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An Interplay Between Quadratic-Phase Fourier Transform and Octonion Algebra

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Abstract: The quadratic-phase Fourier transform (QPFT) has emerged as a versatile five-parameter integral transform, unifying a wide range of unitary transformations, from the classical Fourier transform to the more recent special affine Fourier transform. However, its limitations become evident when applied to the precise representation of non-transient octonion-valued signals. To address this, we introduce the octonion quadratic-phase Fourier transform-a distinct integral transform specifically designed for such signals. In this study, we investigate its fundamental properties, including the energy-preserving relation and the inversion formula. Furthermore, we establish a set of uncertainty principles, such as Heisenberg's and logarithmic uncertainty principles, providing deeper insights into the intricate interplay between octonion algebra and the quadratic-phase Fourier transform.

Keywords: Quadratic-phase Fourier transform; Quaternion Fourier transform; Octonion Fourier transform; Uncertainty principles.

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1 Introduction

Towards the culmination of twentieth century, Saitoh [1] while working on the solution of the heat equation

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{\partial U(x,t)}{\partial t}, \quad x \in \mathbb{R}, t \in \mathbb{R}^+ \tag{1}$$

with the initial condition $U(x,0) = f(x) \in L^2(\mathbb{R})$ derived a typical result for a novel integral transform arising in the framework of the model (1) by using the theory of reproducing kernels. Invoking the classical Fourier transform, it was demonstrated that a solution $U(x,t)$ of (1) has the following representation:

$$U_f(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(\omega) \exp\left\{ -\frac{(x-\omega)^2}{4t} \right\} d\omega. \tag{2}$$

Therefore, for any $t > 0$, it was examined that the resulting integral transform $U \to U_f, f \in L^2(\mathbb{R})$, can be extended analytically to $\mathbb{C}$. Inspired by the work of Saitoh, Castro et al. [2] studied certain possibilities for the quadratic Fourier transform by employing a general quadratic function in the exponent of the transform. Keeping in view the contemporary trends of using different chirps in the analysis of finite energy signals, Castro et al. [3] introduced the notion of quadratic-phase Fourier transform (QPFT) which embodies et variety of integral transforms including the Fourier transform, fractional Fourier transform (FrFT) [4,5], linear canonical transform (LCT) [6,7] and the special affine Fourier transform (SAFT) [8]. Moreover, for real parameters $\Omega = (A,B,C,D,E)$ with $B \neq 0$, the QPFT of any function $f \in L^2(\mathbb{R})$ is given by

$$\mathscr{Q}_\Omega[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)} dt. \tag{3}$$

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Thus, we observe that when $A = C = D = E = 0$ and $B = 1$, the QPFT reduces to the Fourier transform. Moreover, when $D = E = 0$, (3) simplifies to the LCT as well as to the FrFT, up to the choice of certain constant factors that do not affect the properties of the corresponding integral operators. Given the above definitions, it is also evident that the QPFT encompasses the SAFT as a particular case when $\Omega = \big(A/2B, -1/B, D/2B, p/B, -(Dp - Bq)/B\big)$.

In addition to the mathematical objective of constructing a new transform that generalizes several existing mathematical concepts, significant attention was devoted to designing the definition with elements that facilitate the verification of essential and intriguing properties. These properties enhance the utility of this mathematical tool in various applications. The QPFT has shown significant advantages as an integral operator, demonstrating remarkable potential due to the flexibility provided by its five free parameters. This versatility is evident in several recent publications, including [9,10,11,12,13,14,15].

In recent times, hyper-complex Fourier transforms have ignited a surge of interest, presenting a compelling approach for treating multi-channel signals as unified algebraic entities without sacrificing a crucial spectral information. The landscape of hyper-complex Fourier transforms is adorned with various formulations, each offering unique perspectives. Among these, the quaternion Fourier transform (QFT) stand as fundamental and paramount [16,17]. The non-commutativity of quaternion algebra has led to the development of distinct variants of the quaternion Fourier transform, including the quaternion Fourier transform (QFT), quaternion FrFT (QFrFT), quaternion LCT (QLCT), quaternion SAFT (QSAFT), quaternion QPFT (QQPFT) and many more [18,19,20,21,22]. These quaternion-based transforms consider the non-commutative properties of quaternion algebra, providing unique insights into signals characterized by both magnitude and phase components. Moreover, these quaternion-valued Fourier transforms extends traditional Fourier analysis to quaternion-valued signals, offering a unified framework for time-frequency analysis.

Quaternions, also known as the Cayley-Dickson algebra of order 4, have found substantial applications in various fields, particularly in multidimensional and multichannel signal analysis, where traditional Fourier transforms prove inadequate [23,24,25]. Yet, beyond the realm of quaternions lies a domain deserving of equal attention in hyper-complex signal processing: the octonions. Octonions, also known as Cayley-Dickson algebra of order 8, have been captivating modern signal and image processing. In contemporary signal processing, the octonion Fourier transform (OFT) has emerged as a rapidly growing area of interest for researchers. It generalizes the QFT by utilizing an octonion kernel to transform octonion-valued signals into the frequency domain. Its ability to address the limitations of the QFT by extending the analysis to an 8-dimensional framework makes it an essential tool in modern signal processing.

In 2011, Hahn and Snopek [26] introduced the octonion Fourier transform and investigated its fundamental properties. Since then, numerous applications of the octonion Fourier transform in signal processing have been explored [27,28,29]. In 2021, Gao and Li [30] introduced the octonion linear canonical transform (OLCT) as a generalization of the OFT by replacing the Fourier kernel with the LCT kernel. Additionally, Bhat and Dar [31] proposed the octonion short-time linear canonical transform (OSTLCT), whereas Sheikh et al. [32] extended the octonion Fourier transform to the SAFT domain, investigating its fundamental properties and associated uncertainty principles.

Motivated and inspired by these advancements, we propose a novel octonion quadratic-phase Fourier transform (OQPFT) for real-valued functions. Many established integral transforms are special cases of the newly proposed OQPFT, including the Fourier transform, FrFT, LCT, QPFT, QFT, QFrFT, QLCT, QQPFT, OFT, OLCFT, and many others. This study focuses on properties such as linearity, Parseval's formula and the reconstruction formula of the proposed transform. The core contribution of this paper lies in establishing well-known uncertainty inequalities.

The rest of the article is organized as follows: Section 2 deals with essential preliminaries that set the stage for our study. In Section 3, we introduce the novel octonion quadratic-phase Fourier domain, exploring its profound implications. Section 4 is devoted to the formulation of various uncertainty principles intricately linked with the octonion QPFT. Finally, the conclusion is presented in Section 5.

2 Preliminaries

In this section, we revisit the foundational concepts of octonion algebra and introduce the fundamental definitions of the octonion Fourier transform.

2.1 Octonion algebra

The present subsection mainly deals with some basic facts and notations on the octonion algebra, which shall be utilized in the rest of the article. For convenience, we will denote (t_1, t_2, t_3) by $\mathbf{t}$ and $(\omega_1, \omega_2, \omega_3)$ by $\mathbf{w}$ throughout the paper.

In accordance with the Cayley-Dickson construction, the octonion algebra $\mathbb{O}$ is a non associative and non-commutative algebra defined over $\mathbb{R}$ and is generated by the seven imaginary units $e_1, e_2, \ldots, e_7$ given as [33]:

$$e_0 = (1,0), e_1 = (i,0), e_2 = (j,0), e_3 = (k,0), e_4 = (0,1), e_5 = (0,i), e_6 = (0,j), e_7 = (0,k).$$

The multiplication rules for octonion algebra are presented in table 1.

*	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1: Multiplication rules in octonion algebra

An arbitrary octonion $o \in \mathbb{O}$ can be represented as:

$$o = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7, \tag{4}$$

where $o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R}$ represent vectors in $\mathbb{R}^8$. Here, the real part of o is o_0, while the remaining components constitute the imaginary part. This structure is analogous to a complex number but uniquely extends to seven degrees of freedom, resembling a vector in $\mathbb{R}^7$. Within the framework of octonion algebra, characterized by non-associativity and non-commutativity over the real numbers, $e_1, e_2, \ldots, e_7$ represent the seven imaginary standard units. Moreover, the octonion conjugate is given by

$$\overline{o} = o_0 - o_1e_1 - o_2e_2 - o_3e_3 - o_4e_4 - o_5e_5 - o_6e_6 - o_7e_7. \tag{5}$$

Besides, the norm of octonions is defined by $|o| = \sqrt{o\overline{o}} = \sqrt{\overline{o}o}$ and $|o|^2 = \sum_{r=0}^{7} o_r^2$. Furthermore, the norm of octonions satisfy $|o_1o_2| = |o_1||o_2|, \forall o_1, o_2 \in \mathbb{O}$. It is important to mention that each $o \in \mathbb{O}$ can be represented in the quaternionic form as:

$$o = \alpha + \beta e_4, \tag{6}$$

where $\alpha = o_0 + o_1e_1 + o_2e_2 + o_3e_3$ and $\beta = o_4 + o_5e_1 + o_6e_2 + o_7e_3$ are quaternions ($\mathbb{H}$). Thus, an octonion is equal to the direct sum $\mathbb{H} \bigoplus \mathbb{H}$, and the multiplication for any two pairs $(\alpha, \beta), (\gamma, \delta) \in \mathbb{H} \bigoplus \mathbb{H}$ is given by

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma - \overline{\delta}\beta, \delta\alpha + \beta\overline{\gamma}). \tag{7}$$

The following Lemmas are very useful and are frequently used in the subsequent sections.

Lemma 1.[33] *For any $\alpha, \beta \in \mathbb{H}$, the following relations hold:*

$$(i)\, e_4\alpha = \overline{\alpha}e_4; \quad (ii)\, e_4(\alpha e_4) = -\overline{\alpha}; \quad (iii)\, (\alpha e_4)e_4 = -\alpha;$$
$$(iv)\, \alpha(\beta e_4) = (\beta\alpha)e_4; \quad (v)\, (\alpha e_4)\beta = (\alpha\overline{\beta})e_4; \quad (vi)\, (\alpha e_4)(\beta e_4) = -\overline{\beta}\alpha.$$

Remark. It is obvious from Lemma 1 that the quaternionic form $\alpha + \beta e_4$; $\alpha, \beta \in \mathbb{H}$ of any octonion satisfies the following properties:

$$\overline{\alpha + \beta e_4} = \overline{\alpha} - \beta e_4 \tag{8}$$

and

$$|\alpha + \beta e_4|^2 = |\alpha|^2 + |\beta|^2. \tag{9}$$

Lemma 2.[33] *Let o_1 and o_2 be any two arbitrary octonions. Then, we have*

$$e^{o_1} \cdot e^{o_2} = e^{o_1+o_2} \qquad \textit{if and only if} \qquad o_1 \cdot o_2 = o_2 \cdot o_1. \tag{10}$$

An octonion-valued function $f(\mathbf{t})$ can be considered as a mapping from $\mathbb{R}^3$ to $\mathbb{O}$ and has the following explicit form:

$$\begin{aligned} f(\mathbf{t}) &= f_0(\mathbf{t}) + f_1(\mathbf{t})e_1 + \cdots + f_7(\mathbf{t})e_7 \\ &= f_0 + f_1e_1 + (f_2 + f_3e_1)e_2 + (f_4 + f_5e_1 + (f_6 + f_7e_1)e_2)\,e_4 \\ &= g(\mathbf{t}) + h(\mathbf{t})e_4, \end{aligned} \tag{11}$$

where each $f_i(\mathbf{t}), i = 0, 1, \ldots 7$ is a real-valued funcion and $g, h \in \mathbb{H}$ as in (6). For each octonion-valued function f defined over $\mathbb{R}^3$ and $1 \leq p < \infty$, the L^p norm of f is defined by

$$\|f\|_p^p = \int_{\mathbb{R}^3} |f(\mathbf{t})|^p \, d\mathbf{t} < \infty. \tag{12}$$

Thus, $L^p(\mathbb{R}^3, \mathbb{O})$ is a Banach-space consisting of all measurable functions $f(\mathbf{t})$ that have a finite L^p-norm. For $p = \infty$, $L^\infty(\mathbb{R}^3, \mathbb{O})$ is the set of essentially bounded measurable functions with the norm $\|f(\mathbf{t})\|_\infty = \operatorname{ess\,sup}_{\mathbf{t}\in\mathbb{R}^3} |f(\mathbf{t})|$. If $f(\mathbf{t}) \in L^\infty(\mathbb{R}^3, \mathbb{O})$ is continuous, then $\|f(\mathbf{t})\|_\infty = \sup_{(\mathbf{t})\in\mathbb{R}^3} |f(\mathbf{t})|$. For $p = 2$, we can define the inner product $\langle f, g\rangle := \int_{\mathbb{R}^3} f(\mathbf{t})g(\mathbf{t})\,d\mathbf{t}$, where $g(\mathbf{t}) = g_0(\mathbf{t}) + g_1(\mathbf{t})e_1 + g_2(\mathbf{t})e_2 + g_3(\mathbf{t})e_3 + g_4(\mathbf{t})e_4 + g_5(\mathbf{t})e_5 + g_6(\mathbf{t})e_6 + g_7(\mathbf{t})e_7$, which turns $L^2(\mathbb{R}^3, \mathbb{O})$ into a Hilbert space.

2.2 Octonion Fourier transform

Let $e_1, e_2, \ldots, e_7$ be the imaginary units in Cayley-Dickson algebra of octonions, then for any $f \in L^1(\mathbb{R}, \mathbb{O})$, the one-dimensional octonion Fourier transform is defined as [34]:

$$\mathscr{F}_1[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{-e_4 t\omega}\, dt \tag{13}$$

and its inverse transform is given by

$$f(t) = \mathscr{F}_1^{-1}\left(\mathscr{F}_1[f]\right)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}_1[f](\omega)\, e^{e_4 t\omega}\, d\omega. \tag{14}$$

Next, we shall define the 3D OFT of an octonion-valued function $f(\mathbf{t}) \in L^2(\mathbb{R}^3, \mathbb{O})$ as [34]:

$$\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(\mathbf{t})\, e^{-e_1t_1\omega_1}\, e^{-e_2t_2\omega_2}\, e^{-e_4t_3\omega_3}\, d\mathbf{t}. \tag{15}$$

Moreover, the inverse transform of (15) is given by

$$\begin{aligned} f(\mathbf{t}) &= \mathscr{F}^{-1}_{e_1,e_2,e_4}\left(\mathscr{F}_{e_1,e_2,e_4}[f]\right)(\mathbf{t}) \\ &= \int_{\mathbb{R}^3} \mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})\, e^{e_1t_1\omega_1}\, e^{e_2t_2\omega_2}\, e^{e_4t_3\omega_3}\, d\mathbf{w}. \end{aligned} \tag{16}$$

*Remark.*For $f \in L^2\left(\mathbb{R}^3, \mathbb{O}\right)$, we have

$$\begin{aligned} \mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w}) = {} & \mathscr{F}_{e_1,e_2,e_4}[f_0](\mathbf{w}) + \mathscr{F}_{e_1,e_2,e_4}[f_1](\mathbf{w})e_1 + \mathscr{F}_{e_1,e_2,e_4}[f_2](\mathbf{w})e_2 + \mathscr{F}_{e_1,e_2,e_4}[f_3](\mathbf{w})e_3 \\ & + \mathscr{F}_{e_1,e_2,e_4}[f_4](\mathbf{w})e_4 + \mathscr{F}_{e_1,e_2,e_4}[f_5](\mathbf{w})e_5 + \mathscr{F}_{e_1,e_2,e_4}[f_6](\mathbf{w})e_6 + \mathscr{F}_{e_1,e_2,e_4}[f_7](\mathbf{w})e_7. \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})|^2_{\mathbb{O}} = {} & \left|\mathscr{F}_{e_1,e_2,e_4}[f_0](\mathbf{w})\right|^2 + \left|\mathscr{F}_{e_1,e_2,e_4}[f_1](\mathbf{w})\right|^2 e_1 + \left|\mathscr{F}_{e_1,e_2,e_4}[f_2](\mathbf{w})\right|^2 e_2 \\ & + \left|\mathscr{F}_{e_1,e_2,e_4}[f_3](\mathbf{w})\right|^2 e_3 + \left|\mathscr{F}_{e_1,e_2,e_4}[f_4](\mathbf{w})\right|^2 e_4 + \left|\mathscr{F}_{e_1,e_2,e_4}[f_5](\mathbf{w})\right|^2 e_5 \\ & + \left|\mathscr{F}_{e_1,e_2,e_4}[f_6](\mathbf{w})\right|^2 e_6 + \left|\mathscr{F}_{e_1,e_2,e_4}[f_7](\mathbf{w})\right|^2 e_7. \end{aligned}$$

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We define a new L^p-norm as follows:

$$\left\|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})\right\|_{\mathbb{O},p} = \left(\int_{\mathbb{R}^3} |\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})|_{\mathbb{O}}^p \, d\mathbf{w}\right)^{1/p}. \tag{17}$$

It is important to note that the measures $|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})|_{\mathbb{O}}$ and $|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})|$, as well as the norms $\|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})\|_{\mathbb{O},p}$ and $\|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})\|_p$, are distinct. The distinction arises because $\|\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})\|_{\mathbb{O},p}$ depends on the unique structure of the octonion-valued function $f(\mathbf{t})$, requiring each component function $f_k(\mathbf{t})$, where $0 \le k \le 7$, to be real-valued. However, in certain cases, the L^p-norm of $\mathscr{F}_{e_1,e_2,e_4}[f](\mathbf{w})$ may exhibit behavior similar to that of the octonion-valued function $f(\mathbf{t})$ within the L^p-space.

3 Octonion Quadratic-phase Fourier Transform

This section delves into the intricacies of the quadratic-phase Fourier transform. Our primary objective is to develop both one-dimensional and three-dimensional formulations of the OQPFT. Along the way, we systematically explore the fundamental properties that define and govern the three-dimensional OQPFT. From now on, we shall represent the quadratic function $A_i t_i \omega_i + B_i t_i \omega_i + C_i \omega_i^2 + D_i t_i + E_i \omega_i$ by $q_i t_i$, where $i = 1,2$ and 3.

Definition 1.*The one-dimensional OQPFT of any $f \in L^1(\mathbb{R},\mathbb{O})$ with respect to the real parametric set $\Omega = (A,B,C,D,E)$, $B \neq 0$ is denoted by $\mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)$ and is defined as:*

$$\mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega) = \int_{\mathbb{R}} f(t)\, \mathscr{K}_{\Omega}^{-e_4}(t,\omega)\, dt, \tag{18}$$

where

$$\mathscr{K}_{\Omega}^{e_4}(t,\omega) = \frac{1}{\sqrt{2\pi}} \exp\left\{e_4\left(At^2 + Bt\omega + C\omega^2 + Dt + E\omega\right)\right\}. \tag{19}$$

*Remark.*The one-dimensional OQPFT shares an elegant bond with the traditional one-dimensional octonion Fourier transform and is given by

$$\begin{aligned}
\mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{-e_4\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{-e_4\left(At^2+Dt+Bt\omega+C\omega^2+E\omega\right)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(f(t)\, e^{-e_4\left(At^2+Dt\right)}\right) e^{-e_4 Bt\omega} dt\, e^{-e_4\left(C\omega^2+E\omega\right)} \\
&= \mathscr{F}_1\left[f(t)\, e^{-e_4\left(At^2+Dt\right)}\right](B\omega)\, e^{-e_4\left(C\omega^2+E\omega\right)}, \quad B \neq 0.
\end{aligned} \tag{20}$$

The inversion formula corresponding to the OQPFT (18) is given in the following theorem.

Theorem 1.*If $\mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f]$ is the one-dimensional OQPFT of any function $f \in L^1(\mathbb{R},\mathbb{O})$. Then, the following inversion formula holds:*

$$f(t) = \frac{1}{|B|} \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)\, \mathscr{K}_{\Omega}^{e_4}(t,\omega)\, d\omega, \tag{21}$$

where $\mathscr{K}_{\Omega}^{e_4}(t,\omega)$ is given by (19).

*Proof.*By virtue of the relation (20), we have

$$\mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)\, e^{e_4\left(C\omega^2+E\omega\right)} = \mathscr{F}_1\left[f(t)\, e^{-e_4\left(At^2+Dt\right)}\right](B\omega). \tag{22}$$

Employing (14) in (22) yields

$$f(t)\, e^{-e_4\left(At^2+Dt\right)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](B\omega)\, e^{e_4\left(C\omega^2+E\omega\right)}\, e^{e_4 tB\omega}\, d(B\omega). \tag{23}$$

Equation (23) can be recast as:

$$\begin{aligned} f(t) &= \frac{1}{|B|\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)\, e^{e_4\left(C\omega^2+E\omega\right)}\, e^{e_4 t\omega}\, e^{e_4\left(At^2+Dt\right)}\, d\omega \\ &= \frac{1}{|B|\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)\, e^{e_4\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)}\, d\omega \\ &= \frac{1}{|B|} \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{O}}_{1,\Omega}[f](\omega)\, \mathscr{K}^{e_4}_{\Omega}(t,\omega)\, d\omega. \end{aligned} \tag{24}$$

This completes the proof of Theorem 1.

We are now in a position to give the formal definition of three-dimensional OQPFT by replacing the kernel in the conventional QPFT (3) with the quadratic-phase octonion kernel.

Definition 2. *The three-dimensional OQPFT of an octonion-valued function $f \in L^2\left(\mathbb{R}^3, \mathbb{O}\right)$ with respect to a real parametric set $\Omega = (A,B,C,D,E)$, $B \neq 0$ is defined as:*

$$\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) = \int_{\mathbb{R}^3} f(\mathbf{t}) \mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1) \mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2) \mathscr{K}^{-e_4}_{\Omega_3}(t_3,\omega_3)\, d\mathbf{w}, \tag{25}$$

where

$$\left.\begin{aligned} \mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1) &= \frac{1}{\sqrt{2\pi}} e^{e_1 q_1 t_1} = \frac{1}{\sqrt{2\pi}} e^{e_1\left(A_1 t_1^2 + B_1 t_1 \omega_1 + C_1 \omega_1^2 + D_1 t_1 + E_1 \omega_1\right)}, \quad B_1 \neq 0 \\ \mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2) &= \frac{1}{\sqrt{2\pi}} e^{e_2 q_2 t_2} = \frac{1}{\sqrt{2\pi}} e^{e_2\left(A_2 t_2^2 + B_2 t_2 \omega_2 + C_2 \omega_2^2 + D_2 t_2 + E_2 \omega_2\right)}, \quad B_2 \neq 0 \\ \mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3) &= \frac{1}{\sqrt{2\pi}} e^{e_4 q_3 t_3} = \frac{1}{\sqrt{2\pi}} e^{e_4\left(A_3 t_3^2 + B_3 t_3 \omega_3 + C_3 \omega_3^2 + D_3 t_3 + E_3 \omega_3\right)}, \quad B_3 \neq 0 \end{aligned}\right\}. \tag{26}$$

Remark. (i). The octonion quadratic-phase Fourier transform (25) can be represented in the inner product form as:

$$\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) = \left\langle f(\mathbf{t}), \mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3) \mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2) \mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1) \right\rangle, \tag{27}$$

where $\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1), \mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)$ and $\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)$ are given by (26).

(ii). It is worthwhile to mention that the kernels $\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)$, $\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)$ and $\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)$ with complex units e_1, e_2 and e_4, respectively, are octonion-valued and do not reduce to the quaternion cases. As such the present integral transform is more interesting and complicated.

Next, our endeavor is to explore certain mathematical properties of the OQPFT, such as linearity, parity, modulation, and shifting properties. To achieve the goal, we first expand the kernel (26) of the OQPFT (25) in the closed form as:

$$\begin{aligned} &\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1) \mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2) \mathscr{K}^{-e_4}_{\Omega_3}(t_3,\omega_3) \\ &= \frac{1}{(2\pi)^{3/2}} \left[e^{-e_1 q_1 t_1} e^{-e_2 q_2 t_2} e^{-e_4 q_3 t_3} \right] \\ &= \frac{1}{(2\pi)^{3/2}} \Big[\cos(q_1 t_1)\cos(q_2 t_2)\cos(q_3 t_3) - \sin(q_1 t_1)\cos(q_2 t_2)\cos(q_3 t_3) e_1 \\ &\qquad - \cos(q_1 t_1)\sin(q_2 t_2)\cos(q_3 t_3) e_2 + \sin(q_1 t_1)\sin(q_2 t_2)\cos(q_3 t_3) e_3 \\ &\qquad - \cos(q_1 t_1)\cos(q_2 t_2)\sin(q_3 t_3) e_4 + \sin(q_1 t_1)\cos(q_2 t_2)\sin(q_3 t_3) e_5 \\ &\qquad + \cos(q_1 t_1)\sin(q_2 t_2)\sin(q_3 t_3) e_6 - \sin(q_1 t_1)\sin(q_2 t_2)\sin(q_3 t_3) e_7 \Big]. \end{aligned} \tag{28}$$

By virtue of the full octonion form (28) of the kernel, the three-dimensional OQPFT $f(\mathbf{t})$ can be considered as the map from $\mathbb{R}^3$ to $\mathbb{O}$ and is given by

$$\begin{aligned} \mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) &= \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eee}(\mathbf{w}) - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{oee}(\mathbf{w}) e_1 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eoe}(\mathbf{w}) e_2 \\ &+ \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{ooe}(\mathbf{w}) e_3 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eeo}(\mathbf{w}) e_4 + \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{oeo}(\mathbf{w}) e_5 \\ &+ \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eoo}(\mathbf{w}) e_6 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{ooo}(\mathbf{w}) e_7, \end{aligned} \tag{29}$$

where

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eee}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eee}(\mathbf{t})\cos(q_1t_1)\cos(q_2t_2)\cos(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{oee}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{oee}(\mathbf{t})\sin(q_1t_1)\cos(q_2t_2)\cos(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eoe}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eoe}(\mathbf{t})\cos(q_1t_1)\sin(q_2t_2)\cos(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{ooe}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{ooe}(\mathbf{t})\sin(q_1t_1)\sin(q_2t_2)\cos(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eeo}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eeo}(\mathbf{t})\cos(q_1t_1)\cos(q_2t_2)\sin(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{oeo}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{oeo}(\mathbf{t})\sin(q_1t_1)\cos(q_2t_2)\sin(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eoo}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eoo}(\mathbf{t})\cos(q_1t_1)\sin(q_2t_2)\sin(q_3t_3)d\mathbf{t};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{ooo}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{ooo}(\mathbf{t})\sin(q_1t_1)\sin(q_2t_2)\sin(q_3t_3)d\mathbf{t}.$$

Here, the subscripts e and o denotes whether a function is even (e) or odd (o) with respect to an appropriate variable, for instance, $f_{oee}(t_1,t_2,t_3)$ is odd with respect to t_1 and even with respect to the variables t_2 and t_3.

Next, we present a theorem that outlines key features of the octonion quadratic-phase Fourier transform (25).

Theorem 2. *For a pair of functions $f,g\in L^2(\mathbb{R}^3,\mathbb{O})$ and the arbitrary octonion scalars $\boldsymbol{\gamma},\boldsymbol{\delta}$, the OQPFT has the following properties:*

(i). *Linearity:* $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\boldsymbol{\gamma}f+\boldsymbol{\delta}g\right](\mathbf{w}) = \boldsymbol{\gamma}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right](\mathbf{w}) + \boldsymbol{\delta}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[g\right](\mathbf{w})$;

(ii). *Anti-linearity:* $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right](\mathbf{w}) = \bar{\boldsymbol{\gamma}}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right](\mathbf{w}) + \bar{\boldsymbol{\delta}}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[g\right](\mathbf{w})$;

(iii). *Scaling:* $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\mathscr{S}f\right](\mathbf{w}) = \mathscr{Q}^{\mathbb{O}}_{\Omega_1',\Omega_2',\Omega_3'}\left[f\right]\left(\frac{\omega_1}{\alpha_1},\frac{\omega_2}{\alpha_2},\frac{\omega_3}{\alpha_3}\right)$, *where* $\mathscr{S}f(\mathbf{t}) = f(\alpha_1t_1,\alpha_2t_2,\alpha_3t_3), \alpha_i\in\mathbb{R}^+$ *and* $\Omega_i' = \left(A_i/\alpha_i^2, B_i, \alpha_i^2C_i, D_i/\alpha_i, \alpha_iE_i\right)$ *for* $i=1,2,3$;

(iv). *Reflection:* $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\mathscr{P}f\right](\mathbf{w}) = -\mathscr{Q}^{\mathbb{O}}_{\Omega_1^\star,\Omega_2^\star,\Omega_3^\star}\left[f\right](-\mathbf{w})$, *where* $\mathscr{P}f(\mathbf{t}) = f(-\mathbf{t})$ *and* $\Omega_i^\star = (A_i,B_i,C_i,-D_i,-E_i), \quad \forall i=1,2,3$;

(v). *Conjugation:* $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\bar{f}\right](\mathbf{w}) = \overline{\mathscr{Q}^{\mathbb{O}}_{-\Omega_1,-\Omega_2,-\Omega_3}\left[f\right](\mathbf{w})}$.

Proof. For the sake of brevity, we omit the proofs of (i) and (ii).

(ii). To study the effect of scaling on OQPFT, we proceed as:

$$\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\mathscr{S}f\right](\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(\alpha_1t_1,\alpha_2t_2,\alpha_3t_3)e^{-e_1q_1t_1}e^{-e_2q_2t_2}e^{-e_4q_3t_3}d\mathbf{t}.$$

Substituting $z_1=\alpha_1 t_1, z_2=\alpha_2 t_2, z_3=\alpha_3 t_3$ in the above equation, we obtain

$$\begin{aligned}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\mathscr{S}f\right](\mathbf{w}) &= \frac{1}{\alpha_1\alpha_2\alpha_3(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(z_1,z_2,z_3)\\ &\quad\times e^{-e_1\left(A_1(z_1/\alpha_1)^2+B_1(z_1/\alpha_1)\omega_1+C_1\omega_1^2+D_1(z_1/\alpha_1)+E_1\omega_1\right)}\\ &\quad\times e^{-e_2\left(A_2(z_2/\alpha_2)^2+B_2(z_2/\alpha_2)\omega_2+C_2\omega_2^2+D_2(z_2/\alpha_2)+E_2\omega_2\right)}\\ &\quad\times e^{-e_4\left(A_3(z_3/\alpha_3)^2+B_3(z_3/\alpha_3)\omega_3+C_3\omega_3^2+D_3(z_3/\alpha_3)+E_3\omega_3\right)}dz_1dz_2dz_3\\ &= \mathscr{Q}^{\mathbb{O}}_{\Omega_1',\Omega_2',\Omega_3'}\left[f\right]\left(\frac{\omega_1}{\alpha_1},\frac{\omega_2}{\alpha_2},\frac{\omega_3}{\alpha_3}\right),\end{aligned}$$

where $\Omega_i'=\left(A_i/\alpha_i^2, B_i, \alpha_i^2C_i, D_i/\alpha_i, \alpha_iE_i\right)$ for each $i=1,2,3$.

(iv). Plugging $(\alpha_1,\alpha_2,\alpha_3)=(-1,-1,-1)$ into the property stated in (iii), we arrive at the desired result.

(iii). Invoking the Definition 2, we have

$$\begin{aligned}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[\bar{f}\right](\mathbf{w}) &= \int_{\mathbb{R}^3}\overline{f(\mathbf{t})}\,\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{-e_4}_{\Omega_3}(t_3,\omega_3)\,d\mathbf{t}\\ &= \overline{\int_{\mathbb{R}^3} f(\mathbf{t})\,\overline{\mathscr{K}^{-e_1}_{-\Omega_1}(t_1,\omega_1)\,\mathscr{K}^{-e_2}_{-\Omega_2}(t_2,\omega_2)\,\mathscr{K}^{-e_4}_{-\Omega_3}(t_3,\omega_3)}\,d\mathbf{t}}\\ &= \overline{\mathscr{Q}^{\mathbb{O}}_{-\Omega_1,-\Omega_2,-\Omega_3}\left[f\right](\mathbf{w})}.\end{aligned}$$

This completes the proof of Theorem 2.

The shifting property for the OQPFT is studied in the following theorem.

Theorem 3. *If $\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]$, $\mathscr{Q}^{\mathbb{O},x_2}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]$ and $\mathscr{Q}^{\mathbb{O},x_3}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]$ denotes the OQPFT of the octonion-valued functions $f(t_1-x_1,t_2,t_3)$, $f(t_1,t_2-x_2,t_3)$ and $f(t_1,t_2,t_3-x_3)$, respectively. Then, we have*

$$\begin{aligned}\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right](\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\quad\times\nabla_1 f\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &\quad-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\quad\times\Delta_1 f\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right),\end{aligned}$$

where

$$\begin{aligned}\nabla_1 f &= \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eee}-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_oee}e_1-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eoe}e_2\\ &\quad+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_ooe}e_3-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eeo}e_4+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_oeo}e_5\\ &\quad+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eoo}e_6-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_ooo}e_7\end{aligned}$$

and

$$\begin{aligned}\Delta_1 f &= \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eee}+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_oee}e_1-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eoe}e_2\\ &\quad-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_ooe}e_3-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eeo}e_4-\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_oeo}e_5\\ &\quad+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eoo}e_6+\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_ooo}e_7.\end{aligned}$$

Proof. From (29), we have

$$\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eee}(\mathbf{w}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eee}(t_1-x_1,t_2,t_3)\cos(q_1t_1)\cos(q_2t_2)\cos(q_3t_3)d\mathbf{t}.$$

Putting $z_1 = t_1 - x_1, z_2 = t_2, z_3 = t_3$ in the above equation, we obtain

$$\begin{aligned}
&\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eee}(\mathbf{w})\\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eee}(\mathbf{z})\cos\left(A_1(z_1+x_1)^2 + B_1(z_1+x_1)\omega_1 + C_1\omega_1^2 + D_1(z_1+x_1) + E_1\omega_1\right)\\
&\quad\times\cos(q_2z_2)\cos(q_3z_3)\,d\mathbf{z}\\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eee}(\mathbf{z})\\
&\quad\times\cos\left(\left(A_1x_1^2 + D_1x_1 + B_1x_1\omega_1 - \frac{4A_1^2C_1}{B_1^2}x_1^2 - \frac{4A_1C_1}{B_1}\omega_1x_1 - \frac{2A_1}{B_1}x_1\right)\right.\\
&\quad\left.+\left(A_1z_1^2 + B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1 + C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2 + D_1z_1 + E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)\right)\\
&\quad\times\cos(q_2z_2)\cos(q_3z_3)d\mathbf{z}\\
&= \frac{1}{(2\pi)^{3/2}}\cos\left(A_1x_1^2 + D_1x_1 + B_1x_1\omega_1 - \frac{4A_1^2C_1}{B_1^2}x_1^2 - \frac{4A_1C_1}{B_1}\omega_1x_1 - \frac{2A_1}{B_1}x_1\right)\int_{\mathbb{R}^3} f_{eee}(\mathbf{z})\\
&\quad\times\cos\left(A_1z_1^2 + B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1 + C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2 + D_1z_1 + E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)\\
&\quad\times\cos(q_2z_2)\cos(q_3z_3)d\mathbf{z}\\
&\quad-\frac{1}{(2\pi)^{3/2}}\sin\left(A_1x_1^2 + D_1x_1 + B_1x_1\omega_1 - \frac{4A_1^2C_1}{B_1^2}x_1^2 - \frac{4A_1C_1}{B_1}\omega_1x_1 - \frac{2A_1}{B_1}x_1\right)\int_{\mathbb{R}^3} f_{eee}(\mathbf{z})\\
&\quad\times\sin\left(A_1z_1^2 + B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1 + C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2 + D_1z_1 + E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)\\
&\quad\times\cos(q_2z_2)\cos(q_3z_3)d\mathbf{z}
\end{aligned}$$

Moreover, we assume that

$$\begin{aligned}
\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right) &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{eee}(\mathbf{z})\\
&\times\sin\left(A_1z_1^2 + B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1 + C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2 + D_1z_1 + E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)\\
&\times\cos(q_2z_2)\cos(q_3z_3)d\mathbf{z};
\end{aligned}$$

$$\begin{aligned}
\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_oee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right) &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f_{oee}(\mathbf{z})\\
&\times\cos\left(A_1z_1^2 + B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1 + C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2 + D_1z_1 + E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)\\
&\times\cos(q_2z_2)\cos(q_3z_3)d\mathbf{z};
\end{aligned}$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eoe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{eoe}(\mathbf{z})$$
$$\times\sin\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\sin(q_2z_2)\cos(q_3z_3)d\mathbf{z};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eoe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{eoe}(\mathbf{z})$$
$$\times\cos\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\sin(q_2z_2)\cos(q_3z_3)d\mathbf{z};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_oeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{oeo}(\mathbf{z})$$
$$\times\sin\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\cos(q_2z_2)\sin(q_3z_3)d\mathbf{z};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{eeo}(\mathbf{z})$$
$$\times\cos\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\cos(q_2z_2)\sin(q_3z_3)d\mathbf{z};$$

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_ooo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{ooo}(\mathbf{z})$$
$$\times\sin\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\sin(q_2z_2)\sin(q_3z_3)d\mathbf{z};$$

and

$$\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eoo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}f_{eoo}(\mathbf{z})$$
$$\times\cos\left(A_1z_1^2+B_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)z_1+C_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)^2+D_1z_1+E_1\left(\omega_1+\frac{2A_1}{B_1}x_1\right)\right)$$
$$\times\sin(q_2z_2)\sin(q_3z_3)d\mathbf{z}.$$

Then, we have

$$\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{eee}(\mathbf{w})=\cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)$$
$$\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)$$
$$-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)$$
$$\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).$$

On the similar lines, we can show that

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{oee}(\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_oee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &+\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_oee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eee}(\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{oee}(\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_oee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_oee}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eoe}(\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eoe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eoe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{ooe}(\mathbf{w}) &= \cos\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_ooe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_ooe}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eeo}(\mathbf{w}) = \cos&\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{oeo}(\mathbf{w}) = \cos&\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_oeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_oeo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{eoo}(\mathbf{w}) = \cos&\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_eoo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_eoo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

and

$$\begin{aligned}\left(\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{ooo}(\mathbf{w}) = \cos&\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{c_ooo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right)_{s_ooo}\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right).\end{aligned}$$

Invoking (29), we obtain

$$\begin{aligned}\mathscr{Q}^{\mathbb{O},x_1}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right](\mathbf{w}) = \cos&\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\nabla_1 f\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right)\\ &-\sin\left(A_1x_1^2+D_1x_1+B_1x_1\omega_1-\frac{4A_1^2C_1}{B_1^2}x_1^2-\frac{4A_1C_1}{B_1}\omega_1x_1-\frac{2A_1}{B_1}x_1\right)\\ &\times\Delta_1 f\left(\omega_1+\frac{2A_1}{B_1}x_1,\omega_2,\omega_3\right),\end{aligned}$$

where

$$\begin{aligned}\nabla_1 f = &\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eee} - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_oee} e_1 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eoe} e_2 \\ &+ \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_ooe} e_3 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eeo} e_4 + \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_oeo} e_5 \\ &+ \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_eoo} e_6 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_ooo} e_7.\end{aligned}$$

and

$$\begin{aligned}\Delta_1 f = &\left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eee} + \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_oee} e_1 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eoe} e_2 \\ &- \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_ooe} e_3 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eeo} e_4 - \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_oeo} e_5 \\ &+ \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{s_eoo} e_6 + \left(\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right)_{c_ooo} e_7.\end{aligned}$$

Similarly, the translation property with respect to the variables t_2 and t_3 can be studied.

This completes the proof of Theorem 2.

Next, we present the inversion formula for the octonion quadratic-phase Fourier transform (25).

Theorem 4.*If $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]$ is the OQPFT of an octonion-valued function $f \in L^2(\mathbb{R}^3,\mathbb{O})$. Then, f can be recontructed via:*

$$f(\mathbf{t}) = \frac{1}{|B_1B_2B_3|}\int_{\mathbb{R}^3} \mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\,d\mathbf{w}. \tag{30}$$

*Proof.*Invoking the definitions of the QQPFT, the 1-D OQPFT, and the 3-D OQPFT, we have

$$\begin{aligned}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) &= \int_{\mathbb{R}} \mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[f](\omega_1,\omega_2)\mathscr{K}_3^{-e_4}(t_3,\omega_3)dt_3 \\ &= \mathscr{Q}^{\mathbb{O}}_{1,\Omega_3}\left[\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[f]\right](\mathbf{w}).\end{aligned}$$

Using the fact that $f \in L^2(\mathbb{R}^3,\mathbb{O})$, we have $\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[f] \in L^2(\mathbb{R}^3,\mathbb{O})$ and

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\,d\mathbf{w} \\ &\quad= \int_{\mathbb{R}^2} \mathscr{Q}^{\mathbb{O}}_{1,\Omega_3}\left[\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[f]\right](\mathbf{w})\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\,d\omega_1 d\omega_2.\end{aligned}$$

By virtue of (21) and the inversion of QQPFT, we have

$$\begin{aligned}&\frac{1}{|B_1B_2B_3|}\int_{\mathbb{R}^3} \mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\,d\mathbf{w} \\ &\quad= \int_{\mathbb{R}^2} \mathscr{Q}^{\mathbb{O}}_{1,\Omega_3}\left[\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[f]\right](\mathbf{w})\mathscr{K}^{e_4}_{\Omega_3}(t_3,\omega_3)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\,d\omega_1 d\omega_2 \\ &\quad= f(\mathbf{t}).\end{aligned}$$

This concludes the proof.

We denote the even and odd parts of a function $f(t_1,t_2,t_3)$ by $f_e(t_1,t_2,t_3)$ and $f_o(t_1,t_2,t_3)$, respectively. Here, $f_e(t_1,t_2,t_3) = [f(t_1,t_2,t_3)+f(t_1,t_2,-t_3)]/2$, which is only even in the variable t_3 and $f_o(t_1,t_2,t_3) = [f(t_1,t_2,t_3)-f(t_1,t_2,-t_3)]/2$, which is only odd in the variable t_3.

The following lemma shows that the norm of OQPFT splits into four norms of the quaternion functions.

Lemma 3. *Let $f = g + he_4$ be any octonion valued function. Then, we have*

$$\left\|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\right\|^2_{2,\mathbb{O}} = \frac{1}{2\pi}\left(\left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[g_e]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[h_o]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[h_e]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[g_o]\right\|^2_{2,\mathbb{H}}\right). \tag{31}$$

Proof. Since, the octonion valued function f can be represented in the quaternion form as:

$$f = g + he_4, \quad g, h \in \mathbb{H}. \tag{32}$$

Consequently, the OQPFT of f can be represented by

$$\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) = \int_{\mathbb{R}^3} g(\mathbf{t})\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\mathscr{K}^{-e_4}_{\Omega_3}(t_3,\omega_3)\,d\mathbf{t} + \int_{\mathbb{R}^3} h(\mathbf{t})\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)e_4\mathscr{K}^{-e_4}_{\Omega_3}(t_3,\omega_3)\,d\mathbf{t}.$$

By virtue of the even and odd parts of a function, we have

$$\begin{aligned}\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w}) = &\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} g_e(t_1,t_2,t_3)\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\cos(q_3t_3)d\mathbf{t}\\ &+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} h_o(\mathbf{t})\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\sin(q_3t_3)d\mathbf{t}\\ &+\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} h_e(\mathbf{t})\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\cos(q_3t_3)d\mathbf{t}\right.\\ &\left.-\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} g_o(\mathbf{t})\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\sin(q_3t_3)d\mathbf{t}\right)e_4.\end{aligned} \tag{33}$$

From (33), it is clear that the OQPFT can be divided into four QQPFTs. Therefore, the norm of OQPFT splits into four norms of quaternion functions as:

$$\begin{aligned}&\left\|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\right\|^2_{2,\mathbb{O}}\\ &=\frac{1}{2\pi}\left(\left\|\int_{\mathbb{R}^3} g_e(\mathbf{t})\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\cos(q_3t_3)d\mathbf{t} + \int_{\mathbb{R}^3} h_o(\mathbf{t})\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\sin(q_3t_3)d\mathbf{t}\right\|^2_{2,\mathbb{H}}\right.\\ &\left.+\left\|\int_{\mathbb{R}^3} h_e(\mathbf{t})\mathscr{K}^{e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{e_2}_{\Omega_2}(t_2,\omega_2)\cos(q_3t_3)d\mathbf{t} - \int_{\mathbb{R}^3} g_o(\mathbf{t})\mathscr{K}^{-e_1}_{\Omega_1}(t_1,\omega_1)\mathscr{K}^{-e_2}_{\Omega_2}(t_2,\omega_2)\sin(q_3t_3)d\mathbf{t}\right\|^2_{2,\mathbb{H}}\right).\end{aligned}$$

Since $f_e(\mathbf{t})$ and $f_o(\mathbf{t})$ are orthogonal under the L^2-inner product, it follows that

$$\left\|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})\right\|^2_{2,\mathbb{O}} = \frac{1}{2\pi}\left(\left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[g_e]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[h_o]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[h_e]\right\|^2_{2,\mathbb{H}} + \left\|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}[g_o]\right\|^2_{2,\mathbb{H}}\right).$$

This completes the proof of Lemma 3.

Theorem 5. *Let $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f](\mathbf{w})$ be the OQPFT of an octonion-valued function $f \in L^2(\mathbb{R}^3,\mathbb{O})$. Then, we have*

$$\left\|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right\|^2_{2,\mathbb{O}} = 2\pi\left\|f\right\|^2_{2,\mathbb{O}}.$$

*Proof.*Employing Plancherel's theorem for the QQPFT [35], we have

$$\left.\begin{aligned}
\left\|\int_{\mathbb{R}^3} g_e(\mathbf{t})\mathscr{K}_{\Omega_1}^{-e_1}(t_1,\omega_1)\mathscr{K}_{\Omega_2}^{-e_2}(t_2,\omega_2)\cos(q_3t_3)\,d\mathbf{t}\right\|_{2,\mathbb{O}}^2 &= \|g_e\|_{2,\mathbb{H}}^2\\
\left\|\int_{\mathbb{R}^3} h_o(\mathbf{t})\mathscr{K}_{\Omega_1}^{e_1}(t_1,\omega_1)\mathscr{K}_{\Omega_2}^{e_2}(t_2,\omega_2)\sin(q_3t_3)\,d\mathbf{t}\right\|_{2,\mathbb{O}}^2 &= \|h_o\|_{2,\mathbb{H}}^2\\
\left\|\int_{\mathbb{R}^3} h_e(\mathbf{t})\mathscr{K}_{\Omega_1}^{e_1}(t_1,\omega_1)\mathscr{K}_{\Omega_2}^{e_2}(t_2,\omega_2)\cos(q_3t_3)\,d\mathbf{t}\right\|_{2,\mathbb{O}}^2 &= \|h_e\|_{2,\mathbb{H}}^2\\
\left\|\int_{\mathbb{R}^3} g_o(\mathbf{t})\mathscr{K}_{\Omega_1}^{-e_1}(t_1,\omega_1)\mathscr{K}_{\Omega_2}^{-e_2}(t_2,\omega_2)\sin(q_3t_3)\,d\mathbf{t}\right\|_{2,\mathbb{O}}^2 &= \|g_o\|_{2,\mathbb{H}}^2
\end{aligned}\right\}. \tag{34}$$

Invoking (32), the norm of any octonion-valued function f can be written as:

$$\begin{aligned}
\|f\|_{2,\mathbb{O}}^2 &= \|g+he_4\|_{2,\mathbb{O}}^2\\
&= \|g\|_{2,\mathbb{H}}^2 + \|h\|_{2,\mathbb{H}}^2\\
&= \|g_e\|_{2,\mathbb{H}}^2 + \|h_o\|_{2,\mathbb{H}}^2 + \|h_e\|_{2,\mathbb{H}}^2 + \|g_o\|_{2,\mathbb{H}}^2.
\end{aligned} \tag{35}$$

By combining (31), (34) and (35), we have

$$\left\|\mathscr{Q}_{\Omega_1,\Omega_2,\Omega_3}^{\mathbb{O}}[f]\right\|_{2,\mathbb{O}}^2 = 2\pi\left\|f\right\|_{2,\mathbb{O}}^2. \tag{36}$$

This concludes the proof of Theorem 5.

4 Uncertainty Principles Associated with Octonion Quadratic-phase Fourier Transform

The uncertainty principle is a fundamental concept in both harmonic analysis and signal processing [36]. Originally derived from quantum mechanics, this principle asserts that the position and momentum of a particle cannot both be precisely determined. Instead, they can only be described probabilistically, with a certain degree of uncertainty. In essence, greater precision in determining a particle's position leads to increased uncertainty in its momentum, and vice versa. In harmonic analysis, uncertainty principles are interpreted as follows: a nontrivial function cannot be simultaneously well-localized in both the time and frequency domains [37,38]. Given that the proposed transform represents a parameterized continuum of transforms encompassing several widely used types, our objective is to derive a Heisenberg's and logarithmic uncertainty inequalities for the proposed OQPFT (25).

Theorem 6.*Let $\mathscr{Q}_{\Omega_1,\Omega_2,\Omega_3}^{\mathbb{O}}[f]$ be the OQPFT of any octonion-valued signal $f\in L^2(\mathbb{R}^3,\mathbb{O})$, then the following inequality holds:*

$$\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2,\Omega_3}^{\mathbb{O}}[f]\right|^2 d\mathbf{w}\int_{\mathbb{R}^3}|\mathbf{t}|^2|f(\mathbf{t})|^2 d\mathbf{t} \geq \frac{|B_3|}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|f(\mathbf{t})|^2 d\mathbf{t}\right)^{1/2}. \tag{37}$$

*Proof.*By virtue of (31), we have

$$\begin{aligned}
\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2,\Omega_3}^{\mathbb{O}}[f]\right|^2 d\mathbf{w} = \frac{|B_3|}{2\pi}\Bigg(&\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2}^{\mathbb{H}}[g_e]\right|^2 d\mathbf{w} + \int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2}^{\mathbb{H}}[h_o]\right|^2 d\mathbf{w}\\
&+\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2}^{\mathbb{H}}[h_e]\right|^2 d\mathbf{w} + \int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}_{\Omega_1,\Omega_2}^{\mathbb{H}}[g_o]\right|^2 d\mathbf{w}\Bigg).
\end{aligned} \tag{38}$$

Invoking (34) yields

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\mathbf{t}|^2|f(\mathbf{t})|^2 d\mathbf{t}\\
&= \int_{\mathbb{R}^3}|\mathbf{t}|^2|g_e(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3}|\mathbf{t}|^2|h_o(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3}|\mathbf{t}|^2|h_e(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3}|\mathbf{t}|^2|g_o(\mathbf{t})|^2 d\mathbf{t}.
\end{aligned} \tag{39}$$

The Heisenberg's uncertainty principle for the QQPFT is given by [35]

$$\left.\begin{array}{l}\int_{\mathbb{R}^3}|\mathbf{t}|^2|g_e(\mathbf{t})|^2d\mathbf{t}\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[g_e\right]\right|^2d\mathbf{w}\geq\frac{1}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|g_e(\mathbf{t})|^2d\mathbf{t}\right)^{1/2}\\ \int_{\mathbb{R}^3}|\mathbf{t}|^2|h_o(\mathbf{t})|^2d\mathbf{t}\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[h_o\right]\right|^2d\mathbf{w}\geq\frac{1}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|h_o(\mathbf{t})|^2d\mathbf{t}\right)^{1/2}\\ \int_{\mathbb{R}^3}|\mathbf{t}|^2|h_e(\mathbf{t})|^2d\mathbf{t}\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[h_e\right]\right|^2d\mathbf{w}\geq\frac{1}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|h_e(\mathbf{t})|^2d\mathbf{t}\right)^{1/2}\\ \int_{\mathbb{R}^3}|\mathbf{t}|^2|g_o(\mathbf{t})|^2d\mathbf{t}\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[g_o\right]\right|^2d\mathbf{w}\geq\frac{1}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|g_o(\mathbf{t})|^2d\mathbf{t}\right)^{1/2}\end{array}\right\}. \tag{40}$$

Summarizing (38), (39) and (40), we obtain

$$\int_{\mathbb{R}^3}|\mathbf{w}|^2\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right|^2d\mathbf{w}\int_{\mathbb{R}^3}|\mathbf{t}|^2|f(\mathbf{t})|^2d\mathbf{t}\geq\frac{|B_3|}{4|B_1B_2|^2}\left(\int_{\mathbb{R}^3}|f(\mathbf{t})|^2d\mathbf{t}\right)^{1/2}.$$

This completes the proof of Theorem 6.

In continuation, we shall derive the logarithmic uncertainty inequality for the octonion quadratic-phase Fourier transform (25). Prior to that, we have the following definition.

Definition 3. *For any two indices $(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3)\in\mathbb{Z}^3\setminus\{0\}$, the Schwartz space in $L^2(\mathbb{R}^3,\mathbb{O})$ is defined by*

$$\mathscr{S}\left(\mathbb{R}^3,\mathbb{O}\right)=\left\{f\in C^\infty\left(\mathbb{R}^3,\mathbb{O}\right):\sup_{(t_1,t_2,t_3)\in\mathbb{R}^3}\left|t_1^{\alpha_1}t_2^{\alpha_2}t_3^{\alpha_3}\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}f(t_1,t_2,t_3)\right|<\infty\right\},$$

where $C^\infty\left(\mathbb{R}^3,\mathbb{O}\right)$ is the class of octonion-valued smooth functions and ∂ denoted the differential operator in the octonion domain.

Theorem 7. *Let $\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]$ be the OQPFT of any octonion-valued function $f\in L^2(\mathbb{R}^3,\mathbb{O})$. Then, the following logarithmic inequality holds:*

$$\int_{\mathbb{R}^3}\ln|\mathbf{t}|\,|f(\mathbf{t})|^2d\mathbf{t}+\frac{2\pi}{|B_3|}\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right|^2d\mathbf{w}=4\left(N-\ln|B_1B_2|\right)\int_{\mathbb{R}^3}|f(\mathbf{t})|^2d\mathbf{t}, \tag{41}$$

where $N=\frac{1}{2}\left(\frac{\Gamma'(3/4)}{\Gamma(3/4)}\right)$ and $B_1,B_2,B_3\neq 0$.

Proof. For any $f\in\mathscr{S}\left(\mathbb{R}^3,\mathbb{O}\right)\subseteq L^2(\mathbb{R})$, the logarithmic uncertainty principle for the quaternion QPFT reads [35]:

$$\int_{\mathbb{R}^3}\ln|\mathbf{t}||f(\mathbf{t})|^2d\mathbf{t}+\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right|^2d\mathbf{w}\geq\left(N-\ln|B_1B_2|\right)\int_{\mathbb{R}^3}|f(\mathbf{t})|^2d\mathbf{t}. \tag{42}$$

Moreover, relation (31) yields

$$\begin{aligned}\frac{2\pi}{|B_3|}\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}\left[f\right]\right|^2d\mathbf{w}&\\ =\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[g_e\right]\right|^2d\mathbf{w}&+\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[h_o\right]\right|^2d\mathbf{w}\\ &+\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[h_e\right]\right|^2d\mathbf{w}+\int_{\mathbb{R}^3}\ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{H}}_{\Omega_1,\Omega_2}\left[g_o\right]\right|^2d\mathbf{w}.\end{aligned} \tag{43}$$

By employing (34), we have

$$\begin{aligned}\int_{\mathbb{R}^3}\ln|\mathbf{t}|\,|f(\mathbf{t})|^2d\mathbf{t}=\int_{\mathbb{R}^3}\ln|\mathbf{t}||g_e(\mathbf{t})|^2d\mathbf{t}+\int_{\mathbb{R}^3}\ln|\mathbf{t}||h_o(\mathbf{t})|^2d\mathbf{t}+\int_{\mathbb{R}^3}\ln|\mathbf{t}||h_e(\mathbf{t})|^2d\mathbf{t}&\\ +\int_{\mathbb{R}^3}\ln|\mathbf{t}||g_o(\mathbf{t})|^2d\mathbf{t}.&\end{aligned} \tag{44}$$

By virtue of (42), we have

$$\left.\begin{aligned}
\int_{\mathbb{R}^3} \ln|\mathbf{t}||g_e(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3} \ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[g_e]\right|^2 d\mathbf{w} \geq (N - \ln|B_1B_2|)\int_{\mathbb{R}^3} |g_e(\mathbf{t})|^2 d\mathbf{t} \\
\int_{\mathbb{R}^3} \ln|\mathbf{t}||h_o(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3} \ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[h_o]\right|^2 d\mathbf{w} \geq (N - \ln|B_1B_2|)\int_{\mathbb{R}^3} |h_o(\mathbf{t})|^2 d\mathbf{t} \\
\int_{\mathbb{R}^3} \ln|\mathbf{t}||h_e(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3} \ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[h_e]\right|^2 d\mathbf{w} \geq (N - \ln|B_1B_2|)\int_{\mathbb{R}^3} |h_e(\mathbf{t})|^2 d\mathbf{t} \\
\int_{\mathbb{R}^3} \ln|\mathbf{t}||g_e(\mathbf{t})|^2 d\mathbf{t} + \int_{\mathbb{R}^3} \ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[g_e]\right|^2 d\mathbf{w} \geq (N - \ln|B_1B_2|)\int_{\mathbb{R}^3} |g_e(\mathbf{t})|^2 d\mathbf{t}
\end{aligned}\right\}. \tag{45}$$

Summarizing (45), (43) and (44), we obtain

$$\int_{\mathbb{R}^3} \ln|\mathbf{t}|\,|f(\mathbf{t})|^2 d\mathbf{t} + \frac{2\pi}{|B_3|}\int_{\mathbb{R}^3} \ln|\mathbf{w}|\left|\mathscr{Q}^{\mathbb{O}}_{\Omega_1,\Omega_2,\Omega_3}[f]\right|^2 d\mathbf{w} \geq 4(N - \ln|B_1B_2|)\int_{\mathbb{R}^3} |f(\mathbf{t})|^2 d\mathbf{t}.$$

This completes the proof of Theorem 7.

5 Conclusion

In this article, we introduced a novel integral transform within the framework of octonions. We analyzed its fundamental mathematical properties and investigated the effects of translation on the transform. Additionally, we derived uncertainty principles specific to the OQPFT. These results provide valuable insights for the mathematical and signal processing communities, offering both practical and theoretical contributions.

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