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Geometry of Paracontact Manifolds Admitting Conformal Ricci-Yamabe Solitons

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Abstract: This study investigates the classification of conformal Ricci-Yamabe solitons within the framework of paracontact geometry. In particular, we analyze the structural properties of para-Kenmotsu manifolds that satisfy the conditions for conformal Ricci-Yamabe solitons, with special attention to three-dimensional cases exhibiting conformal gradient Ricci-Yamabe solitons. In addition, we provide a comprehensive classification of para-Sasakian and para-cosymplectic manifolds that admit conformal Ricci-Yamabe solitons and its gradient form conformal gradient Ricci-Yamabe solitons. To substantiate the theoretical findings, an explicit example is constructed and discussed in detail.

Keywords: Paracontact manifolds; self-similar solution; conformal Ricci-Yamabe solitons; Einstein manifolds.

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1 Introduction

The study of geometric structures in Riemannian geometry often involves the analysis of geometric flows, which serve as a central tool in understanding the evolution of metrics. Among these, the Ricci flow, introduced by Hamilton [17], is particularly prominent. Hamilton utilized the Ricci flow to establish significant results concerning three-dimensional spheres [16]. This concept has also played a vital role in the proof of Thurston's geometrization conjecture, notably contributing to the resolution of the Poincar*é* conjecture. A Ricci soliton (denoted $\mathscr{RS}$) on a Riemannian manifold (M,g) represents a self-similar solution to the Ricci flow and is characterized by the equation:

$$\pounds_Z g + 2Ric_g = 2\lambda g, \tag{1}$$

where $\pounds_Z g$ denotes the Lie derivative of the metric g with respect to the vector field Z, Ric_g is the Ricci curvature tensor, and λ is a real constant. When $Z = \nabla f$ for some smooth function f, the soliton is called a gradient Ricci soliton. Petersen and Wylie [22] defined a gradient $\mathscr{RS}$ as rigid, if it corresponds to a warped product of a flat space with an Einstein manifold. They also provided a classification for such rigid solitons. The concept of an almost Ricci soliton where λ is allowed to vary smoothly was introduced by Pigola et al. [23], and further rigidity results were established in [2,3,31], showing that such solitons are isometric to either Euclidean space $\mathbb{R}^n$ or a standard sphere.

To address the Yamabe problem for manifolds with positive conformal Yamabe invariant, Hamilton proposed the Yamabe flow. A self-similar solution under this flow is known as a Yamabe soliton ($\mathscr{YS}$), defined by:

$$\pounds_Z g = 2(\tau - \lambda)g, \tag{2}$$

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where τ is the scalar curvature of the manifold. While Ricci and Yamabe solitons exhibit analogous behavior in two dimensions, they differ in higher dimensions. Specifically, $\mathscr{YS}$ preserves the conformal class of the metric, unlike $\mathscr{RS}$. If λ is a smooth function, the soliton is termed an almost Yamabe soliton. Various authors have investigated such solitons, including Alkhaldi et al. [1] and Barbosa and Ribeiro [4]. In the paracontact setting, De and De studied characterizations of almost quasi-Yamabe solitons and their gradient versions [8].

In subsequent work, Güler and Crasmareanu [15] introduced a new geometric flow known as the Ricci-Yamabe flow, defined as a linear combination of the Ricci and Yamabe flows. Referred to as the (α,β)-type Ricci-Yamabe flow, it is governed by the equation:

$$\pounds_Z g + (2\lambda - \beta\tau)g + 2\alpha Ric_g = 0, \tag{3}$$

where $\alpha,\beta \in \mathbb{R}$, and Ric_g, τ, and λ are as previously defined. If $Z = \nabla f$ for a smooth function f, the resulting solution is known as a gradient Ricci-Yamabe soliton, and equation (3) reduces to:

$$2Hess f + (2\lambda - \beta\tau)g + 2\alpha Ric_g = 0, \tag{4}$$

where $Hess(f)$ is the Hessian of f. As a generalization of Ricci solitons and conformal Ricci solitons, Zhang et al. [33] introduced the concept of conformal Ricci-Yamabe solitons (abbreviated $\mathscr{C}$-$\mathscr{RYS}$), defined as follows:

Definition 1. *A Riemannian manifold (M^n, g), $n > 2$, admits a conformal Ricci-Yamabe soliton if it satisfies*

$$\pounds_Z g + 2\alpha Ric_g + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{n}\right)\right] g = 0. \tag{5}$$

When $Z = \nabla f$, the soliton is called a conformal gradient Ricci-Yamabe soliton ($\mathscr{C}$-$\mathscr{GRYS}$), and equation (5) simplifies to:

$$2\text{Hess} f + 2\alpha Ric_g + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{n}\right)\right] g = 0. \tag{6}$$

Depending on the sign of λ, $\mathscr{C}$-$\mathscr{RYS}$ are classified as expanding ($\lambda > 0$), steady ($\lambda = 0$), or shrinking ($\lambda < 0$). Recent investigations have explored various properties and classifications of both $\mathscr{C}$-$\mathscr{RYS}$ and $\mathscr{C}$-$\mathscr{GRYS}$ (see [19, 27,28,33,34]). Although Ricci and Yamabe solitons have been widely studied within the framework of various classes of Riemannian geometry (see [6,8],[12]-[14],[18,20,21],[24]-[26],[29,30] and the references therein), their conformal counterparts-particularly the conformal Ricci-Yamabe soliton ($\mathscr{C}$-$\mathscr{RYS}$) and its gradient form ($\mathscr{C}$-$\mathscr{GRYS}$)-remain relatively underexplored. This gap is especially evident in the setting of paracontact geometry. Motivated by this observation, the present work aims to investigate the interaction of these generalized solitons with specific paracontact structures, including para-Kenmotsu, para-Sasakian, and para-cosymplectic manifolds.
The structure of the paper is as follows: Sections 2, 4, and 6 provides the foundational concepts related to para-Kenmotsu, para-Sasakian, and para-cosymplectic manifolds, respectively. In Section 3, we examine the existence of $\mathscr{C}$-$\mathscr{RYS}$ and $\mathscr{C}$-$\mathscr{GRYS}$ on para-Kenmotsu manifolds. Section 5 is dedicated to the classification of para-Sasakian manifolds admitting such solitons. In Section 7, we investigate their existence on para-cosymplectic manifolds. Finally, an illustrative example is presented to demonstrate the applicability of the theoretical results.

2 Para-Kenmotsu Manifolds

A differentiable manifold M^{2p+1} of dimension $2p+1$ is termed an almost paracontact manifold if there exist ϕ, ξ, η on M^{2p+1}. Here, ϕ is a $(1,1)$ tensor field, ξ is a characteristic vector field, and η is a global 1-form satisfying the conditions:

$$\phi^2 E_1 = E_1 - \eta(E_1)\xi, \; \eta(\xi) = 1. \tag{7}$$

This definition implies that $\phi\xi = 0$, $\eta \circ \phi = 0$, and the rank of ϕ is $2p$. If the Nijenhuis tensor uniformly vanishes, the manifold is termed normal. Additionally, M^{2p+1} is designated as an almost paracontact metric manifold if there exists a semi-Riemannian metric g such that:

$$g(\phi E_1, \phi E_2) = -g(E_1, E_2) + \eta(E_1)\eta(E_2), \tag{8}$$

for all $E_1, E_2 \in \chi(M^{2p+1})$, where $\chi(M^{2p+1})$ denotes the vector fields on M^{2p+1}. Furthermore, an almost paracontact metric manifold M^{2p+1} with a structure (ϕ, ξ, η, g) is termed a paracontact metric manifold if $d\eta(E_1, E_2) = g(E_1, \phi E_2) =$

$\Phi(E_1,E_2)$, where Φ is the fundamental 2-form of M^{2p+1}.
A para-Kenmotsu manifold satisfies [10]

$$(\nabla_{E_1}\phi)(E_2) = g(\phi E_1,E_2)\xi - \eta(E_2)\phi E_1, \quad (9)$$

$$\nabla_{E_1}\xi = E_1 - \eta(E_1)\xi, \quad (10)$$

$$R(E_1,E_2)\xi = \eta(E_1)E_2 - \eta(E_2)E_1, \quad (11)$$

$$R(\xi,E_1)E_2 = -g(E_1,E_2)\xi + \eta(E_2)E_1, \quad (12)$$

$$R(E_1,\xi)E_2 = g(E_1,E_2)\xi - \eta(E_2)E_1, \quad (13)$$

$$\eta(R(E_1,E_2)E_3) = -g(E_2,E_3)\eta(E_1) + g(E_1,E_3)\eta(E_2), \quad (14)$$

$$Ric_g(E_1,\xi) = -2p\eta(E_1). \quad (15)$$

First, we recall the following result, which we use in the proof of our main results.

Lemma 1.*[10] In a three-dimensional para-Kenmotsu manifold M^3,*

$$\xi\tau = -2(\tau+6). \quad (16)$$

In M^3, we also have

$$QE_1 = \left(\frac{\tau}{2}+1\right)E_1 - \left(\frac{\tau}{2}+3\right)\eta(E_1)\xi, \quad (17)$$

which gives

$$Ric_g(E_1,E_2) = \left(\frac{\tau}{2}+1\right)g(E_1,E_2) - \left(\frac{\tau}{2}+3\right)\eta(E_1)\eta(E_2), \quad (18)$$

where Q denotes the Ricci operator defined by $Ric_g(E_1,E_2) = g(QE_1,E_2)$.

3 $\mathscr{C}-\mathscr{RYS}$ on Para-Kenmotsu manifolds

Throughout this section, we denote a $(2p+1)$-dimensional para-Kenmotsu manifold by M_{PK}^{2p+1}. Let us consider a M_{PK}^{2p+1} admits $\mathscr{C}-\mathscr{RYS}$. Then from (5), we have

$$\pounds_Z g(E_1,E_2) + 2\alpha Ric_g(E_1,E_2) + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1}\right)\right]g(E_1,E_2) = 0, \quad (19)$$

which yields

$$g(\nabla_{E_1}\xi,E_2) + g(E_1,\nabla_{E_2}\xi) + 2\alpha Ric_g(E_1,E_2) + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1}\right)\right]g(E_1,E_2) = 0. \quad (20)$$

Utilizing the expression (10) in the preceding equation yields

$$\alpha Ric_g(E_1,E_2) = \eta(E_1)\eta(E_2) - \frac{1}{2}\left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1} + 2\right)\right]g(E_1,E_2). \quad (21)$$

By setting $E_1 = E_2 = \xi$ in the preceding equation, one can determine $\frac{\beta}{2}\tau = -2p\alpha + \lambda - \frac{\left(\pi + \frac{2}{2p+1} + 2\right)}{2}$. Consequently, equation (21) assumes the following expression

$$\alpha Ric_g(E_1,E_2) = \eta(E_1)\eta(E_2) - 2p\alpha g(E_1,E_2).$$

Thus, we can state the following:

Theorem 1.*If M_{PK}^{2p+1} admits a $\mathscr{C}-\mathscr{RYS}$, then it is an η-Einstein manifold.*

Consider, $\{e_i\}_{1\le i\le 2p+1}$ be an orthonormal frame. By taking $E_1 = e_i$, $E_2 = e_i$ in (19) and summing over i, one can obtain

$$divZ = -\tau\left(\alpha - \frac{(2p+1)\beta}{2}\right) - (2p+1)\lambda + \frac{(2p+1)\pi}{2} + 1. \quad (22)$$

Assume, for a smooth function ψ, if Z is of gradient type i.e. $Z = grad(\psi)$, then the equation (22) becomes

$$\Delta(\psi) = -\tau\left(\alpha - \frac{(2p+1)\beta}{2}\right) - (2p+1)\lambda + \frac{(2p+1)\pi}{2} + 1, \quad (23)$$

where $\Delta(\psi)$ is the Laplacian equation satisfied by ψ. Hence, we can state the following:

Theorem 2. *If $(g, Z, \lambda, \alpha, \beta)$ is a $\mathscr{C}-\mathscr{RYS}$ on a $(2p+1)$-dimensional paracontact manifold M^{2p+1} where $Z = grad(\psi)$, then the Laplacian equation satisfied by ψ is given by (23).*

Next, we prove the following Lemma:

Lemma 2. *If $(g, Z, \lambda, \alpha, \beta)$ is a $\mathscr{C}-\mathscr{GRYS}$ on a M^{2p+1}_{PK}, then the Riemannian curvature tensor R satisfies*

$$R(E_1,E_2)Df = -\alpha[(\nabla_{E_1}Q)E_2 - (\nabla_{E_2}Q)E_1] - E_1(\lambda)E_2 + E_2(\lambda)E_1 + \frac{\beta}{2}[E_1(\tau)E_2 - E_2(\tau)E_1]. \tag{24}$$

Proof. Let us consider, M^{2p+1}_{PK} admit a $\mathscr{C}-\mathscr{GRYS}$. Then, equation (6) infers

$$\nabla_{E_1}Df = -\alpha QE_1 - \left(\lambda - \frac{\beta}{2}\tau - \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\right)E_1. \tag{25}$$

Taking covariant differentiation of (25) we obtain

$$\begin{aligned}\nabla_{E_2}\nabla_{E_1}Df =& -\alpha[(\nabla_{E_2}Q)E_1 + Q(\nabla_{E_2}E_1)] - E_2(\lambda)E_1 - \lambda(\nabla_{E_2}E_1)\\ &+\frac{\beta}{2}\tau\nabla_{E_2}E_1 + \frac{\beta}{2}E_2(\tau)E_1 + \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\nabla_{E_2}E_1.\end{aligned} \tag{26}$$

Swapping E_1 and E_2 in (25) implies

$$\begin{aligned}\nabla_{E_1}\nabla_{E_2}Df =& -\alpha[(\nabla_{E_1}Q)E_2 + Q(\nabla_{E_1}E_2)] - E_1(\lambda)E_2 - \lambda(\nabla_{E_1}E_2)\\ &+\frac{\beta}{2}\tau\nabla_{E_1}E_2 + \frac{\beta}{2}E_1(\tau)E_2 + \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\nabla_{E_1}E_2.\end{aligned} \tag{27}$$

Substituting equations (19), (26) and (27) in the definition of Riemannian curvature, we obtain (24).

Now differentiating (17) covariantly with respect to E_2, we have

$$(\nabla_{E_2}Q)E_1 = \frac{E_2(\tau)}{2}[E_1 - \eta(E_1)\xi] - \left(\frac{\tau}{2}+3\right)[g(E_1,E_2) - 2\eta(E_1)\eta(E_2)\xi + \eta(E_1)E_2]. \tag{28}$$

Utilizing (28) in (27), we obtain

$$\begin{aligned}R(E_1,E_2)Df =& -\alpha\left\{\frac{E_1(\tau)}{2}[E_2 - \eta(E_2)\xi] - \frac{E_2(\tau)}{2}[E_1 - \eta(E_1)\xi] - \left(\frac{\tau}{2}+3\right)[\eta(E_2)E_1 - \eta(E_1)E_2]\right\}\\ &-E_1(\lambda)E_2 + E_2(\lambda)E_1 + \frac{\beta}{2}[E_1(\tau)E_2 - E_2(\tau)E_1].\end{aligned} \tag{29}$$

Contracting (29), we have

$$Ric_g(E_2, Df) = \left(\frac{\alpha}{2} - \beta\right)E_2(\tau) + 2E_2(\lambda) - (\alpha+1)(\tau+6)\eta(Y). \tag{30}$$

Substituting E_1 by Df in (18) and comparing with (30), one can easily obtain

$$\left(\frac{\tau}{2}+3\right)\xi(f)\eta(E_2) - \left(\frac{\tau}{2}+1\right)E_2(f) = \left(\frac{\alpha}{2} - \beta\right)E_2(\tau) + 2E_2(\lambda) - (\alpha+1)(\tau+6)\eta(Y). \tag{31}$$

Taking ξ in place of E_2 in the foregoing equation, we have

$$\xi(f) = \left(\alpha - \beta + \frac{1}{2}\right)(\tau+6) - \xi(\lambda). \tag{32}$$

Applying inner product to (29) with ξ yields

$$\eta(E_2)E_1(f) - \eta(E_1)E_2(f) = -E_1(\lambda)\eta(E_2) + E_2(\lambda)\eta(E_1) + \frac{\beta}{2}[E_1(\tau)\eta(E_2) - E_2(\tau)\eta(E_1)]. \tag{33}$$

Plugging $E_2 = \xi$ in (33) and utilizing (32), we obtain

$$E_1(f) = \left(\alpha + \frac{1}{2}\right)(\tau+6)\eta(E_1) + \frac{\beta}{2}E_1(\tau) - E_1(\lambda). \tag{34}$$

Suppose the scalar curvature τ is constant. Then, using (16), we find $\tau = -6$. Consequently, the preceding equation implies

$$E_1(f) = -E_1(\lambda), \tag{35}$$

this means

$$Df = -D\lambda. \tag{36}$$

Utilizing (36) in (25) gives

$$-\nabla_{E_1} D\lambda = -\alpha Q E_1 - \left(\lambda - \frac{\beta}{2}\tau - \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\right) E_1. \tag{37}$$

This shows that M^3_{PK} is a $\mathscr{C}-\mathscr{GRYS}$, whose soliton function is $-\lambda$. Thus, we have

Theorem 3. *A M^3_{PK} with constant scalar curvature admits $\mathscr{C}-\mathscr{GRYS}$ whose soliton function is $-\lambda$.*

4 Para-Sasakian Manifolds

A para-Sasakian manifold is defined as a normal paracontact metric manifold. It's important to emphasize that a para-Sasakian manifold is a subset of general paracontact metric manifolds. Additionally, it should be noted that in three dimensions, a para-Sasakian manifold is equivalent to a k-paracontact manifold and vice versa [5]. In a $(2p+1)$-dimensional para-Sasakian manifold, the following conditions are satisfied [32]:

$$(\nabla_{E_1}\phi)(E_2) = -g(\phi E_1, E_2)\xi + \eta(E_2)\phi E_1, \tag{38}$$
$$\nabla_{E_1}\xi = -\phi E_1, \tag{39}$$
$$R(E_1,E_2)\xi = \eta(E_1)E_2 - \eta(E_2)E_1, \tag{40}$$
$$R(\xi,E_1)E_2 = -g(E_1,E_2)\xi + \eta(E_2)E_1, \tag{41}$$
$$\eta(R(E_1,E_2)E_3) = -(g(E_2,E_3)\eta(E_1) - g(E_1,E_3)\eta(E_2)), \tag{42}$$
$$Ric_g(E_1,\xi) = -2p\eta(E_1). \tag{43}$$

First, we recall the following result, which we use in the proof of our main results.

Lemma 3. *[10] In a three-dimensional para-Sasakian manifold M^3,*

$$\xi\tau = 0. \tag{44}$$

In M^3, we also have

$$QE_1 = \left(\frac{\tau}{2}+1\right)E_1 - \left(\frac{\tau}{2}+3\right)\eta(E_1)\xi, \tag{45}$$

which provides

$$Ric_g(E_1,E_2) = \left(\frac{\tau}{2}+1\right)g(E_1,E_2) - \left(\frac{\tau}{2}+3\right)\eta(E_1)\eta(E_2). \tag{46}$$

5 $\mathscr{C}-\mathscr{RYS}$ on Para-Sasakian manifolds

Throughout this section, we denote a $(2p+1)$-dimensional para-Sasakian manifold by M^{2p+1}_{PS}. Let us consider a M^{2p+1}_{PS} admits $\mathscr{C}-\mathscr{RYS}$. Then from (5), we have

$$\pounds_Z g(E_1,E_2) + 2\alpha Ric_g(E_1,E_2) + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1}\right)\right] g(E_1,E_2) = 0, \tag{47}$$

which yields

$$g(\nabla_{E_1}\xi, E_2) + g(E_1, \nabla_{E_2}\xi) + 2\alpha Ric_g(E_1,E_2) + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1}\right)\right] g(E_1,E_2) = 0. \tag{48}$$

Making use of (39) in the foregoing equation, one can easily obtain

$$\alpha Ric_g(E_1,E_2) = -\frac{1}{2}\left[2\lambda - \beta\tau - \left(\pi + \frac{2}{2p+1}\right)\right] g(E_1,E_2). \tag{49}$$

Setting $E_1 = E_2 = \xi$ in the above equation yields $\frac{\beta}{2}\tau = -(2p)\alpha + \lambda - \frac{\left(\pi + \frac{2}{2p+1}\right)}{2}$. Consequently, equation (49) assumes the following expression

$$Ric_g(E_1, E_2) = -2pg(E_1, E_2).$$

Thus, we can state the following:

Theorem 4. *If M_{PS}^{2p+1} admits a proper $\mathscr{C} - \mathscr{RYS}$, then it is an Einstein manifold.*

Next, we prove the following Lemma:

Lemma 4. *If $(g, Z, \lambda, \alpha, \beta)$ is a $\mathscr{C} - \mathscr{GRYS}$ on a M_{PS}^{2p+1}, then the Riemannian curvature tensor R satisfies*

$$R(E_1, E_2)Df = -\alpha[(\nabla_{E_1} Q)E_2 - (\nabla_{E_2} Q)E_1] - E_1(\lambda)E_2 + E_2(\lambda)E_1 + \frac{\beta}{2}[E_1(\tau)E_2 - E_2(\tau)E_1]. \tag{50}$$

Proof. Let us consider, M_{PS}^{2p+1} admit a $\mathscr{C} - \mathscr{GRYS}$. Then, equation (6) infers

$$\nabla_{E_1} Df = -\alpha QE_1 - \left(\lambda - \frac{\beta}{2}\tau - \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\right)E_1. \tag{51}$$

Taking covariant differentiation of (51) we obtain

$$\begin{aligned}\nabla_{E_2}\nabla_{E_1} Df = & -\alpha[(\nabla_{E_2} Q)E_1 + Q(\nabla_{E_2} E_1)] - E_2(\lambda)E_1 - \lambda(\nabla_{E_2} E_1) \\ & + \frac{\beta}{2}\tau\nabla_{E_2} E_1 + \frac{\beta}{2}E_2(\tau)E_1 + \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\nabla_{E_2} E_1.\end{aligned} \tag{52}$$

Swapping E_1 and E_2 in (51) implies

$$\begin{aligned}\nabla_{E_1}\nabla_{E_2} Df = & -\alpha[(\nabla_{E_1} Q)E_2 + Q(\nabla_{E_1} E_2)] - E_1(\lambda)E_2 - \lambda(\nabla_{E_1} E_2) \\ & + \frac{\beta}{2}\tau\nabla_{E_1} E_2 + \frac{\beta}{2}E_1(\tau)E_2 + \frac{1}{2}\left(\pi + \frac{2}{2p+1}\right)\nabla_{E_1} E_2.\end{aligned} \tag{53}$$

Substituting equations (47), (52) and (53) in the definition of Riemannian curvature, we obtain (50).

Now differentiating (45) covariantly with respect to E_2, we have

$$(\nabla_{E_2} Q)E_1 = \frac{E_2(\tau)}{2}[E_1 - \eta(E_1)\xi] + \left(\frac{\tau}{2} + 3\right)[g(E_1, \phi E_2)\xi + \eta(E_1)\phi E_2]. \tag{54}$$

Utilizing (54) in (50), we obtain

$$\begin{aligned}R(E_1, E_2)Df = & -\alpha\left\{\frac{E_1(\tau)}{2}[E_2 - \eta(E_2)\xi] - \frac{E_2(\tau)}{2}[E_1 - \eta(E_1)\xi] + \left(\frac{\tau}{2} + 3\right)[-2g(E_1, \phi E_2)\xi)\right. \\ & \left. + \eta(E_2)\phi E_1 - \eta(E_1)\phi E_2]\right\} - E_1(\lambda)E_2 + E_2(\lambda)E_1 + \frac{\beta}{2}[E_1(\tau)E_2 - E_2(\tau)E_1].\end{aligned} \tag{55}$$

Contracting (55), we have

$$Ric_g(E_2, Df) = \left(\frac{\alpha}{2} - \beta\right)E_2(\tau) + 2E_2(\lambda). \tag{56}$$

Substituting E_1 by Df in (46) and comparing with (56), one can easily obtain

$$\left(\frac{\tau}{2} + 1\right)E_2(f) - \left(\frac{\tau}{2} + 3\right)\xi(f)\eta(E_2) = \left(\frac{\alpha}{2} - \beta\right)E_2(\tau) + 2E_2(\lambda). \tag{57}$$

Replacing E_2 by ξ in (57) gives

$$\xi(f) = \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\xi(\tau) + \xi(\lambda). \tag{58}$$

Applying inner product to (55) with ξ yields

$$\begin{aligned}\eta(E_2)E_1(f) - \eta(E_1)E_2(f) = & \; 2\alpha\left(\frac{\tau}{2} + 3\right)g(E_1, \phi E_2) - E_1(\lambda)\eta(E_2) + E_2(\lambda)\eta(E_1) \\ & + \frac{\beta}{2}[E_1(\tau)\eta(E_2) - E_2(\tau)\eta(E_1)].\end{aligned} \tag{59}$$

By replacing E_1 and E_2 with ϕE_1 and ϕE_2 respectively, we find that $2\alpha\left(\frac{\tau}{2}+3\right)g(\phi E_1,E_2)=0$. Given that $\alpha\neq 0$ for proper $\mathscr{C}-\mathscr{GRYS}$, the equation mentioned above indicates that $\tau=-6$. Consequently, by utilizing equation (46), we obtain

$$Ric_g(E_1,E_2)=-2g(E_1,E_2). \tag{60}$$

This implies that M^3_{PS} is an Einstein manifold. Considering equation (60) in the definition of curvature tensor R (ref equation (45) of [9]), we obtain

$$R(E_1,E_2)E_3=g(E_1,E_3)E_2-g(E_2,E_3)E_1.$$

This signifies that it is a space characterized by a constant sectional curvature of -1. Thus, we can state the following:

Theorem 5. *A M^3_{PS} admitting $\mathscr{C}-\mathscr{GRYS}$ is Einstein and it is locally isometric to hyperbolic space $H^3(-1)$.*

6 Almost para-cosymplectic manifolds

An almost paracontact metric manifold $M^{2p+1}(\phi,\xi,\eta,g)$ is said to be an almost α-paracosymplectic manifold, if

$$d\eta=0,\ d\Phi=2\alpha\eta\wedge\Phi. \tag{61}$$

Particularly, if $\alpha=0$, then we obtain an almost paracosymplectic manifold. For more details we refer to [7,11]. In a $(2p+1)$-dimensional almost paracosymplectic manifold the following conditions hold:

$$(\nabla_{E_1}\phi)(E_2)=0, \tag{62}$$

$$\nabla_{E_1}\xi=0, \tag{63}$$

$$R(E_1,E_2)\xi=0, \tag{64}$$

$$\eta(R(E_1,E_2)E_3)=-(g(E_2,E_3)\eta(E_1)-g(E_1,E_3)\eta(E_2)), \tag{65}$$

$$Ric_g(E_1,\xi)=0. \tag{66}$$

First, we recall the following result, which we use in the proof of our main results.

Lemma 5. *[10] In a three-dimensional paracosymplectic manifold M^3,*

$$\xi\tau=0. \tag{67}$$

In M^3, we also have

$$QE_1=\frac{\tau}{2}[E_1-\eta(E_1)\xi], \tag{68}$$

which provides

$$Ric_g(E_1,E_2)=\frac{\tau}{2}[g(E_1,E_2)-\eta(E_1)\eta(E_2)]. \tag{69}$$

7 $\mathscr{C}-\mathscr{RYS}$ on almost paracosymplectic manifolds

Throughout this section, we denote a $(2p+1)$-dimensional almost paracosymplectic manifold by M^{2p+1}_{APC}. Let us consider a M^{2p+1}_{APC} admits $\mathscr{C}-\mathscr{RYS}$. Then from (5), we have

$$\pounds_Z g(E_1,E_2)+2\alpha Ric_g(E_1,E_2)+\left[2\lambda-\beta\tau-\left(\pi+\frac{2}{2p+1}\right)\right]g(E_1,E_2)=0, \tag{70}$$

which yields

$$g(\nabla_{E_1}\xi,E_2)+g(E_1,\nabla_{E_2}\xi)+2\alpha Ric_g(E_1,E_2)+\left[2\lambda-\beta\tau-\left(\pi+\frac{2}{2p+1}\right)\right]g(E_1,E_2)=0. \tag{71}$$

Utilizing the expression (63) in the preceding equation yields

$$\alpha Ric_g(E_1,E_2)=-\frac{1}{2}\left[2\lambda-\beta\tau-\left(\pi+\frac{2}{2p+1}\right)\right]g(E_1,E_2).$$

Thus, we can state the following:

Theorem 6. *If M^{2p+1}_{APC} admits a proper $\mathscr{C}-\mathscr{RYS}$, then it is an Einstein manifold.*

Next, we prove the following Lemma:

Lemma 6. *If $(g,Z,\lambda,\alpha,\beta)$ is a $\mathscr{C}-\mathscr{GRYS}$ on a M^{2p+1}_{APC}, then the Riemannian curvature tensor R satisfies*

$$R(E_1,E_2)Df = -\alpha[(\nabla_{E_1}Q)E_2-(\nabla_{E_2}Q)E_1]-E_1(\lambda)E_2+E_2(\lambda)E_1+\frac{\beta}{2}[E_1(\tau)E_2-E_2(\tau)E_1]. \tag{72}$$

Proof. Let us consider, M^{2p+1}_{APC} admit a $\mathscr{C}-\mathscr{GRYS}$. Then, equation (6) infers

$$\nabla_{E_1}Df = -\alpha QE_1-\left(\lambda-\frac{\beta}{2}\tau-\frac{1}{2}\left(\pi+\frac{2}{2p+1}\right)\right)E_1. \tag{73}$$

Taking covariant differentiation of (73) we obtain

$$\begin{aligned}\nabla_{E_2}\nabla_{E_1}Df &= -\alpha[(\nabla_{E_2}Q)E_1+Q(\nabla_{E_2}E_1)]-E_2(\lambda)E_1-\lambda(\nabla_{E_2}E_1)\\ &+\frac{\beta}{2}\tau\nabla_{E_2}E_1+\frac{\beta}{2}E_2(\tau)E_1+\frac{1}{2}\left(\pi+\frac{2}{2p+1}\right)\nabla_{E_2}E_1.\end{aligned} \tag{74}$$

Swapping E_1 and E_2 in (73) implies

$$\begin{aligned}\nabla_{E_1}\nabla_{E_2}Df &= -\alpha[(\nabla_{E_1}Q)E_2+Q(\nabla_{E_1}E_2)]-E_1(\lambda)E_2-\lambda(\nabla_{E_1}E_2)\\ &+\frac{\beta}{2}\tau\nabla_{E_1}E_2+\frac{\beta}{2}E_1(\tau)E_2+\frac{1}{2}\left(\pi+\frac{2}{2p+1}\right)\nabla_{E_1}E_2.\end{aligned} \tag{75}$$

Substituting equations (70), (74) and (75) in the definition of Riemannian curvature, we obtain (72).

Now differentiating (68) covariantly with respect to E_2, we have

$$(\nabla_{E_2}Q)E_1 = \frac{E_2(\tau)}{2}[E_1-\eta(E_1)\xi]. \tag{76}$$

Utilizing (76) in (72), we obtain

$$\begin{aligned}R(E_1,E_2)Df &= -\alpha\left\{\frac{E_1(\tau)}{2}[E_2-\eta(E_2)\xi]-\frac{E_2(\tau)}{2}[E_1-\eta(E_1)\xi]\right\}\\ &-E_1(\lambda)E_2+E_2(\lambda)E_1+\frac{\beta}{2}[E_1(\tau)E_2-E_2(\tau)E_1].\end{aligned} \tag{77}$$

Contracting (77), we have

$$Ric_g(E_2,Df) = \left(\frac{\alpha}{2}-\beta\right)E_2(\tau)+2E_2(\lambda). \tag{78}$$

Substituting E_2 by Df in (69) and comparing with (78), one can easily obtain

$$\frac{\tau}{2}[E_2(f)-\xi(f)\eta(E_2)] = \left(\frac{\alpha}{2}-\beta\right)E_2(\tau)+2E_2(\lambda). \tag{79}$$

Replacing E_2 by ξ in (79) gives

$$\xi(\lambda)=0. \tag{80}$$

Applying inner product to (77) with ξ yields

$$-E_1(\lambda)\eta(E_2)+E_2(\lambda)\eta(E_1)+\frac{\beta}{2}[E_1(\tau)\eta(E_2)-E_2(\tau)\eta(E_1)]=0. \tag{81}$$

Plugging $E_2=\xi$ in (81) and utilizing (80), we obtain

$$E_1(\lambda)=0. \tag{82}$$

Which infers that λ is a constant. Thus, we can state the following:

Theorem 7. *If M^3_{APC} admits a $\mathscr{C}-\mathscr{AGRYS}$, then the soliton becomes a $\mathscr{C}-\mathscr{GRYS}$.*

8 Example

We consider $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates of R^3.
Let us consider three linearly independent vector fields

$$u_1 = e^x \frac{\partial}{\partial y}, \;\; u_2 = e^x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right), \;\; u_3 = -\frac{\partial}{\partial x}.$$

Then

$$[u_1,u_2] = 0, \;\; [u_2,u_3] = u_2, \;\; [u_1,u_3] = u_1.$$

Let g be the Riemannian metric defined by $g(u_i,u_j) = \begin{cases} 1, \text{ if i=j;} \\ 0, \text{ otherwise.} \end{cases}$
Let $\xi = u_3$ and η be the 1-form defined by $\eta(E_1) = g(E_1,u_3)$ for any $E_1 \in \chi(M)$.
Let us define $(1,1)$-tensor field ϕ as

$$\phi u_1 = u_1, \;\; \phi u_2 = u_2, \;\; \phi u_3 = 0.$$

By utilizing the above relations, we obtain the following results:

$$\begin{aligned} \phi^2 E_1 &= E_1 - \eta(E_1)u_3, \\ g(\phi E_1, \phi E_2) &= -g(E_1,E_2) + \eta(E_1)\eta(E_2), \end{aligned}$$

for any $E_1,E_2 \in \chi(M)$.
Consequently, with $u_3 = \xi$, the structure (ϕ,ξ,η,g) forms an almost paracontact structure on the manifold M.
Let ∇ be the Levi-Civita connection of g. Then by using Koszul's formula, we have

$$\begin{aligned} \nabla_{u_1} u_1 &= -u_3, \;\; \nabla_{u_1} u_2 = 0, \;\; \nabla_{u_1} u_3 = u_1, \\ \nabla_{u_2} u_1 &= 0, \;\; \nabla_{u_2} u_2 = -u_3, \;\; \nabla_{u_2} u_3 = u_2, \\ \nabla_{u_3} u_1 &= 0, \;\; \nabla_{u_3} u_2 = 0, \;\; \nabla_{u_3} u_3 = 0. \end{aligned} \tag{83}$$

From the above relations, it becomes evident that the relation (38) is indeed fulfilled. Consequently the considered manifold is para-Sasakian manifold. The components of the Riemannian curvature tensor are given by

$$\begin{aligned} R(u_1,u_2)u_2 &= -u_1, \;\; R(u_1,u_3)u_3 = -u_1, \;\; R(u_2,u_1)u_1 = -u_2, \\ R(u_2,u_3)u_3 &= -u_2, \;\; R(u_3,u_1)u_1 = -u_3, \;\; R(u_3,u_2)u_2 = -u_3, \\ R(u_1,u_2)u_3 &= 0, \qquad R(u_3,u_2)u_3 = u_2, \qquad R(u_3,u_1)u_2 = 0, \\ &\text{and} \\ Ric_g(u_1,u_1) &= -2, \quad Ric_g(u_2,u_2) = -2, \quad Ric_g(u_3,u_3) = -2. \end{aligned}$$

By using the above results, we can easily deduce that the scalar curvature of the manifold is $\tau = -6$. Therefore, we have established the validity of **Theorem 4** for three dimensions.
Now, let us consider a potential vector field Z on M, which can be expressed as $Z = f_1u_1 + f_2u_2 + f_3u_3$, where f_1, f_2, and f_3 are smooth functions. A $\mathscr{C}-\mathscr{RYS}$ equation (5) is written as

$$g(\nabla_{E_1} Z, E_2) + g(E_1, \nabla_{E_2} Z) + 2\alpha Ric_g(E_1,E_2) + \left[2\lambda - \beta\tau - \left(\pi + \frac{2}{n} \right) \right] g(E_1,E_2) = 0. \tag{84}$$

Based on equations (83) and (84), we can deduce the existence of a $\mathscr{C}-\mathscr{RYS}$ on M for the smooth functions f_1, f_2, and f_3 that satisfy the following:

$$\begin{aligned} &u_1(f_1) + f_3 - u_3(f_3) = 0, \qquad u_1(f_2) + u_2(f_1) = 0, \\ &u_1(f_3) + u_3(f_1) - f_1 = 0, \qquad u_2(f_2) + f_3 - u_3(f_3) = 0, \\ &u_2(f_3) + u_3(f_2) - f_2 = 0, \qquad u_3(f_3) = 2\alpha - \lambda + \frac{\tau\beta + \pi}{2} + \frac{1}{(2p+1)}. \end{aligned}$$

9 Conclusion

In this work, we explored the existence and characterization of conformal Ricci-Yamabe solitons ($\mathscr{C}-\mathscr{RYS}$) and their gradient counterparts ($\mathscr{C}-\mathscr{GRYS}$) in the setting of paracontact geometry. Specifically, we analyzed these solitons on para-Kenmotsu, para-Sasakian, and almost paracosymplectic manifolds. Our investigation led to several significant findings:

- For para-Kenmotsu manifolds, we established that the existence of a $\mathscr{C}-\mathscr{RYS}$ implies that the manifold is η-Einstein **Theorem 1**, and when the potential vector field is the gradient of a function, the associated function satisfies a Laplace-type equation **Theorem 2**. Furthermore, in the 3-dimensional case with constant scalar curvature, the soliton function is explicitly determined **Theorem 3**.
- For para-Sasakian manifolds, we demonstrated that any manifold admitting a proper $\mathscr{C}-\mathscr{RYS}$ must be Einstein **Theorem 4**, and in three dimensions, a $\mathscr{C}-\mathscr{GRYS}$-admitting manifold is necessarily Einstein and locally isometric to the hyperbolic space $H^3(-1)$.
- For almost paracosymplectic manifolds, we proved that the existence of a proper $\mathscr{C}-\mathscr{RYS}$ also enforces an Einstein structure **Theorem 6**, and if such a manifold admits an almost gradient soliton $\mathscr{C}-\mathscr{AGRYS}$, then it reduces to a $\mathscr{C}-\mathscr{GRYS}$ **Theorem 7**.

These results contribute to the deeper understanding of geometric flows in pseudo-Riemannian settings, particularly highlighting the strong structural constraints imposed by $\mathscr{C}-\mathscr{RYS}$ and $\mathscr{C}-\mathscr{GRYS}$ on various paracontact manifolds. The study opens avenues for further investigation into soliton theory in more general non-Riemannian or indefinite metric structures.

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