liouville equation. Specifically, we consider equations that can be expressed in the standard self-adjoint differential form, rather than pursuing a functional-analytic framework involving an explicit inner product. This approach is consistent with traditional Sturm–Liouville theory and preserves important properties such as the reality of eigenvalues and orthogonality of eigenfunctions in the conformable setting.

In their work [34], Al-Refai et al. established foundational results for conformable Sturm–Liouville eigenvalue problems, including the existence and properties of eigenvalues and eigenfunctions. Building upon these results, the present study introduces a modified conformable operator to further investigate the self-adjoint structure of Sturm–Liouville problems. This modification enables a deeper analysis of orthogonality conditions and the completeness of eigenfunctions within the conformable framework, thereby extending the theoretical foundation established by Al-Refai et al. [34].

Recent studies [35,36] have introduced nonlocal fractional derivatives, particularly in the form of generalized proportional fractional derivatives, which broaden the applicability of fractional calculus in the context of differential equations. While the present work focuses on the modified conformable derivative, investigating its nonlocal counterpart could offer further insights into more generalized forms of conformable Sturm–Liouville problems. In this paper, we consider a newly defined conformable operator introduced in [37]. This modified conformable operator has been studied in detail. The main objective of this work is to investigate both the homogeneous and nonhomogeneous forms of the self-adjoint modified conformable differential equation:

$$Lu(x) = 0 \quad \text{and} \quad Lu(x) = f(x),$$

where

$$Lu(x) = D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha,x)u(x)\right)\right] + q(x)u(x),$$

with $\alpha \in [0,1]$, D^{α} denoting the modified conformable derivative operator, and $p(x)$, $q(x)$ being continuous functions on $[x_0,\infty)$ such that $p(x) \neq 0$ for all $x \in [x_0,\infty)$. In particular, we examine and analyze the following modified conformable Sturm–Liouville differential equation:

$$D^{\alpha}\left[p(x)\left(D^{\alpha}u(x) - \kappa_1(\alpha,x)u(x)\right)\right] + (\lambda r(x) + q(x))u(x) = 0,$$

where p, q, and r are real-valued continuous functions on $[x_0,\infty)$, with $p(x) \neq 0$ and $r(x) \geq 0$ not identically zero on $[x_0,\infty)$. This equation can be equivalently written in the form

$$Lu(x) = -\lambda r(x)u(x).$$

Finally, as an application, several illustrative examples are provided.

2 Basics of the Modified Conformable Operator

In this section, we examine the modified conformable derivative operator D^α of order α. This operator reduces to the identity operator and the classical differential operator when $\alpha = 0$ and $\alpha = 1$, respectively.

Definition 2(Modified Conformable Differential Operator [37]). *Let $0 < \alpha \leq 1$. The operator D^α is called the modified conformable derivative if it satisfies the following:*

$$D^0 f(x) = f(x) \quad \text{and} \quad D^1 f(x) = \frac{d}{dx} f(x) = f'(x), \quad x \in \mathbb{R}, \tag{1}$$

for any differentiable function $f(x)$. Thus, D^0 and D^1 correspond to the identity and classical differential operators, respectively.

Definition 3(A Class of Modified Conformable Derivatives [37]). *Let $0 < \alpha \leq 1$, and assume that the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous and satisfy the following conditions:*

$$\begin{array}{ccc} \lim\limits_{\alpha \to 0^+} k_1(\alpha, x) = 1, & \lim\limits_{\alpha \to 0^+} k_0(\alpha, x) = 0, & \forall x \in \mathbb{R}, \\ \lim\limits_{\alpha \to 1^-} k_1(\alpha, x) = 0, & \lim\limits_{\alpha \to 1^-} k_0(\alpha, x) = 1, & \forall x \in \mathbb{R}, \\ k_1(\alpha, x) \neq 0,\ \alpha \in [0,1), & k_0(\alpha, x) \neq 0,\ \alpha \in (0,1], & \forall x \in \mathbb{R}. \end{array} \tag{2}$$

Then, the differential operator D^α defined by

$$D^\alpha f(x) = k_1(\alpha, x) f(x) + k_0(\alpha, x) f'(x), \tag{3}$$

is called a modified conformable derivative, provided that $f(x)$ is differentiable and $f'(x) = \frac{d}{dx} f(x)$.

*Example 1.***Examples of Modified Conformable Derivatives**

(1)Let $k_1(\alpha, x) = (1-\alpha)x^\alpha$ and $k_0(\alpha, x) = \alpha x^{1-\alpha}$ for any $x \in (0,\infty)$. Then the operator

$$D^\alpha f(x) = k_1(\alpha, x) f(x) + k_0(\alpha, x) f'(x)$$

defines a valid modified conformable derivative.

(2)A different form of a modified conformable derivative, belonging to a related class, is given by

$$D^\alpha f(x) = \cos\left(\frac{\alpha\pi}{2}\right) x^\alpha f(x) + \sin\left(\frac{\alpha\pi}{2}\right) x^{1-\alpha} f'(x), \quad x \in (0,\infty).$$

Note. In general, the modified conformable derivatives do not commute; that is,

$$D^\alpha D^\beta \neq D^\beta D^\alpha.$$

Definition 4(Partial Conformable Derivatives [37]). *Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be two continuous functions satisfying the conditions in (2). Suppose that $f(x,s) : \mathbb{R}^2 \to \mathbb{R}$ is such that $\frac{\partial}{\partial x} f(x,s)$ exists for each fixed $s \in \mathbb{R}$. Then, the partial modified conformable derivative of f with respect to x, denoted by $D_x^\alpha f$, is defined by*

$$D_x^\alpha f(x,s) = k_1(\alpha, x) f(x,s) + k_0(\alpha, x) \frac{\partial}{\partial x} f(x,s), \tag{4}$$

for all $0 < \alpha \leq 1$.

Definition 5(Modified Conformable Exponential Function [37]). *Suppose $s, x \in \mathbb{R}$ with $s \leq x$, and let $m : [s,x] \to \mathbb{R}$ be a continuous function. Also, assume that $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions satisfying the conditions in (2), and that the functions m/k_0 and k_1/k_0 are Riemann integrable on $[s,x]$. Then, with respect to the modified conformable derivative D^α, the modified conformable exponential function is defined by*

$$e_m(x,s) = \exp\left(\int_x^s \frac{m(\lambda) - k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda\right), \quad e_0(x,s) = \exp\left(\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda\right), \tag{5}$$

for $0 < \alpha \leq 1$.

Based on the previous discussion, together with (3) and (5), the following elementary results can be derived.

Lemma 1(Elementary Results on Modified Conformable Derivatives [37]). *Let (3) define the modified conformable differential operator D^α, where $0<\alpha\leq 1$. Assume that the functions $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ and $m:[s,x]\to\mathbb{R}$ are continuous and satisfy the conditions in (2), with m/k_0 and k_1/k_0 being Riemann integrable on $[s,x]$. Suppose f and g are differentiable wherever necessary. Then, the following properties hold:*

(i)Linearity:

$$D^\alpha[af+bg]=aD^\alpha[f]+bD^\alpha[g],\quad \forall a,b\in\mathbb{R}.$$

(ii)Derivative of a constant:

$$D^\alpha[c]=c\,k_1(\alpha,x),\quad \textit{for any } c\in\mathbb{R},\ x\in\mathbb{R}.$$

(iii)Product rule:

$$D^\alpha[fg]=f\,D^\alpha[g]+g\,D^\alpha[f]-fg\,k_1(\alpha,x),\quad x\in\mathbb{R}.$$

(iv)Exponential rule:

$$D_x^\alpha[e_m(x,s)]=m(x)\,e_m(x,s),\quad \textit{for fixed } s\in\mathbb{R},\ \alpha\in(0,1]. \tag{6}$$

(v)Quotient rule:

$$D^\alpha\left[\frac{f}{g}\right]=\frac{g\,D^\alpha[f]-f\,D^\alpha[g]}{g^2}+\frac{f}{g}\,k_1(\alpha,x),\quad x\in\mathbb{R},\ g(x)\neq 0.$$

(vi)Inverse operation of integration with exponential kernel:

$$D^\alpha\left[\int_a^x\frac{f(s)\,e_0(x,s)}{k_0(\alpha,s)}\,ds\right]=f(x),\quad \alpha\in(0,1]. \tag{7}$$

*Proof.*See [37] for a detailed derivation.

Definition 6(Modified Conformable Integral [37]). *Let $0<\alpha\leq 1$ and $x_0\in\mathbb{R}$. In view of (5) and Lemma 2.6 (v)–(vi), the following antiderivative is defined as*

$$\int D^\alpha f(x)\,d_\alpha x=f(x)+c\,e_0(x,x_0),\quad c\in\mathbb{R}.$$

Similarly, the integral of a function f over a closed interval $[a,b]$ is defined by

$$\int_a^b f(s)\,e_0(x,s)\,d_\alpha s=\int_a^b\frac{f(s)\,e_0(x,s)}{k_0(\alpha,s)}\,ds,\qquad d_\alpha s=\frac{ds}{k_0(\alpha,s)}. \tag{8}$$

Therefore, the exponential function $e_0(x,s)$ can be expressed as

$$e_0(x,s)=\exp\left(\int_x^s\frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}\,d\lambda\right)=\exp\left(\int_x^s k_1(\alpha,\lambda)\,d_\alpha\lambda\right).$$

Lemma 2(Elementary Results on Modified Conformable Integrals [37]). *Suppose D^α is the modified conformable differential operator with $0<\alpha\leq 1$, and let the integral be defined as in (8). Assume that f and g are differentiable as needed, and that $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ are continuous functions satisfying the conditions in (2). Then the following properties hold:*

(i)The derivative of the definite integral of f is given by

$$D^\alpha\left[\int_a^x f(s)\,e_0(x,s)\,d_\alpha s\right]=f(x).$$

(ii)The definite integral of the derivative of f satisfies

$$\int_a^x D^\alpha\left[f(s)\,e_0(x,s)\right]d_\alpha s=f(s)\,e_0(x,s)\big|_{s=a}^x=f(x)-f(a)\,e_0(x,a).$$

(iii)The integration by parts formula is given by

$$\int_a^b f(x)\,D^\alpha[g(x)]\,e_0(b,x)\,d_\alpha x=f(x)g(x)\,e_0(b,x)\big|_{x=a}^b-\int_a^b g(x)\left(D^\alpha[f(x)]-k_1(\alpha,x)f(x)\right)e_0(b,x)\,d_\alpha x.$$

(iv)The Leibniz rule for differentiating under the integral sign is given by

$$D^{\alpha}\left[\int_a^x f(x,s)\,e_0(x,s)\,d_{\alpha}s\right]=\int_a^x\left(D_x^{\alpha}[f(x,s)]-k_1(\alpha,x)f(x,s)\right)e_0(x,s)\,d_{\alpha}s+f(x,x).$$

If the exponential kernel $e_0(x,s)$ is absent, then using (4), we obtain

$$D^{\alpha}\left[\int_a^x f(x,s)\,d_{\alpha}s\right]=\int_a^x D_x^{\alpha}f(x,s)\,d_{\alpha}s+f(x,x).$$

*Proof.*See [37].

3 Self-Adjoint and Sturm–Liouville Modified Conformable Equations

In the first part of this section, we consider the self-adjoint modified conformable equation

$$Lu(x)=0,$$

where

$$Lu(x)=D^{\alpha}\left[p(x)\left(D^{\alpha}u(x)-\kappa_1(\alpha,x)u(x)\right)\right]+q(x)u(x), \tag{9}$$

with D^{α} defined as in (3), $\alpha\in[0,1]$, and $p(x),q(x)$ being continuous functions on $[x_0,\infty)$ such that $p(x)\neq 0$ for all $x\in[x_0,\infty)$.

A function $u(x):[x_0,\infty)\to\mathbb{R}$ is called a solution to (9) if $D^{\alpha}u(x)$ and

$$D^{\alpha}\left[p(x)\left(D^{\alpha}u(x)-\kappa_1(\alpha,x)u(x)\right)\right]$$

are both continuous on $[x_0,\infty)$, and $Lu(x)=0$ holds for all $x\in[x_0,\infty)$.

In the next result, we establish the existence and uniqueness of solutions to the modified conformable initial value problem (MCIVP) associated with the nonhomogeneous self-adjoint equation

$$Lu(x)=f(x).$$

Theorem 1.*Let $\alpha\in[0,1]$ and $u_0,u_1\in\mathbb{R}$, and let D^{α} be defined as in (3). Assume that the functions k_0,k_1 satisfy the conditions in (2). Suppose that p,q,f are continuous functions on $[x_0,\infty)$, with $p(x)\neq 0$ for all $x\in[x_0,\infty)$. Then the modified conformable initial value problem (MCIVP)*

$$Lu(x)=f(x),\quad u(x_0)=u_0,\quad D^{\alpha}u(x_0)=u_1$$

has a unique solution on $[x_0,\infty)$.

*Proof.*We begin by reformulating the equation $Lu(x)=f(x)$ into an equivalent first-order vector system. Let

$$y(x)=p(x)\left(D^{\alpha}u(x)-k_1(\alpha,x)u(x)\right).$$

Then, if $u(x)$ is a solution to $Lu(x)=f(x)$, it follows that

$$D^{\alpha}u(x)=k_1(\alpha,x)u(x)+\frac{y(x)}{p(x)}.$$

Since $Lu(x)=f(x)$ and L is defined as in (9), we also have

$$D^{\alpha}y(x)=-q(x)u(x)+f(x).$$

Now define the vector function

$$z(x)=\begin{bmatrix}u(x)\\ y(x)\end{bmatrix}.$$

Then $z(x)$ satisfies the following modified conformable vector equation:

$$D^{\alpha}z(x)=A(x)z(x)+b(x),$$

where

$$A(x) = \begin{bmatrix} k_1(\alpha,x) & \frac{1}{p(x)} \\ -q(x) & 0 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.$$

By the definition of D^α in (3), we can express this equation as:

$$k_1(\alpha,x)z(x) + k_0(\alpha,x)z'(x) = A(x)z(x) + b(x),$$

which leads to

$$k_0(\alpha,x)z'(x) = [A(x) - k_1(\alpha,x)I]\, z(x) + b(x).$$

Thus,

$$z'(x) = \begin{bmatrix} 0 & \frac{1}{k_0(\alpha,x)p(x)} \\ \frac{-q(x)}{k_0(\alpha,x)} & \frac{-k_1(\alpha,x)}{k_0(\alpha,x)} \end{bmatrix} z(x) + \begin{bmatrix} 0 \\ \frac{f(x)}{k_0(\alpha,x)} \end{bmatrix}.$$

Since all the involved functions are continuous on $[x_0,\infty)$, the existence and uniqueness of the solution follow from standard theory for first-order systems of differential equations (as in the classical case when $\alpha = 1$). Hence, the modified conformable initial value problem has a unique solution.

Corollary 1. *Let $u_1(x)$ and $u_2(x)$ be two solutions of (9). Then:*

$$W(u_1,u_2)(x) = 0 \;\; \forall x \in [x_0,\infty) \quad \Longleftrightarrow \quad u_1(x), u_2(x) \textit{ are linearly dependent on } [x_0,\infty),$$

and

$$W(u_1,u_2)(x) \neq 0 \;\; \forall x \in [x_0,\infty) \quad \Longleftrightarrow \quad u_1(x), u_2(x) \textit{ are linearly independent on } [x_0,\infty).$$

Proof. By Abel's formula, we have

$$W(u_1,u_2)(x) = \frac{c\, e_0(x,x_0)}{p(x)}, \quad \forall x \in [x_0,\infty).$$

If $u_1(x)$ and $u_2(x)$ are linearly dependent, then clearly

$$W(u_1,u_2)(x) = 0, \quad \forall x \in [x_0,\infty).$$

Conversely, assume

$$W(u_1,u_2)(x) = 0, \quad \forall x \in [x_0,\infty).$$

Then,

$$\begin{aligned} W(u_1,u_2)(x) &= u_1(x)D^\alpha[u_2(x)] - u_2(x)D^\alpha[u_1(x)] \\ &= u_1(x)\left(k_0(\alpha,x)u_2'(x) + k_1(\alpha,x)u_2(x)\right) \\ &\quad - u_2(x)\left(k_0(\alpha,x)u_1'(x) + k_1(\alpha,x)u_1(x)\right) \\ &= k_0(\alpha,x)\left(u_1(x)u_2'(x) - u_2(x)u_1'(x)\right) = 0. \end{aligned}$$

Since $k_0(\alpha,x) \neq 0$, we conclude that

$$u_1(x)u_2'(x) - u_2(x)u_1'(x) = 0,$$

which implies that $u_1(x)$ and $u_2(x)$ are linearly dependent on $[x_0,\infty)$.

In the second part of this section, we consider the following Sturm–Liouville modified conformable differential equation:

$$D^\alpha\left[p(x)\left(D^\alpha u(x) - \kappa_1(\alpha,x)u(x)\right)\right] + (\lambda r(x) + q(x))u(x) = 0, \tag{10}$$

where $p(x), q(x)$, and $r(x)$ are real-valued continuous functions on $[x_0,\infty)$, with

$$p(x) \neq 0, \quad x \in [x_0,\infty), \qquad r(x) \geq 0 \text{ and not identically zero on } [x_0,\infty).$$

Equation (10) can be equivalently written in the operator form:

$$Lu(x) = -\lambda r(x)u(x),$$

where the linear operator L is defined by

$$Lu(x) = D^\alpha\left[p(x)\left(D^\alpha u(x) - \kappa_1(\alpha,x)u(x)\right)\right] + q(x)u(x). \tag{11}$$

Definition 7. *Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ be constants satisfying*

$$a_1^2 + a_2^2 > 0, \qquad b_1^2 + b_2^2 > 0.$$

Then the boundary value problem defined by

$$\begin{aligned} &Lu(x) = -\lambda r(x) u(x), \\ &a_1 u(a) + a_2 D^\alpha u(a) = 0, \\ &b_1 u(b) + b_2 D^\alpha u(b) = 0, \end{aligned} \tag{12}$$

is called a modified conformable Sturm–Liouville problem*, where L is the operator defined previously and D^α is the modified conformable derivative.*

Definition 8. *A number λ is called an* eigenvalue *of the modified conformable Sturm–Liouville problem (12) if the problem admits a nontrivial solution $u(x)$, which is referred to as an* eigenfunction *corresponding to the eigenvalue λ.*

Theorem 2. *Let λ be an eigenvalue of the modified conformable Sturm–Liouville problem (12). Then $\lambda \in \mathbb{R}$; that is, all eigenvalues of (12) are real.*

Proof. Let λ be an eigenvalue of (12), and let $\varphi(x)$ be the corresponding eigenfunction. Then,

$$L\varphi(x) = -\lambda r(x) \varphi(x). \tag{13}$$

Taking the complex conjugate of both sides and multiplying by $e_0(x,b)\varphi(x)$, we obtain

$$L\overline{\varphi(x)} \cdot e_0(x,b)\varphi(x) = -\overline{\lambda}\, r(x) \cdot \overline{\varphi(x)} e_0(x,b)\varphi(x). \tag{14}$$

This simplifies to

$$L\overline{\varphi(x)} \cdot e_0(x,b)\varphi(x) = -\overline{\lambda}\, r(x)\, e_0(x,b)|\varphi(x)|^2. \tag{15}$$

Integrating both sides of (15) over the interval $[a,b]$, we get

$$\int_a^b L\overline{\varphi(x)} \cdot e_0(x,b)\varphi(x)\, d_\alpha x = -\overline{\lambda} \int_a^b r(x)\, e_0(x,b)|\varphi(x)|^2\, d_\alpha x.$$

Similarly, multiplying (13) by $e_0(x,b)\overline{\varphi(x)}$ and integrating, we find

$$\int_a^b L\varphi(x) \cdot e_0(x,b)\overline{\varphi(x)}\, d_\alpha x = -\lambda \int_a^b r(x)\, e_0(x,b)|\varphi(x)|^2\, d_\alpha x.$$

Since the operator L is self-adjoint, we have

$$\int_a^b L\varphi(x) \cdot e_0(x,b)\overline{\varphi(x)}\, d_\alpha x = \int_a^b L\overline{\varphi(x)} \cdot e_0(x,b)\varphi(x)\, d_\alpha x.$$

Therefore, by equating the two expressions, we obtain

$$-\lambda \int_a^b r(x)\, e_0(x,b)|\varphi(x)|^2\, d_\alpha x = -\overline{\lambda} \int_a^b r(x)\, e_0(x,b)|\varphi(x)|^2\, d_\alpha x.$$

Since the weight function $r(x) \geq 0$, not identically zero, and $|\varphi(x)|^2 > 0$ for a nontrivial eigenfunction, we conclude that

$$\lambda = \overline{\lambda},$$

which shows that $\lambda \in \mathbb{R}$. This completes the proof.

Theorem 3. *All eigenvalues of the modified conformable Sturm–Liouville problem (12) are simple. Equivalently, if λ is an eigenvalue of (12) and $\varphi_1(x), \varphi_2(x)$ are two corresponding eigenfunctions, then $\varphi_1(x)$ and $\varphi_2(x)$ are linearly dependent.*

Proof. Applying the boundary conditions in (12) to $\varphi_1(x)$ and $\varphi_2(x)$, we have:

$$a_1\varphi_1(a)+a_2D^\alpha\varphi_1(a)=0,$$
$$a_1\varphi_2(a)+a_2D^\alpha\varphi_2(a)=0.$$

Solving each for the modified conformable derivatives, we get:

$$D^\alpha\varphi_1(a)=-\frac{a_1}{a_2}\varphi_1(a), \qquad D^\alpha\varphi_2(a)=-\frac{a_1}{a_2}\varphi_2(a).$$

Now, we compute the modified conformable Wronskian at $x=a$:

$$\begin{aligned}W(\varphi_1,\varphi_2)(a)&=\varphi_1(a)D^\alpha\varphi_2(a)-\varphi_2(a)D^\alpha\varphi_1(a)\\&=\varphi_1(a)\left(-\frac{a_1}{a_2}\varphi_2(a)\right)-\varphi_2(a)\left(-\frac{a_1}{a_2}\varphi_1(a)\right)\\&=-\frac{a_1}{a_2}\varphi_1(a)\varphi_2(a)+\frac{a_1}{a_2}\varphi_2(a)\varphi_1(a)\\&=0.\end{aligned}$$

By Corollary 7, the vanishing Wronskian on an interval implies that the eigenfunctions are linearly dependent. Therefore, all eigenvalues of (12) are simple.

Theorem 4. *Eigenfunctions corresponding to distinct eigenvalues of the modified conformable Sturm–Liouville problem* (12) *are orthogonal.*

Proof. Let λ_1,λ_2 be two distinct eigenvalues of (12) with corresponding eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$, respectively. Then, by (12), we have

$$L\varphi_1(x)+\lambda_1r(x)\varphi_1(x)=0, \tag{16}$$
$$L\varphi_2(x)+\lambda_2r(x)\varphi_2(x)=0. \tag{17}$$

Multiply (16) by $e_0(x,b)\varphi_2(x)$, and (17) by $e_0(x,b)\varphi_1(x)$, to obtain:

$$L\varphi_1(x)\cdot e_0(x,b)\varphi_2(x)+\lambda_1r(x)\varphi_1(x)\cdot e_0(x,b)\varphi_2(x)=0, \tag{18}$$
$$L\varphi_2(x)\cdot e_0(x,b)\varphi_1(x)+\lambda_2r(x)\varphi_2(x)\cdot e_0(x,b)\varphi_1(x)=0. \tag{19}$$

Subtracting (18) from (19) and integrating over the interval $[a,b]$, we obtain:

$$\int_a^b(L\varphi_2(x)\cdot e_0(x,b)\varphi_1(x)-L\varphi_1(x)\cdot e_0(x,b)\varphi_2(x))\,d_\alpha x=(\lambda_1-\lambda_2)\int_a^b r(x)\varphi_1(x)\varphi_2(x)\cdot e_0(x,b)\,d_\alpha x.$$

Since L is self-adjoint, the left-hand side is zero. Therefore,

$$(\lambda_1-\lambda_2)\int_a^b r(x)\varphi_1(x)\varphi_2(x)\cdot e_0(x,b)\,d_\alpha x=0.$$

Given that $\lambda_1\neq\lambda_2$, it follows that

$$\int_a^b r(x)\varphi_1(x)\varphi_2(x)\cdot e_0(x,b)\,d_\alpha x=0.$$

This proves that the eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ are orthogonal with respect to the weight function $r(x)e_0(x,b)$ on $[a,b]$.

4 Applications

In this section, we demonstrate the applicability of the theoretical results developed in the previous sections by applying them to the following illustrative examples.

*Example 2.*Suppose that $k_0(\alpha,x), k_1(\alpha,x) : [0,1]\times\mathbb{R}\to[0,\infty)$ are continuous functions satisfying the conditions in (2), and assume that $k_1(\alpha,x)$ is differentiable on $[0,\infty)$. Let $\alpha\in[0,1]$, and define:

$$p(x) = e_0(x,0), \qquad q(x) = p(x) + D^{\alpha}k_1(\alpha,x)\cdot p(x), \quad x\in[0,\infty).$$

Substituting these into (9) yields the following modified conformable initial value problem (MCIVP):

$$D^{2\alpha}u(x) + u(x) = 0, \qquad u(0) = 2, \quad D^{\alpha}u(0) = 5.$$

Since p and q are continuous and $p(x)\neq 0$ on $[0,\infty)$, Theorem 3.1 guarantees the existence and uniqueness of the solution. The associated characteristic equation is:

$$\lambda^2 + 1 = 0,$$

with complex roots $\lambda = \pm i$. Therefore, the general solution is given by:

$$u(x) = c_1 e_0(x,0)\cos\left(\int_0^x 1\,d_\alpha s\right) + c_2 e_0(x,0)\sin\left(\int_0^x 1\,d_\alpha s\right),$$

where $c_1, c_2\in\mathbb{R}$ are constants determined by the initial conditions. Using the first condition $u(0)=2$ and the identity $e_0(0,0)=1$, we find:

$$u(0) = c_1\cdot 1\cdot\cos(0) + c_2\cdot 1\cdot\sin(0) = c_1\cdot 1 + c_2\cdot 0 = c_1 \Rightarrow c_1 = 2.$$

Now, applying the second condition $D^{\alpha}u(0)=5$, we compute:

$$\begin{aligned} D^{\alpha}u(0) &= D^{\alpha}\left[c_1 e_0(x,0)\cos\left(\int_0^x 1\,d_\alpha s\right) + c_2 e_0(x,0)\sin\left(\int_0^x 1\,d_\alpha s\right)\right]\Big|_{x=0} \\ &= c_1\left[k_1(\alpha,0) + 1 - k_1(\alpha,0)\right] + c_2\cdot 1 = c_1 + c_2. \end{aligned}$$

Substituting $c_1 = 2$, we get:

$$D^{\alpha}u(0) = 2 + c_2 = 5 \quad\Rightarrow\quad c_2 = 3.$$

Therefore, the unique solution of the modified conformable differential equation is:

$$u(x) = 2e_0(x,0)\cos\left(\int_0^x 1\,d_\alpha s\right) + 3e_0(x,0)\sin\left(\int_0^x 1\,d_\alpha s\right).$$

*Example 3.*Let $\kappa_0(\alpha,x), \kappa_1(\alpha,x) : [0,1]\times\mathbb{R}\to[0,\infty)$ be continuous functions satisfying the conditions in (2), and assume that $\kappa_1(\alpha,x)\equiv$ const. We aim to find the eigenvalues and corresponding eigenfunctions of the following modified conformable Sturm–Liouville problem:

$$D^{2\alpha}u(x) - 4lD^{\alpha}u(x) + \lambda u(x) = 0, \qquad u(0) = u(2l) = 0.$$

In this example, the parameter $\alpha\in(0,1]$ determines the order of the conformable derivative. The results are analyzed for different values of α to observe their effect on the eigenvalues and eigenfunctions. As $\alpha\to 1$, the solutions tend toward the classical Sturm–Liouville case. For smaller values of α, deviations from the classical case become evident, reflecting fractional-like behavior in the spectrum and shape of eigenfunctions. To solve the above eigenvalue problem, we first consider the corresponding characteristic equation:

$$m^2 - 4lm + \lambda = 0,$$

which has the roots

$$m = 2l \pm \sqrt{4l^2 - \lambda}.$$

Case 1: If $4l^2 - \lambda > 0$, then the characteristic equation has two distinct real roots:

$$m = 2l \pm \sqrt{4l^2 - \lambda}.$$

According to the constant coefficients modified conformable equation theorem, the general solution is given by:

$$u(x) = c_1\, e_{2l+\sqrt{4l^2-\lambda}}(x,0) + c_2\, e_{2l-\sqrt{4l^2-\lambda}}(x,0),$$

where c_1 and c_2 are arbitrary constants. Applying the boundary condition $u(0) = 0$, we obtain:

$$\begin{aligned} u(0) &= c_1 e_{2l+\sqrt{4l^2-\lambda}}(0,0) + c_2 e_{2l-\sqrt{4l^2-\lambda}}(0,0) \\ &= c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1. \end{aligned}$$

Now, applying the second boundary condition $u(2l) = 0$ gives

$$u(2l) = c_1 \left[e_{2l+\sqrt{4l^2-\lambda}}(2l,0) - e_{2l-\sqrt{4l^2-\lambda}}(2l,0) \right].$$

Since the two exponential functions are linearly independent, we have:

$$e_{2l+\sqrt{4l^2-\lambda}}(2l,0) \neq e_{2l-\sqrt{4l^2-\lambda}}(2l,0),$$

so the expression in brackets is nonzero. Hence, $u(2l) = 0$ implies $c_1 = 0$, and consequently $c_2 = 0$. The only solution in this case is the trivial solution $u(x) \equiv 0$. Therefore, when $\lambda < 4l^2$, there are no nontrivial solutions, and such values of λ are not eigenvalues of the problem.

Case 2: If $4l^2 - \lambda = 0$, then the characteristic equation has a real repeated root $m = 2l$. According to the constant coefficients modified conformable equation theorem, the general solution takes the form:

$$u(x) = c_1 e_{2l}(x,0) + c_2 e_{2l}(x,0) \int_0^x 1\, d_\alpha s,$$

where c_1 and c_2 are constants. Applying the boundary condition $u(0) = 0$ yields

$$\begin{aligned} & u(0) = c_1 e_{2l}(0,0) + c_2 e_{2l}(0,0) \int_0^0 1\, d_\alpha s = c_1 \cdot 1 + c_2 \cdot 0 = c_1, \\ \Rightarrow \quad & c_1 = 0. \end{aligned}$$

Again, applying the second boundary condition $u(2l) = 0$ yields

$$u(2l) = c_2 e_{2l}(2l,0) \int_0^{2l} 1\, d_\alpha s.$$

Since both $e_{2l}(2l,0) \neq 0$ and $\int_0^{2l} 1\, d_\alpha s \neq 0$, it follows that:

$$u(2l) = 0 \quad \Rightarrow \quad c_2 = 0.$$

The only solution is the trivial one $u(x) \equiv 0$. Therefore, the value $\lambda = 4l^2$ is not an eigenvalue of the problem.

Case 3: If $4l^2 - \lambda < 0$, then the characteristic equation has complex conjugate roots:

$$m = 2l \pm i\sqrt{\lambda - 4l^2}.$$

According to the constant coefficients modified conformable equation theorem, the general solution is:

$$u(x) = c_1 e_{2l}(x,0) \cos\left(\int_0^x \sqrt{\lambda - 4l^2}\, d_\alpha s \right) + c_2 e_{2l}(x,0) \sin\left(\int_0^x \sqrt{\lambda - 4l^2}\, d_\alpha s \right),$$

where c_1, c_2 are constants. Applying the boundary condition $u(0) = 0$ gives

$$\begin{aligned} & u(0) = c_1 e_{2l}(0,0) \cdot \cos(0) + c_2 e_{2l}(0,0) \cdot \sin(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1, \\ \Rightarrow \quad & c_1 = 0. \end{aligned}$$

Now, applying the second boundary condition $u(2l) = 0$ gives

$$u(2l) = c_2 e_{2l}(2l,0) \cdot \sin\left(\int_0^{2l} \sqrt{\lambda - 4l^2}\, d_\alpha s \right).$$

To obtain a nontrivial solution, we require:

$$\sin\left(\int_0^{2l}\sqrt{\lambda-4l^2}\,d_\alpha s\right)=0 \quad\Rightarrow\quad \int_0^{2l}\sqrt{\lambda-4l^2}\,d_\alpha s=n\pi,\quad n\in\mathbb{N}.$$

Solving for λ, we get the eigenvalues:

$$\lambda_n=4l^2+\left(\frac{n\pi}{\int_0^{2l}1\,d_\alpha s}\right)^2,\quad n\in\mathbb{N}.$$

The eigenfunctions corresponding to the eigenvalues λ_n are given by:

$$u_n(x)=e_{2l}(x,0)\cdot\sin\left(\frac{n\pi}{\int_0^{2l}1\,d_\alpha s}\int_0^x 1\,d_\alpha s\right),\quad n\in\mathbb{N}.$$

*Remark.*The above analysis could be extended to sequential conformable problems, where multiple conformable derivatives of different orders appear in the equation.

*Example 4.*Let $\kappa_0(\alpha,x),\kappa_1(\alpha,x):[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous functions satisfying the conditions in (2), and assume that $\kappa_1(\alpha,x)\equiv$ const. We aim to find the eigenvalues and corresponding eigenfunctions of the following modified conformable Sturm–Liouville problem:

$$D^\alpha D^\alpha u(x)-\lambda u(x)=0,\qquad 0<x<L,$$

subject to the boundary conditions:

$$u(0)=u(L)=0.$$

To solve the above problem, we first consider the associated characteristic equation:

$$r^2-\lambda=0,$$

which has the roots:

$$r=\pm\sqrt{\lambda}.$$

Case 1: If $\lambda>0$, then the characteristic equation has two distinct real roots:

$$r=\pm\sqrt{\lambda}.$$

According to the constant coefficients modified conformable equation theorem, the general solution is given by:

$$u(x)=c_1\,e_{+\sqrt{\lambda}}(x,0)+c_2\,e_{-\sqrt{\lambda}}(x,0),$$

where c_1 and c_2 are arbitrary constants. Applying the boundary condition $u(0)=0$, we get:

$$u(0)=c_1\,e_{+\sqrt{\lambda}}(0,0)+c_2\,e_{-\sqrt{\lambda}}(0,0)=c_1+c_2=0,$$
$$\Rightarrow\quad c_2=-c_1.$$

Now, applying the second boundary condition $u(L)=0$ gives

$$u(L)=c_1\left(e_{+\sqrt{\lambda}}(L,0)-e_{-\sqrt{\lambda}}(L,0)\right).$$

Since the two exponential-type functions are linearly independent, we have:

$$e_{+\sqrt{\lambda}}(L,0)\neq e_{-\sqrt{\lambda}}(L,0),$$

and thus the expression in parentheses is nonzero. Therefore,

$$u(L)=0\quad\Rightarrow\quad c_1=0\quad\Rightarrow\quad c_2=0.$$

The only solution in this case is the trivial one, $u(x)\equiv 0$. Hence, $\lambda>0$ does not yield any eigenvalues for the problem.

Case 2: If $\lambda = 0$, then the characteristic equation has a repeated real root:

$$r = 0.$$

By the constant coefficients modified conformable equation theorem, the general solution is given by:

$$u(x) = c_1\, e_0(x,0) + c_2\, e_0(x,0)\int_0^x 1\, d_\alpha s,$$

where c_1 and c_2 are constants. Applying the first boundary condition $u(0) = 0$, we have:

$$\begin{aligned} u(0) &= c_1\, e_0(0,0) + c_2\, e_0(0,0)\int_0^0 1\, d_\alpha s = c_1 \cdot 1 + c_2 \cdot 0 = c_1, \\ \Rightarrow \quad c_1 &= 0. \end{aligned}$$

Now, applying the second boundary condition $u(L) = 0$ yields

$$u(L) = c_2\, e_0(L,0)\int_0^L 1\, d_\alpha s.$$

Since $e_0(L,0) \neq 0$ and $\int_0^L 1\, d_\alpha s \neq 0$, we obtain:

$$c_2 = 0.$$

The only solution in this case is the trivial solution $u(x) \equiv 0$. Therefore, $\lambda = 0$ is not an eigenvalue of the problem.

Case 3: If $\lambda < 0$, then the characteristic equation has complex roots:

$$r = \pm i\sqrt{|\lambda|}.$$

By the constant coefficients modified conformable equation theorem, the general solution is:

$$u(x) = c_1\, e_0(x,0)\cos\left(\int_0^x \sqrt{|\lambda|}\, d_\alpha s\right) + c_2\, e_0(x,0)\sin\left(\int_0^x \sqrt{|\lambda|}\, d_\alpha s\right),$$

where c_1 and c_2 are arbitrary constants. Applying the first boundary condition $u(0) = 0$, we get:

$$\begin{aligned} u(0) &= c_1\, e_0(0,0)\cos\left(\int_0^0 \sqrt{|\lambda|}\, d_\alpha s\right) + c_2\, e_0(0,0)\sin\left(\int_0^0 \sqrt{|\lambda|}\, d_\alpha s\right) \\ &= c_1 \cdot 1 + c_2 \cdot 0 = c_1, \\ \Rightarrow \quad c_1 &= 0. \end{aligned}$$

Now, applying the second boundary condition $u(L) = 0$ gives

$$u(L) = c_2\, e_0(L,0)\sin\left(\int_0^L \sqrt{|\lambda|}\, d_\alpha s\right) = 0.$$

If $c_2 \neq 0$, then we must have:

$$\sin\left(\int_0^L \sqrt{|\lambda|}\, d_\alpha s\right) = 0.$$

Thus, we have

$$\int_0^L \sqrt{|\lambda|}\, d_\alpha s = n\pi, \quad n \in \mathbb{N}.$$

Solving for λ, we obtain the eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{\int_0^L 1\, d_\alpha s}\right)^2, \quad n \in \mathbb{N}.$$

The eigenvalues of the problem are λ_n, and the corresponding eigenfunctions are given by:

$$u_n(x) = e_0(x,0)\sin\left(\frac{n\pi \int_0^x 1\, d_\alpha s}{\int_0^L 1\, d_\alpha s}\right), \quad n \in \mathbb{N}.$$

5 Conclusion and Future Work

After analyzing and studying the modified conformable self-adjoint equation and associated Sturm–Liouville problems, we conclude our work with the following key classifications and observations:

– The modified conformable Sturm–Liouville equation

$$D^{\alpha}\left[p(x)\left(D^{\alpha}u(x)-\kappa_1(\alpha,x)u(x)\right)\right]+q(x)u(x)+\lambda r(x)u(x)=0$$

is a second-order linear homogeneous modified conformable differential equation.

– The corresponding modified conformable Sturm–Liouville operator is defined by:

$$L=D^{\alpha}\left[p(x)\left(D^{\alpha}-\kappa_1(\alpha,x)\right)\right]+q(x).$$

– The operator L is said to be self-adjoint under the modified conformable integral if:

$$\int_a^b f(x)L[g(x)]\,d_\alpha x=\int_a^b g(x)L[f(x)]\,d_\alpha x.$$

– The modified conformable Sturm–Liouville equation is called *regular* if:

$$p(x)>0 \quad \text{and} \quad q(x)>0 \quad \text{for all} \quad x\in[a,b].$$

– A modified conformable Sturm–Liouville system is called *periodic* if its boundary conditions satisfy:

$$u(a)=u(b) \quad \text{and} \quad D^{\alpha}u(a)=D^{\alpha}u(b).$$

– The system is referred to as *singular* if:

$$p(x)>0 \text{ on } (a,b), \quad r(x)\geq 0 \text{ on } [a,b], \quad \text{and} \quad p(a)=p(b)=0.$$

As a future work, we will study and analyze in detail the mentioned classifications. In addition, we will study the ability of the Rayleigh quotient for getting estimates of eigenvalues. Also, the eigenfunction expansion method and the Fredholm alternative theorem via the modified conformable operator. In the same regard, future work may extend our analysis to nonlocal fractional operators, as in [35,36], to explore their effects on eigenvalues, orthogonality, and spectral completeness in Sturm–Liouville problems. This could provide deeper insights into boundary value problems with memory effects.

Declarations

Competing interests: The authors declare that they have no competing interests.
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