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Some Classes Related to the Set of Generalized Drazin Invertible Linear Relations

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Abstract: In this paper, we introduce and investigate several classes of linear relations on Banach spaces related to generalized Drazin invertibility. First, we focus on a subclass of the generalized Drazin invertible linear relations, namely the class of generalized strongly Drazin invertible linear relations. In particular, we derive two distinct characterizations: one in terms of a bounded projection and a quasi-nilpotent operator, and another based on a specific relationship between T and T^2. This, in turn, leads us to the definition and study of the class of Hirano invertible linear relations. We then introduce the concept of weakly generalized Drazin invertibility and establish a connection between this notion and invertibility in the sense of Hirano. As an application, we present new criteria ensuring the existence of a generalized Drazin inverse for a given linear relation T.

Keywords: Linear relation; Generalized inverse; Drazin inverse; Hirano inverse.

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1 Introduction

Let T be a continuous linear relation (multivalued linear operator) defined on a Banach space X. It is well known (see, e.g., [4]) that T is invertible if $B := T^{-1}$ is a bounded linear operator. In this particular case we encounter $TB = I + T - T$ and $BT = I$. To be invertible, a linear relation must be non-singular in some sense. For example it must be one to one and onto. If one of these two conditions is not satisfied, then B is no longer a bounded operator, T ceases to be invertible and we say that T is singular. However, there are several cases of types of singularities for which one can define an inverse in some generalized sense. The study of theses various kinds of generalized inverses have received a lot of attentions by several mathematicians (see, for instance [3,4,5,6]). More specifically, the authors of [4] were the first to introduce the notion of left (resp. right) invertibility of an injective (resp. surjective) but not surjective (resp. non injective) linear relation T, that is there exists $A \in L(X)$, the set of bounded linear operators, such that $AT = I$ (resp. $TA = I + T - T$). We say that A is the left (resp. right) inverse of T. In [3], R. Cross initiated the theory of semi-Fredholm linear relations. It has defined and studied the class of the relations which are such that the co-dimension of the range is finite or the dimension of the kernel is finite. He has shown that if the kernel is of finite dimension of and the range of T is closed then, there exists $A \in L(X)$ (left Fredholm inverse) such that $AT = I - P$ where P is a finite rank projection. Then, various other classes were introduced. We recall as examples and various others, the class of Drazin invertible linear relations introduced in [5]. Recall that a continuous everywhere defined linear relation T is called Drazin invertible, if there exist $D \in L(X)$ and $k \in \mathbb{N}^*$ satisfying

$$\begin{aligned} &TD = DT + T - T \\ &DTD = D \\ &T^{k+1}D = T^k + T^{k+1} - T^{k+1}. \end{aligned}$$

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Inspired by several studies done in the context of bounded operators and more generally of rings, this concept was generalized by Mnif and Lajnef in [6] where they introduced the generalized Drazin invertibility. We say that T is a generalized Drazin invertible linear relation (denoted by $GDILR(X)$) if there exists $B \in L(X)$ such that:

$$\begin{aligned} TB &= BT + T(0) \\ B &= BTB \\ T - TBT &= N + T(0) \end{aligned}$$

where $N \in QN(X)$ (The set of bounded quasi-nilpotent operators) with $T^2(0) \subset ker(N)$.
In the light of this work and along the same lines, we introduce and study in this paper, some new concept of invertibility in the framework of linear relations. To accurately describe the results found let us start by briefly recalling some basic concepts of linear relations.
In the sequel, we denote the range and the kernel subspaces of a relation T, respectively by $Im(T)$ and $ker(T)$, where $Im(T) = \cup_{x\in D(T)} Tx$ and $ker(T) = T^{-1}(0)$. If $ker(T) = \{0\}$, then T is said to be injective and we say that T is surjective if $Im(T) = X$ and bijective if T is both surjective and injective. Where $T(0) = T^2(0)$ we say that T is stabilized. The singular chain of T is denoted and defined by:

$$R_c(T) := (\bigcup_{i=1}^{\infty} ker(T^i)) \bigcap (\bigcup_{i=1}^{\infty} T^i(0)).$$

Let Q_T denote the natural quotient transformation from X onto $X/\overline{T(0)}$. Clearly that $Q_T T$ is single valued. For $x \in D(T)$, we denote $\|Tx\| := \|Q_T Tx\|$ and we set $\|T\| := \|Q_T T\|$. We say that T is continuous if $\|T\| < +\infty$. Everywhere defined and continuous linear relation is said to be bounded. The class of all closed and bounded linear relations is denoted by $BCR(X)$.
Let $T \in BCR(X)$. We denote by $commu(T)$ and $commu^2(T)$ the sets given respectively by:

$$commu(T) = \{B \in L(X); TB = BT + T(0)\}$$

$$\begin{aligned} commu^2(T) = \{B \in L(X); RB = BR + R(0) \text{ for all } R \in BR(X) \\ \text{satisfying } RT = TR \text{ and } RT(0) = R(0) = T(0)\}. \end{aligned}$$

We will start this paper by presenting the class of generalized strongly Drazin invertible linear relations. It is the set of all $T \in BCR(X)$ for which there exists $C \in L(X)$ such that

$$C \in commu^2(T), CTC = C \text{ and } T - TC = N + T(0)$$

where $N \in QN(X)$ with $T(0) \subset ker(N)$. The properties of this class are studied in Section 2. Theorems 1 and 2 provide new characterizations of generalized strongly Drazin invertible relations through bounded projections and quasi-nilpotent operators, refining the foundational work of Mnif and Lajnef [6] on $GDILR(X)$ and explicitly showing this class forms a subclass (Corollary 2). In Section 3, we introduce the class of Hirano invertible linear relations. These are the relations $T \in BCR(X)$ for which there exist $B \in L(X)$ and $N \in QN(X)$ such that:

$$B \in commu^2(T), BTB = B, T^2 - TB = N + T^2(0) \text{ and } T^2(0) \subset ker(N).$$

Different characterizations of this class were then given. It is shown that this concept is related to weakly generalized Drazin invertibility (Lemma 2) and tripotents (Theorem 4), extending operator-theoretic concepts from [2] to multivalued settings. Finally in Section 4, the previous results are used to establish a new sufficient condition which ensures that a relation T be in $GDILR(X)$, complementing the abstract framework of [6].

2 Generalized strongly Drazin invertibility

Let us initiate this section by providing the definition of the generalized strongly Drazin invertibility.

Definition 1. *Let T in $BCR(X)$. T is said to be generalized strongly Drazin invertible (gs-Drazin invertible), if there exists $C \in L(X)$ such that*

$$C \in commu^2(T), CTC = C \text{ and } T - TC = N + T(0),$$

where $N \in QN(X)$ with $T(0) \subset ker(N)$ (C is said a gs-Drazin inverse of T).

We will see at the end of this section that this class forms a subclass of the set $GDILR(X)$ introduced and studied in [6]. To do this, we start by stating two technical lemmas, then we provide two characterizations of this class which will allow us to conclude.

Lemma 1.*Let T in $BCR(X)$ and $K \in QN(X)$ such that $T(0) \subset kerK$. Then, we have the implication*

$$\text{If } TK = KT + T(0), \text{ then } KT \in QN(X).$$

*Proof.*As $T(0) \subset kerK$ then KT is an operator. By hypothesis we have $TK = KT + T(0)$ then $KTK = K^2T$. Therefore, $(KT)^2 = K^2T^2$. By induction, we shall prove that for all $n \in \mathbb{N}$, $(KT)^n = K^nT^n$. Therefore,

$$\|(KT)^n\|^{\frac{1}{n}} = \|K^nT^n\|^{\frac{1}{n}} \leq \|K^n\|^{\frac{1}{n}}\|T^n\|^{\frac{1}{n}}.$$

But, as $K \in QN(X)$ then KT is also in $QN(X)$, as asserted.

In the following we will give two characterizations of the set of gs-Drazin invertible linear relations that will be useful in the sequel. The first characterization is expressed in term of a bounded projection and a quasi-nilpotent operator.

Theorem 1.*Let $T \in BCR(X)$. Then, the following statements are equivalent:*
i) T is gs-Drazin invertible;
ii) There exist $P,Q \in L(X)$ such that

$$T = P + Q + T(0)$$

$$P^2 = P,\ P \in commu^2(T),\ T(0) \subset ker(Q) \cap ker(P) \text{ and } Q \text{ is quasi-nilpotent.}$$

*Proof.*For the first implication let C be a gs-Drazin inverse of T. Then

$$C \in commu^2(T), CTC = C \text{ and } T - TC = Q + T(0)$$

where $Q \in QN(X)$ with $T(0) \subset ker(Q)$. First, we claim that $T^2(0) = T(0)$. Indeed, we have $T - TC = Q + T(0)$. Then, $(T - TC)T = (Q + T(0))T$. On another hand, let $x \in T(0)$. Then, $TCx \subset TCT(0) \subset T(0)$. Thus $TCx = T(0) = TC(0)$. So, $T(0) \subset ker(TC)$. Therefore, $T^2 - TCT = QT + T(0)$. This implies that $T^2(0) = T(0)$. Set $P = CT$. We have $P(0) = CT(0) = 0$. Then $P \in L(X)$. Furthermore $P^2 = CTCT = CT = P$ and $T - P = T - CT = T - TC = Q + T(0)$. Now, $PT(0) = CT^2(0) = CT(0) = 0$. Hence, $T(0) \subset ker(P) \cap ker(Q)$.
Let $R \in BR(X)$ be such that $RT = TR$ and $RT(0) = R(0) = T(0)$.
Then,

$$RP = RCT = (CR + R(0))T = CTR + R(0) = PR + R(0).$$

Hence, $P \in commu^2(T)$.
For the second implication, let us write $C = (I+Q)^{-1}P$. We claim that C is a gs-Drazin inverse of T. Indeed, $CTC = (I+Q)^{-1}PT(I+Q)^{-1}P$. On another hand, as $T = P + Q + T(0)$, then

$$\begin{aligned} CTC &= (I+Q)^{-1}P(P+Q+T(0))(I+Q)^{-1}P \\ &= (I+Q)^{-1}P(P+PQ+PT(0))(I+Q)^{-1}P \\ &= C. \end{aligned}$$

Now, we prove that $T - TC = Q + T(0)$. To do this we start by proving that, for all $x \in X$

$$P(I+Q)^{-1}x - (I+Q)^{-1}P(x) \in T(0)$$

and

$$P(I+Q)^{-1} = P(I+Q)^{-1}P.$$

We have $P \in commu^2(T)$ and $T(0) \subset ker(P) \cap ker(Q)$. Thus, $TP = PT + T(0)$. Therefore $(P+Q+T(0))P = P(P+Q+T(0)) + T(0)$. Hence,

$$QP + P + T(0) = P + PQ + T(0).$$

So for all $x \in X$, $(Q+I)Px - P(Q+I)x \in T(0)$. Hence, $P(x) - (I+Q)^{-1}P(I+Q)x \in (I+Q)^{-1}T(0)$. On another hand $(I+Q)^{-1}T(0) = T(0)$. Therefore $P(x) - (I+Q)^{-1}P(I+Q)x \in T(0)$. This implies that

$$P(I+Q)^{-1}x - (I+Q)^{-1}Px \in T(0).$$

Thus $P(I+Q)^{-1}x - P(I+Q)^{-1}Px = 0$. Therefore $P(I+Q)^{-1} = P(I+Q)^{-1}P$ as claimed. Then, we compute $T-TC$ as follows:

$$\begin{aligned} T-TC &= P+Q+T(0)-(P+Q+T(0))(I+Q)^{-1}P \\ &= Q+T(0). \end{aligned}$$

We still need to demonstrate that $C \in commu^2(T)$. Let $R \in BR(X)$, be such that $RT = TR$ and $R(T(0)) = R(0) = T(0)$. We claim that $RC = CR + R(0)$. Indeed, we have seen that for all $x \in X$, $(I+Q)^{-1}Px - P(I+Q)^{-1}x \in T(0)$. So,

$$\begin{aligned} RCx &= R(I+Q)^{-1}Px \\ &= R(P(I+Q)^{-1}x+\alpha), \text{ with } \alpha \in T(0) \\ &= RP(I+Q)^{-1}x + R(\alpha). \end{aligned}$$

As, $P \in commu^2(T)$, then $RP = PR + R(0)$. So, $RCx = PR(I+Q)^{-1}x + R(\alpha) + R(0)$. On another hand, $R(\alpha) \subset RT(0) = R(0)$. Thus, $R(\alpha) = R(0)$. Therefore, $RCx = PR(I+Q)^{-1}x + R(0)$. Now we prove that $R(I+Q)^{-1} = (I+Q)^{-1}R$. To do this, we show that $RQ = QR + R(0)$. We have

$$R(Q+T(0)) = RT - RP = TR - PR + R(0).$$

But $R(0) = T(0) \subset ker(T)$, then, $TR - PR = (T-P)R + R$. Hence,

$$R(Q+T(0)) = (T-P)R + R(0) = (Q+T(0))R + R(0) = QR + R(0).$$

So, $RQ = RQ + RT(0) = QR + R(0)$ as desired. Now as $R(0) = T(0) \subset ker(Q)$, then $R(I+Q) = R + RQ = R + QR = (I+Q)R$. This implies that $(I+Q)^{-1}R = R(I+Q)^{-1}$ as claimed. Therefore,

$$RCx = P(I+Q)^{-1}Rx + R(0) = (I+Q)^{-1}PRx + R(0) = CRx + R(0).$$

Hence, $C \in commu^2(T)$.

Remark.i) We note that, in *ii*) of Theorem 1, the operators P and Q satisfy $Im(PQ-QP) \subset T(0)$.
ii) If we further assume that $T(0)$ is complemented, that is, $T(0) \oplus F = X$ for some closed subspace F, then in Theorem 1, we may suppose, without loss of generality, that $\mathrm{Im}(Q) \subset F$. To see this, let V be the bounded projection onto F along $T(0)$. Then, $Im(VQ-Q) \subset Im(V-I) \subset T(0)$. Moreover, we note that $VQ \in QN(X)$. In fact, since $T(0) \subset ker(Q)$, it follows that $QV = Q$ and $(VQ)^n = VQ^n$ for all $n \in \mathbb{N}$. This shows that VQ is quasi-nilpotent. Therefore, by replacing Q with VQ in Theorem 1, we can assume that $\mathrm{Im}(Q) \subset F$.

Presenting now an alternative characterization of the gs-Drazin invertibility in terms of a relationship between T and T^2.

Theorem 2.*Let T be in $BCR(X)$. Suppose that $T(0)$ is complemented and let F denote its complement. Then, T is gs-Drazin invertible if and only if there exists $N \in QN(X)$ such that $T - T^2 = N + T(0)$ with $T(0) \subset ker(N)$ and $Im(N) \subset F$.*

*Proof.*For the first implication, we consider $T \in BCR(X)$ be gs-Drazin invertible. Then, by Theorem 1, there exist two operators $P, N \in L(X)$ with $N \in QN(X)$,

$$T = P + N + T(0),$$

$$P \in commu^2(T),\ P^2 = P,\ T(0) \subset ker(N) \cap ker(P) \text{ and } Im(N) \subset F.$$

On another hand, as T is gs-Drazin invertible then $T^2(0) = T(0)$ and so $T(0) \subset ker(T)$. Hence, by [1, Lemma 2.5] and Remark 2 ii), we have

$$T - T^2 = N(I-N-2P) + T(0).$$

Using Lemma 2.1 in [2], we get that $N(I-N-2P)$ is in $QN(X)$ and we have $T(0) \subset ker(N(I-N-2P))$ and $Im(N(I-N-2P)) \subset F$ as desired.

For the reverse implication, we consider $T \in BCR(X)$ satisfying $T - T^2 = N + T(0)$ where $N \in QN(X)$ with $T(0) \subset ker(N)$ and $Im(N) \subset F$. We claim that T is gs-Drazin invertible. Indeed, let Q be a bounded projection such that $ker(Q) =$

$T(0)$ and $Im(Q) = F$. Then, $A = QT$ is a bounded selection of T and $T - T^2 = A - A^2 + T(0)$. Hence, for every $x \in X$, $(A - A^2 - N)x \in T(0)$. On another hand by Lemma 2.1 in [2], we have $N(4N - I)^{-1}$ is in $QN(X)$. Set

$$B = I - 2\sum_{k=1}^{\infty} \frac{k(2k-2)!}{(2^k k!)^2}[N(4N-I)^{-1}]^k.$$

As,

$$[\frac{k(2k-2)!}{(2^k k!)^2}||[N(4N-I)^{-1}]^k||^{1/k} \longrightarrow 0$$

then the series converges absolutely and $B \in L(X)$ and by Mertens Theorem about Cauchy product of series we can see that $B^2 = I - 4N(4N - T)^{-1}$. Now, if we put $D = \frac{1}{2}I - \frac{1}{2}B$, then we have $D^2 - D = N(4N - T)^{-1}$ and we claim that D is in $QN(X)$. Indeed, we have

$$\begin{aligned} D &= \sum_{k=1}^{\infty} \frac{k(2k-2)!}{2^{2k}(k!)^2}[N(4N-I)^{-1}]^k \\ &= [N(4N-I)^{-1}]\sum_{k=1}^{\infty} \frac{k(2k-2)!}{2^{2k}(k!)^2}[N(4N-I)^{-1}]^{k-1}. \end{aligned}$$

As, $N \in QN(X)$ and $(4N-I)^{-1}\sum_{k=1}^{\infty} \frac{k(2k-2)!}{2^{2k}(k!)^2}[N(4N-I)^{-1}]^{k-1}$ is in $L(X)$, then by Lemma 2.1 in [2], we get that D is in $QN(X)$. On another hand D commutes with every bounded operator that commutes with N. Let $P = A - (2A - I)D$. We claim that

$$T(0) \subset ker(P) \cap ker(2A - I)D,$$

$$P \in commu^2(T), Im(P^2 - P) \subset T(0) \text{ and } Im(P) \subset F.$$

Indeed, we have $T(0) \subset ker(N)$ then from the expression of D it is clear that $T(0) \subset ker(D)$ and so $T(0) \subset ker(2A - I)D$. On another hand $A = QT$. So, $AT(0) = QT^2(0) = QT(0)$. Then, $T(0) \subset ker(P) \cap ker(2A - I)D$ as claimed. To prove that $P \in commu^2(T)$, let $R \in BCR(X)$ be such that $RT = TR$, and $R(T(0)) = R(0) = T(0)$. First we note that $RT = TR$. Then, $R(A + T(0)) = (A + T(0))R$. Thus, $RA = AR + T(0)$. On another hand

$$RN = (T^2 - T)R = NR + R(0).$$

Then, $R(4N - I) = (4N - I)R$ and hence, $R(4N - I)^{-1} = (4N - I)^{-1}R$. Thus,

$$RD = DR + R(0). \tag{1}$$

Therefore,

$$\begin{aligned} RP &= R(A - (2A - I)D) \\ &= AR + R(0) - ((2A - I)R + R(0))D \\ &= AR - (2A - I)DR + R(0) \\ &= PR + R(0). \end{aligned}$$

Now, we prove that $Im(P^2 - P) \subset T(0)$. We have

$$\begin{aligned} P^2 - P &= P(P - I) \\ &= A^2 - A - (2A - I)^2 D + (2A - I)D(2A - I)D. \end{aligned}$$

On another hand, by (1) we have $TD = DT + T(0)$. So, $AD + T(0) = DA + T(0)$. Hence, $Im(AD - DA) \subset T(0)$. Thus, $(I - 2A)D(I - 2A)D = (I - 2A)^2 D^2 + K$ with $Im(K) \subset T(0)$. Therefore,

$$\begin{aligned} P^2 - P &= A^2 - A + (2A - I)^2(D^2 - D) + K \\ &= A^2 - A - (2A - I)^2(4N - I)^{-1}N. \end{aligned}$$

In addition, we have $Im(A-A^2-N) \subset T(0)$. Then,

$$(2A-I)^2 = 4A^2 - 4A + I = 4(A^2 - A) + I = -4N + I + K'$$

with $K' \in L(X)$ satisfying $Im(K') \subset T(0)$. Thus, $P^2 - P = -N + N + K''$ with $Im(K'') \subset T(0)$. Hence, $Im(P^2 - P) \subset T(0)$. Now we prove that $Im(P) \subset F$. We have

$$P = A - (2A-I)D = A(I-2D) + D.$$

As $A = QT$, $Im(A) \subset F$. Hence, $Im(A(I-2D)) \subset F$. It remains to see that $Im(D) \subset F$. To do this we note that

$$D = N(4N-I)^{-1} \sum_{k=1}^{\infty} \frac{k(2k-2)!}{(2^k k!)^2} [N(4N-I)^{-1}]^{k-1}$$

and we have $Im(N) \subset F$. Hence $Im(D) \subset F$. This proves that $Im(P) \subset F$. Therefore, $Im(P^2 - P) \subset F$. So, $Im(P^2 - P) \subset T(0) \cap F = \{0\}$. Thus, $P^2 = P$.
To conclude, we have $A = P + (2A-I)D$. Then, $T = P + (2A-I)D + T(0)$ where P is in $commu^2(T)$ satisfying $P^2 = P$, $T(0) \subset ker(P)$, and the operator $(2A-I)D$ is in $QN(X)$ satisfying $T(0) \subset ker((2A-I)D)$. So, by Theorem 1, T is gs-Drazin invertible as attested.

A simple adaptation of the proofs of Theorems 1 and 2 allows us to conclude the following.

Corollary 1. *Let $T \in BCR(X)$ be stabilized such as T^2 has a gs-inverse $C \in commu^2(T)$. Then*
i) There exist $P, N \in L(X)$ such as $T^2 = P + N + T(0)$ with $P^2 = P$, $P \in commu^2(T)$, $T(0) \subset ker(N) \cap ker(P)$ and N is in $QN(X)$.
ii) If moreover $T(0)$ is complemented and F be its complement, then, there exists $N' \in QN(X)$ in a manner that $T^2 - T^4 = N' + T(0)$ with $T(0) \subset ker(N')$ and $Im(N') \subset F$.

Now, we conclude that the class of gs-Drazin invertible relations is a subclass of the set $GDILR(X)$.

Corollary 2. *Let T in $BCR(X)$ be in a manner that $T(0)$ is complemented. If T is gs-Drazin invertible relation, then it is in $GDILR(X)$.*

3 Hirano invertibility

We start this section by presenting the concept of weakly generalized Drazin invertibility.

Definition 2. *A relation T in $BCR(X)$ is considered to be generalized weakly Drazin invertible (gw-Drazin invertible) if T is stabilized and there exist $C \in L(X)$ and $N \in QN(X)$ such that*

$$CTC = C, \; T - TC = N + T(0) \text{ and } T(0) \subset ker(N),$$

and the operator C is called a gw-Drazin inverse of T.

Analogously to the proof of Theorem 1, we obtain the following result.

Proposition 1. *Let T in $BCR(X)$ be stabilized. Then:*
If there exist $P, N \in L(X)$ such that $T^2 = P + N + T(0)$, $P^2 = P, P \in commu^2(T), T(0) \subset ker(N) \cap ker(P)$ and N quasi-nilpotent then T^2 has a gw-Drazin inverse $C \in commu^2(T)$.

Now we intoduce the definition of the invertibility of a relation $T \in BCR(X)$ in the sense of Hirano.

Definition 3. *Let T be in $BCR(X)$. We say that T is Hirano invertible, if there exist $B \in L(X)$ and $N \in QN(X)$ such that*

$$B \in commu^2(T), BTB = B, \; T^2 - TB = N + T^2(0) \text{ and } T^2(0) \subset ker(N),$$

and the operator B is called a Hirano inverse of T.

The concept of invertibility defined above has some properties that we will study in this section. In the following we give the relationship between the invertibility in the sense of Hirano and the weakly generalized Drazin invertibility.

Lemma 2. *Let T in $BCR(X)$ be stabilized. Then, the following affirmations are equivalent:*
i) T Hirano invertible and T^H is a Hirano inverse of T;
ii) T^2 has a gw-Drazin inverse $C \in commu^2(T)$;
iii) There exists G in $L(X)$ in a manner that:

$$G \in commu^2(T), G = (GT)^2 \text{ and } T^2 - TGT = N + T^2(0)$$

where $N \in QN(X)$ with $T(0) \subset ker(N) \cap ker(G)$.

Proof. $ii) \Rightarrow i)$ T^2 has a gw-Drazin inverse $C \in L(X)$ such that

$$C \in commu^2(T), CT^2C = C \text{ and } T^2 - T^2C = N + T^2(0)$$

where $N \in QN(X)$ satisfying $T^2(0) \subset ker(N)$.
Let $G = CT$. We claim that G is a Hirano inverse of T. First we note that $G \in L(X)$. Indeed, as $CT^2C(0) = C(0) = 0$ then $CT(0) = G(0) = 0$. On another hand, CT is bounded, then $G \in L(X)$. We prove now that $G \in commu^2(T)$. Let R in $CR(X)$ be such that $RT = TR$ and $R(T(0)) = R(0) = T(0)$. As $C \in commu^2(T)$ then $RC = CR + R(0)$.
That is

$$RG = RCT = [CR + R(0)]T = GR + R(0).$$

Therefore, $G \in commu^2(T)$. We claim now that $GTG = G$. Indeed,

$$GTG = CT^2CT = CT = G.$$

Then, we compute $T^2 - TG$. Using the fact that $T^2(0) = T(0)$ and $C \in commu^2(T)$ we get

$$T^2 - TG = N + T^2(0),$$

where $N \in QN(X)$ satisfying $T^2(0) \subset ker(N)$ as desired.
$i) \Rightarrow iii)$ T has a Hirano inverse. Then, there exists $T^H \in L(X)$ such that

$$T^H \in commu^2(T), T^H T T^H = T^H \text{ and } T^2 - TT^H = N + T(0)$$

where $N \in QN(X)$ such that $T^2(0) \subset ker(N)$. Set $G = (T^H)^2$. We have $G \in L(X)$. We claim that $G \in commu^2(T)$. Indeed, let $R \in BCR(X)$ be such that $RT = TR$ and $R(T(0)) = R(0) = T(0)$. As, $RT^H = T^H R + R(0)$ and $T^H R(0) \subset R(0)$ then

$$RG = (T^H R + R(0))T^H = T^H(T^H R + R(0)) + R(0) = GR + R(0).$$

Furthermore, we have

$$(GT)^2 = (T^H)^2 T (T^H)^2 T = (T^H)^2 [T^H T + T(0)] T^H T = (T^H)^2 = G.$$

Now,

$$T^2 - TGT = T^2 - [T^H T + T(0)] T^H T = T^2 - T^H T T^H T = T^2 - T^H T = N + T^2(0)$$

where $N \in QN(X)$ such that $T^2(0) \subset ker(N)$.
On another hand $BT(0) = (T^H)^2 T(0) = 0$. Hence, $T(0) \subset ker(B)$.
$iii) \Rightarrow ii)$ By hypotheses, there exists $G \in L(X)$ such that

$$G \in commu^2(T), G = (GT)^2 \text{ and } T^2 - TGT = N + T^2(0).$$

We claim that G is a gw-Drazin inverse of T^2 and that $G \in commu^2(T)$. For this we begin by proving that $GT^2G = G$. We have

$$GT^2G = GT(GT + T(0) = GTGT + GT^2(0) = G + GT^2(0) = G.$$

Now, to conclude it remains the following.

$$T^2 - T^2G = T^2 - T[GT + T(0)] = N + T^2(0).$$

Now, we give a first characterization of the Hirano invertibility of a relation T in terms of a relationship between T and T^3.

Theorem 3. *Let T in $BCR(X)$ be stabilized, $T(0)$ is complemented and F be a closed complement of $T(0)$. Then, the following are equivalent:*
i) T has a Hirano inverse.
ii) $T-T^3=N+T(0)$ where $T(0)\subset ker(N)$, $Im(N)\subset F$ and $N\in QN(X)$.

Proof. $i)\Rightarrow ii)$ By Lemma 2, T^2 has a gw-Drazin inverse $C\in commu^2(T)$. It follows, by Corollary 1, ii), that there exists $N\in QN(X)$ in such a way that $T^2-T^4=N+T(0)$ with $T(0)\subset ker(N)$ and $Im(N)\subset F$. Thus,

$$T(T-T^3)=N+T(0).$$

Moreover, we have:

$$\begin{aligned}(T^3-T)^2 &= T(T-T^3)(I-T^2)\\ &= (I-(PT)^2+T(0))(N+T(0))\\ &= (I-(PT)^2)N+T(0),\end{aligned}$$

where P is the corresponding projection with kernel $T(0)$ and range F.
We claim that $N'=(I-(PT)^2)N=N-(PT)^2N$ is in $QN(X)$ such that $T(0)\subset ker(N')$ and $Im(N')\subset F$. Indeed, first, we show that

$$TP=PT+T(0).$$

Let $x=x_0+x_1\in T(0)\oplus F=X$. Then, $TPx=TPx_1=Tx_1$. On another hand, as $T(0)\subset ker(P)$ then $T=(P+(I-P))T=PT+(I-P)T$. So, $TPx=PTx_1+(I-P)Tx_1$. Thus, $TPx=TPx+T(0)=PTx_1+(I-P)Tx_1+T(0)$. But, $(I-P)Tx_1\subset Im(I-P)=ker(P)=T(0)$, hence, $TPx=PTx_1+T(0)$. In addition, $PTx=P(Tx_0+Tx_1)$. Since, $x_0\in T(0)$ and T is stabilized, then $Tx_0=T(0)$. Thus, $PTx=PTx_1$. Therefore $TPx=PTx+T(0)$ as desired.
Now, we put $a=N$ and $b=(PT)^2N$. We claim that $a^2b=aba$ and $b^2a=bab$. Indeed, we have $TP=PT+T(0)$. Then,

$$T^2P=T^2+T(0)$$

and likewise we have $T^4P=PT^4+T(0)$. Then, $(T^2-T^4)P=P(T^2-T^4)+T(0)$. Hence, $(N+T(0))P=P(N+T(0))+T(0)$. Therefore,

$$NP+T(0)=PN+T(0).$$

We have $T(T^2-T^4)=(T^2-T^4)T$. This implies $T(N+T(0))=(N+T(0))T$. Hence, $TN=NT+T(0)$. So,

$$NPT+T(0)=P(NT+T(0))+T(0)=PTN+T(0).$$

Thus, $N(PT)=(PT)N+K$ with $Im(K)\subset T(0)$. The previous equalities give

$$a^2b=N^2(PT)^2N=NPT(PTN+T(0))N=aba.$$

and

$$b^2a=(PT)^2N(PT)^2NN=bab.$$

Thus, by Lemma 2.1 in [2], we deduce that $a+b=N'$ is in $QN(X)$.
Now, we prove that $(PT-(PT)^3)^2=N'$. We have $(T-T^3)^2=N'+T(0)$. So, $(PT-(PT)^3+T(0))^2=N'+T(0)$. Hence, $(PT-(PT)^3)^2+T(0))^2=N'+T(0)$. Then, $Im(PT-(PT)^3-N')\subset T(0)\cap F=\{0\}$. Therefore,

$$(PT-(PT)^3)^2=N'.$$

Let $\lambda\in\mathbb{C}^*$. Then, $I-\lambda^2(PT-(PT)^3)^2$ is invertible. Thus,

$$I-\lambda(PT-(PT)^3)][I+\lambda(PT-(PT)^3)$$

$$=(I+\lambda(PT-(PT)^3)(I-\lambda(PT-(PT)^3)$$

is invertible and so is $(I-\lambda(PT-(PT)^3)$ for all $\lambda\in C^*$. Thus, $PT-(PT)^3$ is in $QN(X)$.
Hence,

$$T-T^3=PT-(PT)^3+T(0)$$

where $PT-(PT)^3$ is in $QN(X)$ as desired with $T(0)\subset ker(PT-(PT)^3)$ and $Im(PT-(PT)^3)\subset F$.

ii) $\Rightarrow$ i) Suppose that $T - T^3 = N + T(0)$ where $N \in QN(X)$ satisfying $T(0) \subset ker(N)$ and $Im(N) \subset F$. We have

$$T^2 - T^4 = T(T - T^3) = T(N + T(0)) = TN + T(0).$$

Let P be the continuous projection with kernel $T(0)$ and range F. Then, $T = PT + T(0)$ and so, $T^2 - T^4 = PTN + T(0)$. We claim that PTN is in $QN(X)$. Indeed, as $N \in QN(X)$ and PT is in $L(X)$ and noting that $(PT)^2N = (PT)N(PT)$ and $N^2PT = NPTN$ then, the desired result follows immediately from Lemma 2.1 in [2]. Furthermore we have $T(0) \subset ker(PTN)$ and $Im(PTN) \subset F$. Now, using Theorem 2, we get that T^2 is gs-Drazin invertible and by Lemma 2 we deduce that T is Hirano invertible.

Corollary 3. *Let T in $BCR(X)$ be stabilized such that $T(0)$ is complemented. Then, if T has a Hirano inverse, then for all $n \in \mathbb{N}$, T^n has a Hirano inverse.*

Proof. From Theorem 3, we have $T - T^3 = N + T(0)$ where $N \in QN(X)$ satisfying $T(0) \subset ker(N)$ and $Im(N) \subset F$, where F be its complement. We have

$$T^n - (T^n)^3 = (T - T^3)(\sum_{k=0}^{n-1} T^{3n-3-2k}).$$

Hence, if P be the projection with kernel $T(0)$ and range F, then

$$T^n - (T^n)^3 = N\sum_{k=0}^{n-1}(PT)^{3n-2k-3} + T(0).$$

Let $N' = N\sum_{k=0}^{n-1}(PT)^{3n-3-2k}$. It is evident that $N' \in QN(X)$ and $Im(N') \subset Im(N) \subset F$ and $T(0) \subset ker(N')$. So, Theorem 3 allows as to see that T^n is a Hirano invertible linear relation.

Lemma 3. *Let T in $BCR(X)$ be stabilized and $T(0)$ is complemented. If T has a Hirano inverse, then $B := \frac{1}{2}(T^2 + T)$ and $B' := \frac{1}{2}(T^2 - T)$ have gs-Drazin inverses.*

Proof. Using the stabilization of T and [1, Lemma 2.5], we get that

$$B^2 - B = \frac{1}{4}(T - 2I)(T^3 - T).$$

Let P be the bounded projection with $ker(P) = T(0)$ and $Im(P) = F$ a closed complement of $T(0)$ and set $A = PT$. Then,

$$B^2 - B = \frac{1}{4}(A + 2I)(A^3 - A) + T(0).$$

On another hand, by assumption, we have that T has a Hirano inverse, so by Theorem 3, there exists $N \in QN(X)$ satisfying $T(0) \subset ker(N)$ and $Im(N) \subset F$ such that $T^3 - T = N + T(0)$. Hence, $A^3 - A + T(0) = N + T(0)$. Thus, $A^3 - A = N + K$ where K is in $L(X)$ satisfying $Im(K) \subset T(0)$. Then we check that

$$B^2 - B = \frac{1}{4}(A + 2I)(N + K) + T(0).$$

But $Im(K) \subset T(0)$ and $T(0) \subset ker(N)$, then we get

$$B^2 - B = \frac{1}{4}(A + 2I)N + T(0).$$

In light of Theorem 2, to see that B has a gs-Drazin inverse, it is sufficing to be seen that $N' := \frac{1}{4}(A + 2I)N$ is in $QN(X)$ such that $B(0) = T(0) \subset ker(N')$ and $Im(N') \subset F$. To do this, if we put $a = (A - 2I)$ and $b = N$ then we can verify that $a^2b = aba$ and $b^2a = bab$. So, using [2], we get that $N' \in QN(X)$. Moreover we have $T(0) \subset ker(N')$ and $Im(N') \subset Im(A) + Im(N) \subset F$ as desired. This shows that B has a gs-Drazin inverse. Likewise, $B' = \frac{1}{2}(T^2 - T)$ has a gs-Drazin inverse, as desired.

We observe in the succeeding theorem that the concept of Hirano inverse is related to tripotents.

Theorem 4. *Let T in $BCR(X)$ be stabilized and $T(0)$ complemented. Then the following statements are equivalent:*
i) T has a Hirano inverse.
ii) There exists a quasi-tripotent bounded operator $K \in commu^2(T)$ (that is $Im(K^3-K) \subset T(0)$) and a $Q \in QN(X)$ such that $T = K + Q + T(0)$ with $T(0) \subset ker(Q) \cap ker(K)$.

Proof. For the first implication, set $B := \frac{1}{2}(T^2+T)$ and $B' := \frac{1}{2}(T^2-T)$. In light of Lemma 3, B and B' have gs-Drazin inverses. Then, by Theorem 1 and Remark 2 there exist $P, P', N, N' \in L(X)$ such that $B = P+N+T(0)$ and $B' = P'+N'+T(0)$ with $P^2 = P$, $P \in commu^2(B)$, $P'^2 = P'$, $P' \in commu^2(B')$, $T(0) \subset ker(N) \cap ker(P) \cap ker(N') \cap ker(P')$ and N, N' are in $QN(X)$ verifying $Im(N) \cup Im(N') \subset F$ where F be a complement of $T(0)$. As $P \in commu^2(B)$, $P' \in commu^2(B')$, $TB = BT$, $TB' = B'T$ and $TB(0) = B(0) = B'(0)$ then $TP = PT + T(0)$ and $TP' = P'T + T(0)$. This gives immediately that

$$BP' = P'B + T(0) \text{ and } B'P = PB' + T(0). \tag{2}$$

Now we claim that $PP' + T(0) = P'P + T(0)$. Indeed, if we set $R = P' + T(0)$ then, we have $TR = T(P'+T(0)) = (P'+T(0))T = RT$ and $RT(0) = T(0) = R(0)$. So, as $P \in commu^2(T)$, then $RP = PR + T(0)$. Hence,

$$P'P - PP' \subset T(0) \tag{3}$$

as claimed. Finally we also show that

$$NN' = N'N. \tag{4}$$

Indeed, by (2) and (3) it follows that $(B-P)$ and $(B'-P')$ commute. Thus we have $(N+T(0))(N'+T(0)) = (N'+T(0))(N+T(0))$. Therefore, $Im(NN'-N'N) \subset F \cap T(0) = \{0\}$. So, $NN' = N'N$ as desired. On another hand we have $T = B - B' = P - P' + N - N' + T(0)$. If we take $K = P - P'$ and $Q = N - N'$ then by (4) and Lemma 2.1 in [2] we have $Q \in QN(X)$, and by (3) we get $Im(K^3-K) \subset T(0)$ and $T(0) \subset ker(Q) \cap ker(K)$. In addition as $P \in commu^2(B)$, and $P' \in commu^2(B')$ then $K \in commu^2(T)$.

To prove the converse, suppose that $T = K + Q + T(0)$ with $T(0) \subset ker(Q) \cap ker(K)$, $K \in commu^2(T)$, $Im(K^3-K) \subset T(0)$ and $Q \in QN(X)$ So, $T^2 = K^2 + KQ + QK + Q^2 + T(0)$. Now multiplying the equality $T - K = Q + T(0)$ on left and then on right by K and as $K \in commu^2(T)$ then, we deduce that $KQ + T(0) = QK + T(0)$. Hence, $T^2 = K^2 + (2K+Q)Q + T(0)$. On another hand we have $(K^2)^2 = KK^3 = K(K+M)$ where M is in $L(X)$ verifying $Im(M) \subset T(0)$. Thus, $(K^2)^2 = K^2$ and so, K^2 is a bounded projection in $commu^2(T)$. Furthermore, $Q^2(2K+Q) = Q(2QK+Q^2) = Q(2KQ+M+Q^2)$ where $M \in L(X)$ satisfying $Im(M) \subset T(0)$. Thus, as $T(0) \subset ker(Q)$, we get $Q^2(2K+Q) = Q(2K+Q)Q$. Similarly we show that $(2k+Q)^2Q = (2K+Q)Q(2K+Q)$. Using Lemma 2.1 in [2], it follows that $(2K+Q)Q$ is in $QN(X)$. Finally, the use of Proposition 1 implies that T^2 has a gw-Drazin inverse $C \in commu^2(T)$ and by Lemma 2 we conclude that T has a Hirano inverse as attested.

We now present a non-trivial example of a linear relation admitting a Hirano inverse.

Example 1. Consider the Hilbert space

$$X = \ell^2(\mathbb{N}) = \{(x_1, x_2, \dots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

with canonical orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let us consider the left shift operator $S_g : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$ and the bounded operator $A : (x_1, x_2, x_3, \dots) \mapsto (x_1, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \dots)$. Let us define the following bounded and closed linear relation in $\ell^2(\mathbb{N})$:

$$T := S_g^{-1} A S_g : (x_1, x_2, x_3, \dots) \mapsto (0, x_2, \frac{x_2}{2}, \frac{x_3}{3}, \dots) + span\{e_1\}.$$

We claim that T has a Hirano inverse. Indeed, define $K \in L(X)$ by its action on the basis vectors:

$$K(e_1) = 0, \quad K(e_2) = e_2, \quad K(e_n) = 0 \quad \text{for } n \geq 3.$$

Note that

$$\text{Im}(K^3 - K) = \{0\} \subseteq T(0) = span\{e_1\}.$$

Thus, K is quasi-tripotent and satisfies $\text{Im}(K^3-K) \subseteq T(0)$.
Define now $Q \in L(X)$ by

$$Q(e_1) = 0, \quad Q(e_n) = \frac{1}{n} e_{n+1} \quad \text{for } n \geq 2.$$

We will prove that Q is quasi-nilpotent. For $k \geq 1$ and $n \geq 2$,

$$Q^k(e_n) = \frac{1}{n(n+1)\cdots(n+k-1)} e_{n+k}.$$

This gives

$$\|Q^k\| = \sup_{n\geq 2} \frac{1}{n(n+1)\cdots(n+k-1)} = \frac{1}{(k+1)!} \leq \frac{1}{k!}.$$

Using Stirling's approximation, we conclude that

$$\|Q^k\|^{1/k} \leq \left(\frac{1}{k!}\right)^{1/k} \sim \frac{e}{k} \to 0 \quad \text{as } k \to \infty.$$

Therefore, Q is quasi-nilpotent.
On another hand, we notice that:

$$T = K + Q + T(0) \text{ on } \ell^2(\mathbb{N}).$$

Let us demonstrate that $K \in commu^2(T)$. Let $S \in BCR(\ell^2(\mathbb{N}))$ be such that $ST = TS$ and $ST(0) = S(0) = T(0) = span\{e_1\}$. Then, we claim that $SK = KS + S(0)$. Let $x \in \ell^2(\mathbb{N})$. Since K acts nontrivially only on e_2, we have $Kx = \langle x, e_2\rangle e_2$. So, $S(Kx) = \langle x, e_2\rangle S(e_2)$ and $K(Sx) = \langle Sx, e_2\rangle e_2 = a < x, e_1 > e_2$ for some scalar a. Using now the assumption that S commutes with T with $ST(0) = S(0) = span\{e_1\}$, we get that $S(e_2) \subset span\{e_1\}$ and $(SK - KS)(x) \subset span\{e_1\}$. Thus, $SK = KS + S(0)$. In conclusion it has been proven that

–T is stabilized ($T^2(0) = T(0)$),
–$T(0) = span\{e_1\}$ complemented,
–$T = K + Q + T(0)$, with:
  –K quasi-tripotent ($\text{Im}(K^3 - K) \subseteq T(0)$),
  –Q quasi-nilpotent,
  –$T(0) \subseteq \ker(Q) \cap \ker(K)$.
  –$K \in commu^2(T)$.

Thus, by Theorem 4, we infer that T has a Hirano inverse as claimed.

4 New description of the Generalized Drazin invertibility

In the sequel, we mainly aim to find a sufficient condition for a relation to be in $GDILR(X)$. To do this, let us start by stating the next technical lemma.

Lemma 4.*Let $B, N \in L(X)$ be such that $N \in QN(X)$. Then, we have*
i) If Im $(BN - NB) \subset ker(B)$ then $BN \in QN(X)$
ii) If Im $(BN - NB) \subset ker(N)$ then $NB \in QN(X)$.

Proof.i) By hypothesis we have Im $(BN - NB) \subset ker(B)$. Then, we have

$$BBN = BNB. \tag{5}$$

Which implies that $B^2N^2 = (BN)^2$. By induction, we shall prove that for $n \in \mathbb{N}$, $B^nN^n = (BN)^n$. Therefore,

$$\|(BN)^n\|^{\frac{1}{n}} = \|B^nN^n\|^{\frac{1}{n}} \leq \|B^n\|^{\frac{1}{n}} \|N^n\|^{\frac{1}{n}}.$$

ii) Similarly as in *i*), we may prove that $NB \in QN(X)$.

Now, we are ready to state the principal result of this section.

Theorem 5.*Let T in $BCR(X)$ be such that $T(0)$ is complemented and $R_c(T) = \{0\}$. If there exist $B \in commu(T)$ and $N \in QN(X)$ in a manner that*

$$\begin{cases} T(0) \subset kerB \;\text{ and }\; T^2(0) \subset ker(N), \\ T - TBT = N + T(0), \\ commu((TB)^2) \subset commu(T), \end{cases}$$

then T is in $GDILR(X)$.

*Proof.*Let F be a complement of $T(0)$ in X and let $\tilde{P}$ be the bounded projection onto F with kernel $T(0)$. Then, for all $x \in X$,

$$Tx - TBTx = Nx + T(0) = \tilde{P}Nx + T(0).$$

On another hand, we have $N^2\tilde{P} = N\tilde{P}N$ and $\tilde{P}^2N = \tilde{P}N\tilde{P}$. Hence, Lemma 2.1 in [2] entails that $\tilde{P}N$ is in $QN(X)$ Consequently, $T - TBT = \tilde{P}N + T(0)$, where $\tilde{P}N$ is $QN(X)$ with $T(0) \subset ker(\tilde{P}N)$ and $\operatorname{Im}(\tilde{P}N) \subset F$. Now, put $\tilde{N} = \tilde{P}N$ and $S = BTB$. Evidently, S is an operator verifying $T(0) \subset kerS$. Show that there is a projection P' and $N' \in QN(X)$ such that $TP' = P'T + T(0)$, $TP' = N' + T(0)$ and $T + P'$ is invertible. We divide the proof into six steps.
First step: Show that there exists $W_1 \in QN(X)$ verifying $T - TST = W_1 + T(0)$ and $T(0) \subset kerW_1$.
We have $T - TST = T - TBTBT$. Since $T(0) \subset kerB$ then, $T(0) \subset ker(TB)$. Therefore, by [1, Lemma 2.5], we get

$$\begin{aligned} T - TST &= (I + TB)T - (I + TB)TBT \\ &= \tilde{N} + TB\tilde{N} + T(0) \\ &= \tilde{N} + BT\tilde{N} + T(0). \end{aligned}$$

Moreover, we have $BTT - BTTBT + T(0) = BT\tilde{N} + BT^2(0) + T(0)$. Furthermore, as $BT(0) = \{0\}$ and $TB = BT + T(0)$, we deduce that $BT^2(0) \subset T(0)$. Therefore, $(BT + T(0))T - (BT + T(0))TBT = BT\tilde{N} + T(0)$ and

$$TBT - TBTBT = BT\tilde{N} + T(0). \tag{6}$$

On another hand, using again the equality $T - TBT = \tilde{N} + T(0)$, we obtain $T(BT) - TBT(BT) = \tilde{N}(BT) + T(0)$. Hence, by (6), we obtain

$$\tilde{N}BT + T(0) = BT\tilde{N} + T(0), \tag{7}$$

Consequently, we get that $T - TST = \tilde{N} + \tilde{N}BT + T(0)$. As $T(0) \subset ker\tilde{N} \cap kerB$, then by Lemma 4 we get that $\tilde{N}(BT)$ is in $QN(X)$. Using (7), we get that $\tilde{N}^2(\tilde{N}BT) = \tilde{N}^2(BT\tilde{N})$ and $(\tilde{N}BT)\tilde{N}(\tilde{N}BT) = (\tilde{N}BT)\tilde{N}(BT\tilde{N})$. Whence,

$$\tilde{N}^2(\tilde{N}BT) = \tilde{N}(\tilde{N}BT)\tilde{N} \text{ and } (\tilde{N}BT)\tilde{N}(\tilde{N}BT) = (\tilde{N}BT)^2\tilde{N}.$$

Then, by virtue of Lemma 2.1 in [2], we get that $W_1 := \tilde{N} + \tilde{N}BT$ is in $QN(X)$ with $W_1(T(0)) = \{0\}$. Consequently, $T - TST = W_1 + T(0)$, with W_1 is in $QN(X)$ and $T(0) \subset kerW_1$.
Second step: Show that there exists $W_2 \in QN(X)$ such that $S - S^2T = W_2$ and $T(0) \subset ker(W_2)$.
We have by [1, Lemma 2.5], $ST + T(0) = (BT + T(0))BT = TB(BT + T(0)) = TBTB$. Then

$$ST + T(0) = TS. \tag{8}$$

Whence, $S - S^2T = S - S^2T - ST(0) = S - S(ST + T(0)) = B(T - TBT)B + BTB(T - TBT)B$. Therefore, $S - S^2T = B(\tilde{N} + T(0))B + BTB(\tilde{N} + T(0))B = B\tilde{N}B + BTB\tilde{N}B$. Hence, it remains to verify that $B\tilde{N}B + BTB\tilde{N}B$ is in $QN(X)$ First, we show that $B\tilde{N}B$ is in $QN(X)$. To do this, we claim that

$$B\tilde{N} + T(0) = \tilde{N}B + T(0). \tag{9}$$

Indeed, we have $T - TBT = \tilde{N} + T(0)$. Then, $(T - TBT)B = \tilde{N}B + T(0)$ and $BT - BTBT = B\tilde{N}$. Whence,

$$\begin{aligned} \tilde{N}B + T(0) &= TB - TBTB \\ &= TB - TB(BT + T(0)) \\ &= TB - TBBT - T(0) \\ &= BT - BTBT + T(0) \\ &= B\tilde{N} + T(0) \end{aligned}$$

Then, by Lemma 4, $\tilde{N}B$ is in $QN(X)$. By using (9) we get that $B(\tilde{N}B)B = B^2(\tilde{N}B)$ and $(\tilde{N}B)B(\tilde{N}B) = (\tilde{N}B)^2B$. Then, by virtue of Lemma 2.1 in [2], we get that $B\tilde{N}B$ is in $QN(X)$ Using again Lemma 2.1 in [2] and (9), we shall prove that $BTB\tilde{N}B$ is in $QN(X)$. By Lemma 2.1 in [2] and (9), we get that $W_2 := B\tilde{N}B + BTB\tilde{N}B$ is in $QN(X)$. Additionally, we have $W_2T(0) = \{0\}$.
Third step: Show that TS is gs-Drazin invertible. By using the first step, we get that

$$TS - (TS)^2 = (T - TST)S = (W_1 + T(0))S = W_1S + TSTS(0).$$

We claim that $W_1S+T(0)=SW_1+T(0)$. Indeed, we have $W_1S+T(0)=\tilde{N}BTB+\tilde{N}BTBTB+T(0)$. On another hand, we have $SW_1+T(0)=BTB\tilde{N}+BTB\tilde{N}BT+T(0)$. Using (7) and (9) we get that $\tilde{N}BT(B)+T(0)=BTB\tilde{N}+T(0)$. Therefore, $(\tilde{N}BTB+T(0))TB=(BTB\tilde{N}+T(0))TB$. So, $\tilde{N}BTBTB+T(0)=BTB\tilde{N}BT+T(0)$. Thus,

$$W_1S+T(0)=SW_1+T(0). \tag{10}$$

Since $T(0)\subset kerW_1$ then, by Lemma 4, we get that $W_1S\in QN(X)$ satisfying $T(0)\subset kerW_1S$. Furthermore, we have $\mathrm{Im}(W_1S)\subset F$, since $Im(\tilde{N})\subset F$. By virtue of Theorem 2 we get that TS is gs-Drazin invertible. Then, by Theorem 1, there exists a projection $P\in commu^2(TS)$ and $N'\in QN(X)$ such that

$$TS-P=N'+T(0) \text{ and } T(0)\subset kerN'\cap kerP.$$

In addition, we have

$$N'P+T(0)=(N'+T(0))P=TSP-P=PTS-P+T(0)=PN'+T(0). \tag{11}$$

Fourth step : We construct an invertible bounded operator R satisfying

$$(T+I-TS)(S+I-ST)=R+T(0)=(S+I-ST)(T+I-TS).$$

Using (8) we get

$$\begin{aligned}
&(T+I-TS)(S+I-ST)\\
&=TS+S-(TS)S+T+I-TS-TST-ST+TSST\\
&=I+(TS-TS)+S-(ST+T(0))S+T-TST-ST+TSST\\
&=I+(S-S^2T-ST(0))+(T-TS-TST+TSTS)\\
&=I+(S-S^2T)+(T-TST)(I-S).
\end{aligned}$$

A similar calculation shows that $(S+I-ST)(T+I-TS)=I+(S-S^2T)+(T-TST)(I-S)$. By using the first and the second steps, we get that $(T+I-TS)(S+I-ST)=I+W_2+W_1(I-S)+T(0)$. We note that $W_3:=W_1(I-S)$ is in $QN(X)$. In fact, by (10) we get $W_1^2(I-S)=W_1^2-W_1^2S=W_1^2-W_1(W_1S+T(0)=W_1^2-W_1SW_1=W_1(I-S)W_1$. Now, we shall show that

$$W_3^2W_2=W_3W_2W_3 \text{ and } W_2^2W_3=W_2W_3W_2.$$

To do this, we must demonstrate that $W_3W_2+T(0)=W_2W_3+T(0)$. In fact, by (10) and (9) we get that $W_3W_2+T(0)=W_3S+W_3S\tilde{N}B+T(0)=W_3S+(W_3S+T(0))\tilde{N}B=W_3S+SW_3\tilde{N}B+T(0)$. On another hand, we have $W_2W_3+T(0)=SW_3+T(0)+S\tilde{N}BW_3=W_3S+T(0)+S\tilde{N}BW_3$. Now, by using (7) and (9), it is not hard to prove that $W_3\tilde{N}B+T(0)=\tilde{N}BW_3+T(0)$. Consequently,

$$W_3W_2+T(0)=W_2W_3+T(0).$$

Since, $T(0)\subset kerW_3$, then $W_3^2W_2=W_3(W_3W_2+T(0))=W_3(W_2W_3+T(0))=W_3W_2W_3$. As $T(0)\subset ker(W_2)$, then $W_2^2W_3=W_2(W_2W_3+T(0))=W_2W_3W_2$. Consequently, by virtue of Lemma 2.1 in [2], we get that W_3+W_2 is in $QN(X)$. Thus, there exists an invertible operator $R:=I+W_2+W_3$ such that $(T+I-TS)(S+I-ST)=(S+I-ST)(T+I-TS)$.

Fifth step : Show that $R'=:T+I-TS$ is invertible.

By the fourth step, one can deduce that $T+I-TS$ is surjective. As $R_c(T)=\{0\}$, then it is adequate to prove that $\ker(T+I-TS)\subset T(0)\cap ker(T)$. First, we show that $\ker(T+I-TS)\subset T(0)$. Let $x\in ker(T+I-TS)$. Then, $(T+I-TS)(0)=T(0)$. On another hand, we have $(S+I+ST)(T+I-TS)x=Rx+T(0)$. Then, $ST(0)+T(0)+ST^2(0)=Rx+T(0)$. In addition, we have $ST^2(0)\subset T(0)$ then $Rx\in T(0)$. Moreover, we have $R^{-1}(T(0))=T(0)$. Consequently, $x\in T(0)$. Hence, it remains to prove that $x\in ker(T)$. Since $x\in ker(T+I-TS)$ then, $0\in Tx+x-TSx$ and so, $x\in -Tx+TSx$. But we have $x\in ker(T+I-TS)\subset T(0)\subset ker(S)$ then $x\in -Tx\cap T(0)$. Therefore, $x\in T(0)\cap ker(T)$, as desired.

Final step : Show that T is in $GDILR(X)$.

Put $P'=I-P$. Then, P' is a bounded projection. We show that $T+P'$ is injective. As $T+P'=(T+I-TS)+(TS-P)$, then by the fifth step we get that $T+P'=R''+T(0)$, where $R''=R'+N'$. As $R_c(T)=\{0\}$, then it suffices to prove that $\ker(T+P')\subset T(0)\cap ker(T)$. Let $x\in ker(T+P')$. Then, $(T+P')(x)=T(0)=R''x+T(0)$ and so, $R''x\in T(0)$. Thus, $x\in R''^{-1}(T(0))\subset T(0)$. On another hand, we have TS is gs-Drazin invertible then $CTSC=C$, with C is a gs-Drazin inverse of TS. Whence, $TS(0)=T(0)\subset ker(C)$. Since $x\in ker(T+P')$ and $P=CTS$, then $0\in Tx+x-CTSx$. Now, as

$\ker(T+P') \subset T(0)$, then $x \in -Tx + CT(0)$. Hence, $x \in T(0) \cap T(-x)$ and so, $x \in ker(T)$, as required. Hence, it only remains to be proved that

$$\begin{cases} TP' = P'T + T(0); \\ T + P' \text{ is surjective;} \\ TP' = Q + T(0), \text{ where Q is quasi-nilpotent and } T^2(0) \subset kerQ. \end{cases}$$

As P in $commu^2(TS)$, P in $commu(TS)$. Whence, $TP = PT + T(0)$. Since $T(0) \subset ker(P)$, we get

$$T(I-P) = T - TP = (I - P + T(0))T = (I-P)T + T(0).$$

Note also that $TN' = N'T + T(0)$. In fact, we have $N'(TS-P) = N'^2$. Then

$$N'TS - N'P + T(0) = N'^2 + T(0).$$

On another hand, we have $(TS-P)N' = N'^2 + T(0)$. Then, $TSN' - PN' = N'^2 + T(0)$. Using (11), we get the following:

$$TSN' - N'P = N'^2 + T(0).$$

Consequently, $TSN' - N'P = N'TS - N'P + T(0)$ and $TSN' = N'TS + T(0)$. Which implies that $TBTBN' = N'TBTB + T(0)$. Thus, $N' \in commu((TB)^2) \subset commu(T)$ and hence, $TN' = N'T + T(0)$. Now, we have that

$$R'N' + T(0) = N'R' + T(0).$$

Then, R'' is invertible and so $T + P'$ is surjective. Using the first step and the inclusion $T(0) \subset ker(TS) \cap ker(T-T)$, we get that $TP' = T(I-P) = T - TST + TST - PT = (T - TST) + (TS-P)T = W_1 + T(0) + (N' + T(0))T = W_1 + N'T + T(0)$. Consequently, $TP' = W_1 + N'T + T(0)$. By the construction of W_1 and N', we can show that $T^2(0) \subset ker(W_1 + N'T)$. Hence, it remains to be seen that $Q := N'T + W_1$ is in $QN(X)$. As $T(0) \subset ker(N')$ and $TN' = N'T + T(0)$, then by Lemma 1 we get that $N'T$ is in $QN(X)$. Using Lemma 2.1 in [2], we shall prove that $Q \in QN(X)$.
By virtue of Theorem 7 provided in [6], we get that T is in $GDILR(X)$.

Theorem 5 provides some sufficient conditions ensuring that a linear relation $T \in GDILR(X)$.

*Example 2.*Consider the Hilbert space $X = \ell^2(\mathbb{N})$, $\{e_n\}_{n=1}^{\infty}$ the standard orthonormal basis. Let $M = \text{span}\{e_1\} \subset X$, and write P for the orthogonal projection onto M, $Q = I - P$. Define the (single-valued) eight-shift operator

$$A(e_1) = 0, \qquad A(e_n) = \frac{1}{n-1} e_{n+1} \quad (n \geq 2),$$

which is quasinilpotent. Now, define the linear relation

$$T : X \to X, \qquad T(x) = \{Ax + \alpha e_1 : \alpha \in \mathbb{C}\}.$$

We claim that T is in $GDILR(X)$.
To do this set $B = Q = I - P$, the projection onto $\text{span}\{e_2, e_3, \dots\}$. Then,

$$B \in commu(T),\ T(0) = M \subset \ker B, \tag{12}$$

and

$$T - TBT = \{(x,y) : y - (A - A^2)x \in M\}. \tag{13}$$

Define the single-valued operator

$$N = A - A^2. \tag{14}$$

Since A is quasinilpotent, then $A - A^2$ is again quasinilpotent. Thus $T - TBT = N + T(0)$, with

$$N \text{ is quasi-nilpotent, } T^2(0) = T(M) = \{A(\alpha e_1) + M : \alpha\} = M \subset \ker N. \tag{15}$$

We have also

$$commu((TB)^2) \subset commu(T). \tag{16}$$

Using (12), (13), (14), (15) and (16) and applying Theorem 5, we conclude that $T \in GDILR(X)$, as claimed.

Conclusion

In this work, we have advanced the theory of generalized invertibility for linear relations on Banach spaces through three main contributions:

1. We established a complete characterization of generalized strongly Drazin invertible relations, demonstrating their decomposition into a projection and quasi-nilpotent component under natural commutativity conditions.
2. We introduced and analyzed the class of Hirano invertible relations, revealing their fundamental connection with tripotent operators and providing spectral-theoretic criteria for their identification.
3. We developed new sufficient conditions for generalized Drazin invertibility that significantly extend previous results, particularly through our investigation of the singular chain condition and quasi-nilpotent perturbations.

These results not only unify and generalize existing operator-theoretic concepts to the multivalued setting but also open new directions for research in spectral theory and functional calculus for linear relations. The techniques developed here suggest promising applications to perturbation theory, operator semigroups, and the study of differential inclusions in Banach spaces.

Declarations

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