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Improper Injective Coloring Parameters of Certain Cycle-Related Graphs

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Abstract: Any vertex coloring of a graph can be viewed as a random experiment of assigning colors to the vertices, whose random variable is defined based on the number of vertices assigned a particular color in that coloring. Based on this, the statistical parameters of mean and variance have been extended to chromatic mean and chromatic variance for various vertex colorings of graphs in the literature. In this article, the ideas of chromatic mean and chromatic variance of cycle-related graphs with respect to their improper injective coloring are investigated.

Keywords: Improper coloring; injective coloring; chromatic sum; chromatic mean; chromatic variance.

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1 Introduction

By a graph G, we always mean a simple, undirected and finite graph having its vertex set $V(G)$ and edge set $E(G)$. For terminology in graph theory, the reader may refer to [14], and for concepts in statistics, see [11].

The *open neighbourhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex $v \in V(G)$, denoted by $d(v)$, is the number of vertices adjacent with v in G; that is, $\deg(v) = |N(v)|$ and $\Delta(G) = \max\{|N(v)| : v \in V(G)\}$ is the maximum degree of a graph G. A vertex $v \in V(G)$ with $|N(v)| = 1$ is called a *leaf* or a *pendant vertex* and a vertex $v \in V(G)$ with $d(v) = |V(G)| - 1$ is called a *universal vertex* of G.

Graph coloring is the assignment of colors or labels to the vertices or edges of the graph, according to specific rules, where the set of all entities assigned the same color is called a *color class*. A *proper vertex coloring* of a graph G is the assignment of colors to the vertices of G such that no two adjacent vertices are assigned the same color. An *improper vertex coloring* of a graph G assigns colors to the vertices of G such that adjacent vertices can be assigned the same color. Different vertex coloring protocols are introduced in the literature, as the idea of graph coloring emerged as a tool to model real-life problems, and according to the problems that are to be modelled and solved, these coloring protocols are defined (see [2,3,4,7,8]).

Consider any optimal vertex coloring $c : V(G) \to \{c_i : 1 \le i \le k\}$ of a graph G with $V_i = \{v \in V(G) : c(v) = c_i\}$; $1 \le i \le k$. This can be viewed as a random experiment, having the discrete random variable (r.v.) X as i, and accordingly, a probability mass function (p.m.f.) for this random variable X is also defined as follows.

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$$f(i) = \begin{cases} \frac{|V_i|}{|V(G)|}, & 1 \leq i \leq k; \\ 0, & elsewhere. \end{cases}$$

This idea of viewing the optimal assignment of colors to the vertices as a random experiment was given in [13], where the *chromatic mean* $\mu_c(G)$ and the *chromatic variance* $\sigma_c^2(G)$ of any optimal vertex coloring c of a graph G was defined analogous to the mean and variance of random variables, based on the above mentioned p.m.f. $f(i)$, as $\mu_c(G) = \frac{\omega_c(G)}{|V(G)|}$ and $\sigma_c^2(G) = \sum\limits_{i=1}^{k} i^2 f(i) - \left[\frac{\omega_c(G)}{|V(G)|}\right]^2$, respectively, where $\omega_c(G) = \sum\limits_{i=1}^{k} i|V_i|$ is the *chromatic sum* or the *coloring sum* of G with respect to c.

On introducing these statistical parameters with respect to any optimal vertex coloring of graphs, these protocol-specific chromatic mean and variance were obtained for certain standard graph classes [1,6,5,9,12]. These ideas were extended to improper coloring of graphs by defining and determining these statistical chromatic parameters for improper injective coloring of graphs in [10], where an improper injective coloring of graphs and the notions of chromatic sum, chromatic mean and the chromatic variance for this improper injective coloring of graphs that were defined in [10] are given below. In this article, we determine the chromatic mean and chromatic variance of graphs with respect to their improper injective coloring for certain cycle-related graphs.

Definition 1.[10]. A vertex coloring of a graph G in which no vertex of G is adjacent to more than one vertex of the same color class is called an *injective coloring* of G and the minimum number of colors used to obtain such a coloring is called the *injective chromatic number* of G, denoted by $\chi_i(G)$. An injective coloring of G that uses $\chi_i(G)$ colors is called a χ_i*-coloring* of G.

(a) A proper coloring of a graph. **(b)** A χ_i-coloring of a graph.

Fig. 1 A comparison between the proper coloring and χ_i-coloring of graphs.

Definition 2.[10] Let $c : V(G) \to \{c_i : 1 \leq i \leq \chi_i(G)\}$ be a χ_i-coloring of G, where each color class is denoted by $\mathscr{C}_i$. That is, $\mathscr{C}_i = \{v \in V(G) : c(v) = c_i\}$; $1 \leq i \leq \chi_i(G)$. The *injective coloring sum* or the *injective chromatic sum* of the coloring c of G is defined as $\omega_c^i = \sum\limits_{i=1}^{\chi_i(G)} i|\mathscr{C}_i|$.

Definition 3.[10] The χ_i*-chromatic mean* (resp. χ_i^+*-chromatic mean*) of G, denoted by $\mu_{\chi_i}(G)$ (resp. $\mu_{\chi_i^+}(G)$), is the coloring mean of G with respect to an optimal injective coloring c of G, which has the minimum (resp. maximum) injective coloring sum.

Definition 4.[10] The χ_i*-chromatic variance* (resp. χ_i^+*-chromatic variance*) of G, denoted by $\sigma^2_{\chi_i}(G)$ (resp. $\sigma^2_{\chi_i^+}(G)$), is the coloring variance of G with respect to an optimal injective coloring c of G, which has the minimum (resp. maximum) injective coloring sum.

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2 Results and Discussion

In this section, we discuss the χ_i-chromatic mean and χ_i-chromatic variance of certain cycle-related graphs, where we first begin by obtaining the χ_i-chromatic mean and χ_i-chromatic variance of sunlet graphs.

A *sunlet graph* Sl_t of order $2t$ is a graph obtained by adjoining a pendant vertex to each vertex of the cycle C_t.

Theorem 1. *For a sunlet graph Sl_t; $t \geq 4$, of order $n = 2t$,*

$$\mu_{\chi_i}(Sl_t) = \begin{cases} \frac{7}{4}, & t \equiv 0 \pmod 4; \\ \frac{7t+6}{4t}, & t \equiv 2 \pmod 4; \\ \frac{7t+3}{4t}, & otherwise. \end{cases}$$

and

$$\sigma^2_{\chi_i}(Sl_t) = \begin{cases} \frac{11}{16}, & t \equiv 0 \pmod 4; \\ \frac{11t^2+4t-36}{16t^2}, & t \equiv 2 \pmod 4; \\ \frac{11t^2+2t-9}{16t^2}, & otherwise. \end{cases}$$

Proof. Let Sl_t be a sunlet graph of order $2t$ such that v_i; $1 \leq i \leq t$, are the vertices of the cycle C_t in Sl_t, and u_i; $1 \leq i \leq t$, are the pendant vertices adjoined to the corresponding v_i's of C_t in Sl_t. Since Sl_t has a vertex of degree 3, $\chi_i(Sl_t) \geq 3$, for all t. As the assignment of colors depends on the values of t, we obtain the χ_i-chromatic mean and variance of the graph Sl_t in the following cases.

Case 1: Let $t \equiv 0 \pmod 4$. Consider the coloring $c : V(Sl_t) \to \{c_1, c_2, c_3\}$ such that $c(v_i) = c_3; i \equiv 1,2 \pmod 4$, $c(v_i) = c_2; i \equiv 3,0 \pmod 4$, $c(u_i) = c_1$, for all $1 \leq i \leq t$. It is a χ_i-coloring of Sl_t having its p.m.f. as follows.

$$f(i) = \begin{cases} \frac{1}{2}, & i = 1; \\ \frac{1}{4}, & i = 2,3; \\ 0, & \text{otherwise.} \end{cases}$$

It is a χ_i-coloring of Sl_t with minimum chromatic sum as any χ_i-coloring of C_t requires two colors, and therefore, all the pendant vertices of Sl_t must be colored with the third color. Also, as $t \equiv 0 \pmod 4$, the two colors used to color the vertices of C_t are assigned to exactly the same number vertices of C_t. Hence, the chromatic sum is minimised by coloring all the pendant vertices with the color c_1. Therefore, the χ_i-chromatic mean and the χ_i-chromatic variance $Sl_t; t \equiv 0 \pmod 4$, are $\mu_{\chi_i}(Sl_t) = \frac{7}{4}$ and $\sigma^2_{\chi_i}(Sl_t) = \frac{11}{16}$, respectively.

Case 2: When $t \equiv 2 \pmod 4$, we require all three colors, say c_1, c_2 and c_3, to color the vertices of C_t, in any χ_i-coloring of the sunlet graph Sl_t because, in this case, as we proceed coloring with just 2 colors, the vertices v_{t-1} and v_t will be assigned the color c_1. As the vertices v_1 and v_2 are already assigned the color c_1, the vertices v_t and v_1 have two neighbours colored using the color c_1. As the vertices v_{t-2} and v_{t-3} are assigned the color c_2, the vertex v_{t-1} cannot be assigned the color c_1 nor c_2. Therefore, it has to be assigned the color c_3. Similarly, the vertex v_t can also be given only the color c_3, based on the colors of the vertices v_{t-2} and v_2. Note that the choice of these vertices in a C_t can be any consecutive vertices in the C_t, and hence without loss of generality, we had considered the vertices v_{t-2} to v_1, for the discussion. Therefore, in any χ_i-coloring of Sl_t, when $t \equiv 2 \pmod 4$, all u_i; $1 \leq i \leq t$, also require three colors to be colored. Therefore, we minimise the coloring sum by assigning the maximum number of pendant vertices the color c_1, which is possible when the minimum number of v_i's are assigned the color c_1.

Based on this, we obtain a χ_i-coloring $c : V(C_n) \to \{c_1, c_2, c_3\}$ such that $c(v_i) = c(u_{t-1}) = c(u_{t-2}) = c_3; i \equiv 1,2 \pmod 4$, $1 \leq i \leq t-2$, $c(v_i) = c(u_t) = c(u_1) = c_2; i \equiv 3,0 \pmod 4$, $c(v_{t-1}) = c(v_t) = c(u_i) = c_1; 2 \leq i \leq t-3$, having minimum chromatic sum, whose p.m.f. is given as follows.

$$f(i) = \begin{cases} \frac{t-2}{n}, & i = 1; \\ \frac{t+2}{2n}, & i = 2,3; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the χ_i-chromatic mean $\mu_{\chi_i}(Sl_t) = \frac{t-2}{n} + \frac{5(t+2)}{2n} = \frac{7t+6}{2n}$ and the χ_i-chromatic variance $\sigma^2_{\chi_i}(Sl_t) = \frac{t-2}{n} + \frac{13(t+2)}{2n} - (\frac{7t+6}{2n})^2 = \frac{11t^2+4t-36}{16t^2}$.

Case 3: Let $t \equiv 1,3 \pmod 4$. In this case, there is a χ_i-coloring of C_t that uses one out of the three colors exactly to one vertex of C_t. Based on the value of t, the position of the vertex assigned this color changes, and hence, we obtain such a χ_i-coloring of Sl_t in two cases as follows.

Subcase 3a: When $t \equiv 1 \pmod 4$, the coloring $c : V(Sl_t) \to \{c_1, c_2, c_3\}$ such that $c(v_i) = c(u_{t-1}) = c_3$; $i \equiv 1,2 \pmod 4$, $i \neq t$; $c(v_i) = c(u_1) = c_2$; $i \equiv 3,0 \pmod 4$, and $c(v_t) = c(u_t) = c(u_i) = c_1$; $1 \leq i \leq t-2$, is such a χ_i-coloring of Sl_t.

Subcase 3b: When $t \equiv 3 \pmod 4$, the coloring $c : V(Sl_t) \to \{c_1, c_2, c_3\}$ such that $c(v_i) = c(u_{t-2}) = c_3$; $i \equiv 1,2 \pmod 4$, $i \neq t-1$; $c(v_i) = c(v_t) = c(u_t) = c_2$; $i \equiv 3,0 \pmod 4$, and $c(v_{t-1}) = c(u_i) = c(u_{t-1}) = c_1$; $1 \leq i \leq t-3$, is a χ_i-coloring of Sl_t, as described before.

In both cases, the coloring c gives the minimum chromatic sum as assigning the color c_1 to one of the v_i's is inevitable, and this leads to the situation where two u_i's cannot be assigned the color c_1. Hence, we assign them with the colors c_2 and c_3, one each. In this assignment, the color c_1 is assigned to the maximum number of vertices and the other two colors are assigned to the equal number of vertices, yielding the following p.m.f.

$$f(i) = \begin{cases} \frac{t-1}{n}, & i = 1; \\ \frac{t+1}{2n}; & i = 2,3; \\ 0, & otherwise. \end{cases}$$

Hence, the χ_i-chromatic mean $\mu_{\chi_i}(Sl_t) = \frac{t-1}{n} + \frac{5(t+1)}{2n} = \frac{7t+3}{2n}$ and the χ_i-chromatic variance $\sigma^2_{\chi_i}(Sl_t) = \frac{t-1}{n} + \frac{13(t+1)}{2n} - (\frac{7t+3}{2n})^2 = \frac{11t^2+2t-9}{16t^2}$.

Theorem 2. *For a sunlet graph $Sl_t; t \geq 6$, of order $n = 2t$,*

$$\mu_{\chi_{inj^+}}(Sl_t) = \begin{cases} \frac{9}{4}, & t \equiv 0 \pmod 4; \\ \frac{9t-6}{4t}, & t \equiv 2 \pmod 4; \\ \frac{9t-3}{4t}, & otherwise. \end{cases}$$

and

$$\sigma^2_{\chi_{inj^+}}(Sl_t) = \begin{cases} \frac{11}{16}, & t \equiv 0 \pmod 4; \\ \frac{11t^2+4t-36}{16t^2}, & t \equiv 2 \pmod 4; \\ \frac{11t^2+2t-9}{16t^2}, & otherwise. \end{cases}$$

Proof. Let Sl_t be a sunlet graph as described in Theorem 1. We obtain a χ_i-coloring c' of the graph Sl_t based on the χ_i-coloring c of the graph Sl_t given by swapping the colors c_1 to c_3, in its respective cases of t. The minimality of the chromatic sum of Sl_t in Theorem 1 assures that the chromatic sum for this coloring obtained by the swapping of labels is maximum, from which we obtain the χ_i^+-chromatic mean and χ_i^+-chromatic variance of the graph Sl_t.

Case 1: Let $t \equiv 0 \pmod 4$. The p.m.f. for the coloring c' is obtained as follows.

$$f(i) = \begin{cases} \frac{1}{4}, & i = 1,2; \\ \frac{1}{2}, & i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the χ_i^+-chromatic mean and the χ_i^+-chromatic variance of the graph $Sl_t; t \equiv 0 \pmod 4$, $\mu_{\chi_i^+}(Sl_t) = \frac{9}{4}$ and $\sigma^2_{\chi_i^+}(Sl_t) = \frac{11}{16}$.

Case 2: When $t \equiv 2 \pmod 4$, we obtain a p.m.f. of the χ_i-coloring c' with a maximum chromatic sum as follows.

$$f(i) = \begin{cases} \frac{t+2}{2n}, & i = 1,2; \\ \frac{t-2}{n}, & i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the χ_i^+-chromatic mean $\mu_{\chi_i^+}(Sl_t) = \frac{3(t+2)}{2n} + \frac{3(t-2)}{n} = \frac{9t-6}{2n}$ and the χ_i^+-chromatic variance $\sigma^2_{\chi_i^+}(Sl_t) = \frac{5(t+2)}{2n} + \frac{9(t-2)}{n} - (\frac{9t-6}{2n})^2 = \frac{11t^2+4t-36}{16t^2}$.

Case 3: When $t \equiv 1,3 \pmod 4$, the χ_i-coloring c' gives the maximum chromatic sum as assigning the color c_3 to one of the v_i's and $t-2$ u_i's, which is the maximum possible number of vertices of Sl_t to which a color can be assigned. In this assignment, the other two colors are assigned to an equal number of vertices, yielding the following p.m.f.

$$f(i) = \begin{cases} \frac{t+1}{2n}; & i = 1,2; \\ \frac{t-1}{n}, & i = 3; \\ 0, & otherwise. \end{cases}$$

Therefore, the χ_i^+-chromatic mean $\mu_{\chi_i^+}(Sl_t) = \frac{3(t+1)}{2n} + \frac{3(t-1)}{n} = \frac{9t-3}{2n}$ and the χ_i^+-chromatic variance $\sigma^2_{\chi_i^+}(Sl_t) = \frac{5(t+1)}{2n} + \frac{9(t-1)}{n} - (\frac{9t-3}{2n})^2 = \frac{11t^2+2t-9}{16t^2}$.

A *wheel graph* of order $n = t+1$, denoted by $W_{1,t}$, is a graph obtained by making a vertex, say v, adjacent to all the vertices of the cycle C_t. A *helm graph* of order $n = 2s+1$, denoted by $H_{1,t,t}$, is obtained by attaching a leaf to each vertex of degree 3 in a wheel graph $W_{1,t}$.

Theorem 3. *For a helm graph $H_{1,t,t}$; $t \geq 3$, of order $n = 2t+1$,*

(i) *the χ_i-chromatic mean,* $\mu_{\chi_i}(H_{1,t,t}) = \begin{cases} \frac{16}{7}, & t = 3; \\ \frac{(t+2)(t+3)}{2n}, & t \geq 4. \end{cases}$

(ii) *the χ_i-chromatic variance,* $\sigma^2_{\chi_i}(H_{1,t,t}) = . \begin{cases} \frac{44}{7}, & t = 3; \\ \frac{t^4-8t^3-67t^2-96t-108}{12n^2}, & t \geq 4. \end{cases}$

Proof. Let $H_{1,t,t}$; $t \geq 3$, be a helm graph such that $V(H_{1,t,t}) = \{v, v_i, u_i : 1 \leq i \leq t\}$, where the subgraph induced by the vertices $\{v\} \cup \{v_i; 1 \leq i \leq t\}$ is the wheel graph $W_{1,t}$ with v as the vertex with degree t, and u_i's are the pendant vertices that are adjacent to the corresponding v_i's. Any χ_i-coloring of the wheel graph $W_{1,t}$ requires $t+1$ colors. As the helm graph $H_{1,t,t}$ contains a wheel graph $W_{1,t}$ as an induced subgraph, $\chi_i(H_{1,t,t}) \geq t+1$.

Consider the coloring $c : V(H_{1,t,t}) \to \{c_i : 1 \leq i \leq t+1\}$ such that $c(v) = c_{t+1}$, $c(v_i) = c_i$; $1 \leq i \leq t$, $c(u_i) = c_1$; $3 \leq i \leq t-1$, $c(u_1) = c_1$, $c(u_t) = c(u_2) = c_2$. It can be seen that c is a χ_i-coloring of $H_{1,t,t}$ with $t+1$ colors and hence $\chi_i(H_{1,t,t}) = t+1$. We obtain the result in the following cases as $W_{1,3} \equiv K_4$.

Case 1. When $t = 3$, the assignment $c(u_3) = c_2$ as mentioned above violates the injective coloring protocol as v_3 is adjacent to v_2. Hence, in this case, the only possibility is the assignment $c(u_3) = c_3$. Based on this assignment, the p.m.f. of the graph $H_{1,3,3}$ is given as follows.

$$f(i) = \begin{cases} \frac{2}{7}; & i = 1,2,3; \\ \frac{1}{7}; & i = 4; \\ 0, & otherwise. \end{cases}$$

Hence, the χ_i-chromatic mean $\mu_{\chi_i}(H_{1,3,3}) = \frac{16}{7}$ and the χ_i-chromatic variance $\sigma^2_{\chi_i}(H_{1,3,3}) = \frac{44}{7}$.

Case 2. The p.m.f. with respect to the χ_i-coloring c of the graph $H_{1,t,t}$; $t \geq 4$, is given as follows.

$$f(i) = \begin{cases} \frac{t-1}{n}, & i = 1; \\ \frac{3}{n}; & i = 2; \\ \frac{1}{n}; & i = 3,4,\ldots,t+1; \\ 0, & otherwise. \end{cases}$$

As all the v_i's in $H_{1,t,t}$ must be assigned unique colors in any χ_i-coloring of the graph $H_{1,t,t}$, the only way to minimise the injective chromatic sum of any χ_i-coloring of a helm graph $H_{1,t,t}$ is by assigning the maximum number of pendant vertices with the colors c_1 and c_2. As c_1 is assigned to one of the v_i's of $H_{1,t,t}$, it can be assigned to a maximum of $(t-2)$ u_i's of $H_{1,t,t}$, in order to adhere to the injective coloring protocol. To get a minimum injective coloring sum, these remaining two u_i's must be assigned the color c_2. Based on this strategy, we observe that the given χ_i-coloring c of $H_{1,t,t}$ has the minimum chromatic sum. Therefore, based on the assignment c of $H_{1,t,t}$, the χ_i-chromatic mean and the

χ_i-chromatic variance of the graph $H_{1,t,t}$ are determined using the p.m.f. with respect to the χ_i-coloring c of the graph $H_{1,t,t}$ as follows.

$$\mu_{\chi_i}(H_{1,t,t}) = \frac{t-1}{n} + \frac{6}{n} + \frac{1}{n}\sum_{i=3}^{t+1} i = \frac{(t+2)(t+3)}{2n}.$$

$$\sigma^2_{\chi_i}(H_{1,t,t}) = \frac{t-1}{n} + \frac{12}{n}) + \frac{1}{n}\sum_{i=3}^{t+1} i^2 - (\frac{(t+2)(t+3)}{2(n)})^2 = \frac{t^4-8t^3-67t^2-96t-108}{12n^2}.$$

Theorem 4. *For a helm graph $H_{1,t,t}$; $t \geq 3$, of order $n = 2t+1$,*

(i) the χ_i^+-chromatic mean, $\mu_{\chi_i^+}(H_{1,t,t}) = \begin{cases} \frac{19}{7}, & t = 3; \\ \frac{(t+6)(t-1)}{2n}, & t \geq 4. \end{cases}$

(ii) the χ_i^+-chromatic variance, $\sigma^2_{\chi_i^+}(H_{1,t,t}) = \begin{cases} \frac{30}{7}, & t = 3; \\ \frac{35t^4+130t^3+61t^2-106t+78}{12n^2}, & t \geq 4. \end{cases}$

Proof. For the helm graph $H_{1,t,t}$; $t \geq 4$, as described in Theorem 3, consider the injective coloring $c' : V(H_{1,t,t}) \to \{c_i : 1 \leq i \leq t+1\}$ such that $c'(v) = c_1$, $c'(v_i) = c_{i+1}$; $1 \leq i \leq t$, $c'(u_i) = c_{t+1}$; $2 \leq i \leq t-2$, and $c'(u_1) = c'(u_{t-1}) = c_t$, for $t \geq 4$. We can see that the χ_i-coloring c' of $H_{1,t,t}$ is obtained by swapping the colors of the vertices assigned c_1 and c_2 in the assignment c given in Theorem 3 with the colors c_{t+1} and c_t, respectively, and vice-versa; in order to obtain the maximum injective chromatic sum of the graph $H_{1,t,t}$. Therefore, based on the χ_i-coloring c' of $H_{1,t,t}$, the χ_i^+-chromatic mean and the χ_i^+-chromatic variance of $H_{1,t,t}$ are determined in two cases, as given below.

Case 1. When $t = 3$, the assignment $c'(v) = c_1$, $c'(v_i) = c'(u_i) = c_i$; $2 \leq i \leq 4$, is a χ_i-coloring with maximum chromatic sum. Hence, the p.m.f. of the graph $H_{1,3,3}$ corresponding to this assignment is given as follows.

$$f(i) = \begin{cases} \frac{1}{7}; & i = 1; \\ \frac{2}{7}; & i = 2,3,4; \\ 0, & otherwise. \end{cases}$$

Hence, the χ_i^+-chromatic mean $\mu_{\chi_i^+}(H_{1,3,3}) = \frac{19}{7}$ and the χ_i^+-chromatic variance $\sigma^2_{\chi_i^+}(H_{1,3,3}) = \frac{30}{7}$.

Case 2. The p.m.f. with respect to the χ_i-coloring c' of the graph $H_{1,t,t}$; $t \geq 4$, is given as follows.

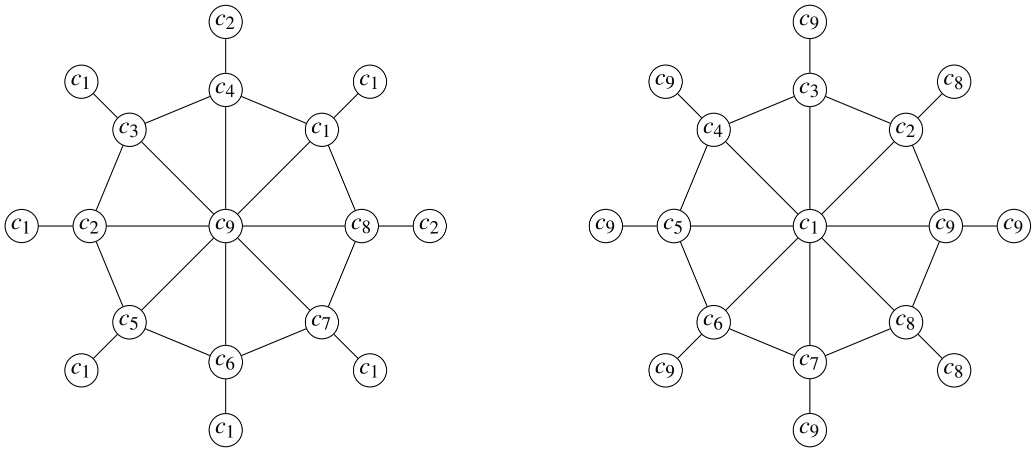
$$f(i) = \begin{cases} \frac{1}{n}, & 1 \leq i \leq t-1; \\ \frac{3}{n}, & i = t; \\ \frac{t-1}{n}, & i = t+1; \\ 0, & otherwise. \end{cases}$$

Therefore, the χ_i^+-chromatic mean $\mu_{\chi_i^+}(H_{1,t,t}) = \frac{(t+1)(t-1)}{n} + \frac{3t}{n} + \frac{1}{n}\sum_{i=1}^{t-1} i = \frac{(t+6)(t-1)}{2n}$ and the χ_i^+-chromatic variance $\sigma^2_{\chi_i^+}(H_{1,t,t}) = \frac{(t+1)^2(t-1)}{n} + \frac{3t^2}{n} + \frac{1}{n}\sum_{i=1}^{t-1} i^2 - (\frac{(t+6)(t-1)}{2n})^2 = \frac{35t^4+130t^3+61t^2-106t+78}{12n^2}$.

A *closed helm graph* $CH_{1,t,t}$ is obtained by making the each pendant vertex u_i; $1 \leq i \leq t$, of the helm graph $H_{1,t,t}$ adjacent to the vertices u_{i+1} and u_{i-1}, where the suffixes are taken modulo t. We obtain the injective chromatic mean and variance for the closed helm graph in the following results.

Theorem 5. *For a closed helm graph $CH_{1,t,t}$; $t \geq 3$, of order $n = 2t+1$,*

(i) $\mu_{\chi_i}(CH_{1,t,t}) = \begin{cases} \frac{16}{7}, & t = 3; \\ \frac{25}{9}, & t = 4; \\ \frac{t^2+6t+8}{2n}, & t \geq 5 \text{ and } t \text{ is even}; \\ \frac{t^2+6t+5}{2n}, & t \geq 5 \text{ and } t \text{ is odd}. \end{cases}$



(a) χ_i-coloring of $H_{1,8,8}$ with minimum injective chromatic sum. **(b)** χ_i-coloring of $H_{1,8,8}$ with maximum injective chromatic sum.

Fig. 2 χ_i-colorings of the graph $H_{1,8,8}$.

$$\textit{(ii)}\ \ \mu_{\chi_i^+}(CH_{1,t,t}) = \begin{cases} \frac{19}{7}, & t=3; \\ \frac{29}{9}, & t=4; \\ \frac{3t^2+4t-4}{2n}, & t \geq 5 \textit{ and } t \textit{ is even}; \\ \frac{3t^2+4t-1}{2n}, & t \geq 5 \textit{ and } t \textit{ is odd}. \end{cases}$$

$$\textit{(iii)}\ \ \sigma^2_{\chi_i}(CH_{1,t,t}) = \sigma^2_{\chi_i^+}(CH_{1,t,t}) = \begin{cases} \frac{52}{49}, & t=3; \\ \frac{140}{81}, & t=4; \\ \frac{5t^4+4t^3-26t^2+104t-24}{12n^2}, & t \geq 5 \textit{ and } t \textit{ is even}; \\ \frac{5t^4+4t^3-8t^2+56t+15}{12n^2}, & t \geq 5 \textit{ and } t \textit{ is odd}. \end{cases}$$

Proof. Let $CH_{1,t,t}$; $t \geq 3$, be a closed helm graph such that $V(H_{1,t,t}) = \{v, v_i, u_i : 1 \leq i \leq t\}$, where the subgraph induced by the vertices $\{v\} \cup \{v_i : 1 \leq i \leq t\}$ is the wheel graph $W_{1,t}$ with v as the vertex with degree t, and each u_i being adjacent to the corresponding v_i's and the vertices u_{i+1} and u_{i-1}, where the suffixes are taken modulo t. Any χ_i-coloring of the wheel graph $W_{1,t}$ requires $t+1$ colors. Since the closed helm graph $H_{1,t,t}$ contains a wheel graph $W_{1,t}$ as an induced subgraph, $\chi_i(H_{1,t,t}) \geq t+1$.

Consider the graph $CH_{1,3,3}$. As the vertices v_i's and u_i's induce a K_3, no color can be assigned to more than one u_i and one v_i. Hence, the coloring $c : V(CH_{1,3,3}) \to \{c_1, c_2, c_3, c_4\}$ such that $c(v_i) = c(u_i) = c_i$; $1 \leq i \leq 3$, and $c(v) = c_4$, gives the injective coloring of $CH_{1,3,3}$ having the minimum chromatic sum. Therefore, the χ_i-chromatic mean and the χ_i-chromatic variance are obtained based on the p.m.f. of this coloring c given below.

$$f(i) = \begin{cases} \frac{2}{7}, & i = 1,2,3; \\ \frac{1}{7}, & i = 4; \\ 0, & otherwise. \end{cases}$$

Therefore, $\mu_{\chi_i}(CH_{1,3,3}) = \frac{19}{7}$ and $\sigma^2_{\chi_i}(CH_{1,3,3}) = \frac{52}{49}$.

To obtain a χ_i-coloring having the maximum injective chromatic sum, we obtain the coloring c' of $CH_{1,3,3}$ by swapping the colors c_1 and c_4 as $c(v_i) = c(u_i) = c_{i+1}$; $1 \leq i \leq 3$, and $c(v) = c_1$, whose p.m.f. is given as follows.

$$f(i) = \begin{cases} \frac{1}{7}, & i = 1; \\ \frac{2}{7}, & i = 2,3,4; \\ 0, & otherwise. \end{cases}$$

Therefore, based on the p.m.f. obtained, $\mu_{\chi_i^+}(CH_{1,3,3}) = \frac{19}{7}$ and $\sigma^2_{\chi_i^+}(CH_{1,3,3}) = \frac{52}{49}$.

When $t = 4$, consider the χ_i-coloring $c : V(CH_{1,4,4}) \to \{c_1, c_2, c_3, c_4, c_5\}$ such that $c(v_i) = c(u_i) = c_i$; $1 \leq i \leq 4$, and $c(v) = c_5$ and the χ_i-coloring c' of $CH_{1,4,4}$ by swapping the colors c_1 and c_5 as $c(v_i) = c(u_i) = c_{i+1}$; $1 \leq i \leq 4$, and

$c(v) = c_1$. As we know, all the v_i's must be assigned one distinct color, and as the u_i's induce a cycle C_t, we need at least two colors to color the u_i's.

If a vertex v_i, for some $1 \leq i \leq 4$, is assigned a color c_j, for some $1 \leq j \leq 4$, then the vertices u_{i-1} and u_{i+1} cannot be assigned c_j. Hence, either u_i or u_{i+2} can be assigned the color c_j. As $t = 4$, the vertices u_{i+1} and u_{i-1} shall be adjacent to both u_i and u_{i+2}, which prohibits in assigning the color c_j to 2 u_j's. Therefore, except for the color assigned for v, all the remaining colors can be given exactly to 2 vertices of $CH_{1,4,4}$. Based on this, we deduce that the coloring c gives the minimum chromatic sum and the coloring c' gives the maximum chromatic sum with the p.m.f. (s) f and f' as given below.

$$f(i) = \begin{cases} \frac{2}{9}, & i = 1,2,3,4; \\ \frac{1}{9}, & i = 5; \\ 0, & otherwise. \end{cases}$$

$$f'(i) = \begin{cases} \frac{1}{9}, & i = 1; \\ \frac{2}{7}, & i = 2,3,4,5; \\ 0, & otherwise. \end{cases}$$

Therefore, based on the p.m.f.(s), we determine the injective chromatic mean(s) and variance(s) of the graph $CH_{1,4,4}$ as, $\mu_{\chi_i}(CH_{1,4,4}) = \frac{25}{9}$, $\mu_{\chi_i^+}(CH_{1,4,4}) = \frac{29}{9}$ and $\sigma^2_{\chi_i}(CH_{1,4,4}) = \sigma^2_{\chi_i^+}(CH_{1,4,4}) = \frac{140}{81}$.

When $t \geq 5$, we consider two cases to obtain the injective chromatic mean and variance of the closed helm graph based on the parity of t as follows.

Case 1. When t is even, we subdivide the case into subcases to determine the χ_i-coloring of $CH_{1,t,t}$ using $t+1$ colors which yields the minimum and maximum chromatic sum as follows.

Subcase 1a: When $t \equiv 0 \pmod 4$, consider the χ_i-coloring $c : V(CH_{1,t,t}) \to \{c_i : 1 \leq i \leq t+1\}$ such that $c(v) = c_{t+1}$, $c(v_i) = c_i$; $1 \leq i \leq t$, $c(u_i) = c_1$; $i \equiv 0,1 \pmod 4$, $1 \leq i \leq t-1$, $c(u_i) = c_2$; $i \equiv 2,3 \pmod 4$, $4 \leq i \leq t-1$, $c(u_2) = c_2$ and $c(u_3) = c(u_t) = c_3$. The p.m.f. of this coloring is given as follows.

$$f(i) = \begin{cases} \frac{t}{2n}, & i = 1,2; \\ \frac{3}{n}, & i = 3; \\ \frac{1}{n}, & 4 \leq i \leq t+1; \\ 0, & otherwise. \end{cases}$$

We require two colors, say c_1 and c_2, to color the u_i's as they induce a C_t. If we color this C_t with only two colors, in the pattern $c_1, c_1, c_2, c_2, \ldots, c_2, c_2$, there will exist two u_i whose all three neighbours are colored using only two colors because the colors c_1 and c_2 must be assigned to one of the v_i's. Hence, one neighbour of these two vertices must be colored with the third color. In this regard, as c maximises the use of c_1, c_2 and c_3, it gives the minimum chromatic sum. Therefore, based on the p.m.f., the injective chromatic mean and variance of the graph $CH_{1,t,t}$ are given as $\mu_{\chi_i}(CH_{1,t,t}) = \frac{t^2+6t+8}{2n}$, and $\sigma^2_{\chi_i}(CH_{1,t,t}) = \frac{5t^4+4t^3-8t^2+56t+15}{12n^2}$.

Now, swapping the colors c_1, c_2 and c_3 with the colors c_{t+1}, c_t and c_{t-1}, respectively, we obtain a χ_i-coloring of $CH_{1,t,t}$ that yields the maximum chromatic sum and the p.m.f. of such a coloring is given as follows.

$$f(i) = \begin{cases} \frac{1}{n}, & 1 \leq i \leq t-2; \\ \frac{3}{n}, & i = t-1; \\ \frac{t}{2n}, & i = t, t+1; \\ 0, & otherwise. \end{cases}$$

Based on this p.m.f., the χ_i^+-chromatic mean and variance of the graph $CH_{1,t,t}$ are given as $\mu_{\chi_i^+}(CH_{1,t,t}) = \frac{3t^2+4t-4}{2n}$, and $\sigma^2_{\chi_i^+}(CH_{1,t,t}) = \frac{5t^4+4t^3-26t^2+104t-24}{12n^2}$.

Subcase 1b: When $t \equiv 2 \pmod 4$, consider the χ_i-coloring $c : V(CH_{1,t,t}) \to \{c_i : 1 \leq i \leq t+1\}$ such that $c(v) = c_{t+1}$, $c(v_i) = c_{i+1}$; $1 \leq i \leq t$, $c(u_i) = c_1$; $1 \equiv 1,2 \pmod 4$, $1 \leq i \leq t-2$, $c(u_i) = c_2$; $1 \equiv 3,0 \pmod 4$, $1 \leq i \leq t-2$, $c(u_{t-1}) = c(u_t) = c_3$, where the suffixes are taken modulo t.

As we need at least three colors to color the C_t induced by u_i's, according to the injective coloring protocol, we assign the maximum number of vertices the color c_1 and c_2 and only 2 u_i's are assigned the color c_3, which is inevitable to yield the minimum chromatic sum (see Theorem 1). Therefore, the p.m.f. of this coloring is the same as the p.m.f. obtained for the coloring c of the graph $CH_{1,t,t}$, when $t \equiv 1 \pmod 4$ given in *Subcase 1a*. Similar color swapping is possible in this case; we get the same injective chromatic mean and variances as given in *Subcase 1a*.

Case 2. When t is odd, we know that the C_t induced by the u_i's must be colored with three colors, where one of them is used to color exactly one u_i and based on whether $t \equiv 1 \pmod 4$ or $t \equiv 3 \pmod 4$, the position of the vertex having the 3rd color differs. Therefore, we obtain the assignment χ_i-coloring of $CH_{1,t,t}$ using $t+1$ colors, which yields the minimum and maximum chromatic sum based on the value of t in two cases.

Subcase 2a: When $t \equiv 1 \pmod 4$, let $c : V(CH_{1,t,t}) \to \{c_i : 1 \le i \le t+1\}$ be a χ_i-coloring of $CH_{1,t,t}$ such that $c(v) = c_{t+1}$,

$$c(v_i) = \begin{cases} c_1, & i = t-1; \\ c_2, & i = t; \\ c_3, & i = 1; \\ c_{i+2}, & 2 \le i \le t-2. \end{cases}$$

and

$$c(u_i) = \begin{cases} c_1, & i \equiv 1,2 \pmod 4, i \neq t; \\ c_2, & i \equiv 3,0 \pmod 4; \\ c_3, & i = t. \end{cases}$$

Subcase 2b: When $t \equiv 3 \pmod 4$, let $c : V(CH_{1,t,t}) \to \{c_i : 1 \le i \le t+1\}$ be a χ_i-coloring of $CH_{1,t,t}$ such that $c(v) = c_{t+1}$,

$$c(v_i) = \begin{cases} c_1, & i = t-2; \\ c_2, & i = t; \\ c_3, & i = t-1; \\ c_{i+3}, & 1 \le i \le t-3. \end{cases}$$

and

$$c(u_i) = \begin{cases} c_1, & i \equiv 1,2 \pmod 4, i \neq t-1; \\ c_2, & i \equiv 3,0 \pmod 4; \\ c_3, & i = t-1. \end{cases}$$

The assignment c in both the cases yield the following p.m.f.

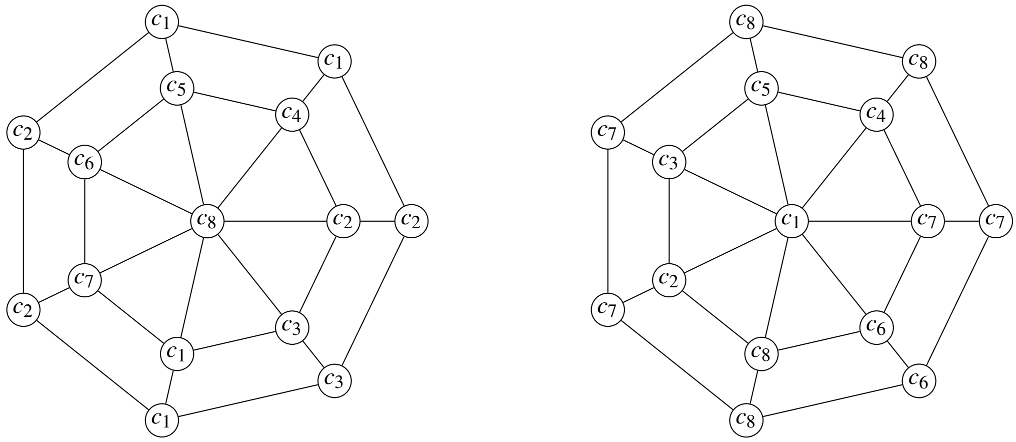
$$f(i) = \begin{cases} \frac{t+1}{2n}, & i = 1,2; \\ \frac{2}{n}, & i = 3; \\ \frac{1}{n}, & 4 \le i \le t+1; \\ 0, & otherwise. \end{cases}$$

As we can see from the p.m.f., the maximum pendant vertices are assigned the colors c_1 and c_2, thereby yielding the minimum chromatic sum. Therefore, the χ_i-chromatic mean and the χ_i-chromatic variance of $CH_{1,t,t}$, when t is odd, are $\mu_{\chi_i}(CH_{1,t,t}) = \frac{t^2+6t+5}{2n}$, and $\sigma^2_{\chi_i}(CH_{1,t,t}) = \frac{5t^4+4t^3-8t^2+56t+15}{12n^2}$.

As mentioned earlier in *Case 1*, swapping the colors c_1, c_2 and c_3 with the colors c_{t+1}, c_t and c_{t-1}, we obtain a χ_i-coloring of the closed helm graph with the following p.m.f., that yields the maximum chromatic sum.

$$f(i) = \begin{cases} \frac{1}{n}, & 1 \le i \le t-2; \\ \frac{2}{n}, & i = t-1; \\ \frac{t+1}{2n}, & i = t, t+1; \\ 0, & otherwise. \end{cases}$$

Hence, the χ_i^+-chromatic mean and the χ_i^+-chromatic variance of $CH_{1,t,t}$, when t is odd, are $\mu_{\chi_i^+}(CH_{1,t,t}) = \frac{3t^2+4t-1}{2n}$, and $\sigma^2_{\chi_i^+}(CH_{1,t,t}) = \frac{5t^4+4t^3-8t^2+56t+15}{12n^2}$.



(a) χ_i-coloring of $CH_{1,7,7}$ with minimum injective chromatic sum. **(b)** χ_i-coloring of $CH_{1,7,7}$ with maximum injective chromatic sum.

Fig. 3 χ_i-colorings of the graph $CH_{1,7,7}$.

Proposition 1. *If G is a graph of order n having $\Delta(G) = n-1$ and $\delta(G) \geq 2$, then $\mu_{\chi_i}(G) = \mu_{\chi_i^+}(G) = \frac{n+1}{2}$ and $\sigma^2_{\chi_i}(G) = \sigma^2_{\chi_i^+}(G) = \frac{n^2-1}{12}$.*

Proof. If at least one universal vertex exists in a graph G, then all the vertices of G must be given unique colors in any injective coloring of the graph G. If v is a universal vertex of G, all the vertices $u \in N(v)$ must be colored with distinct colors. Suppose the color assigned to v is assigned to any other vertex $u \in V(G)$. In that case, there exists a vertex $w \in N(u) \cap N(v)$ whose two neighbours are assigned the same color, violating the injective coloring protocol. Hence, for a graph G having $\Delta(G) = n-1$ and $\delta(G) \geq 2$, $\chi_i(G) = n$. Therefore, the p.m.f. for a χ_i-coloring of G will be the same for any χ_i-coloring yielding the chromatic sum, which is given as follows,

$$f(i) = \begin{cases} \frac{1}{n}, & 1 \leq i \leq n; \\ 0, & otherwise. \end{cases}$$

Based on this p.m.f., the injective chromatic mean and variance are $\mu_{\chi_i}(G) = \mu_{\chi_i^+}(G) = \frac{n+1}{2}$ and $\sigma^2_{\chi_i}(G) = \sigma^2_{\chi_i^+}(G) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$.

The graph obtained by making the central vertex v of a helm graph $H_{1,t,t}$ adjacent to all of its pendant vertices is called a *flower graph* $Fl_{1,t,t}$ and a double wheel graph DW_t is a graph obtained by taking two copies of a cycle C_t and making all the $2t$ vertices of these cycles adjacent to an external vertex v.

Corollary 1. *For a wheel graph $W_{1,t}$; $t \geq 3$, $\mu_{\chi_i}(W_{1,t}) = \mu_{\chi_i^+}(W_{1,t}) = \frac{t+2}{2}$ and $\sigma^2_{\chi_i}(W_{1,t}) = \sigma^2_{\chi_i^+}(W_{1,t}) = \frac{t(t+1)}{12}$.*

Corollary 2. *For a flower graph $F_{1,t,t}$; $t \geq 3$, and a double wheel graph $DW_{1,t,t}$, $\mu_{\chi_i}(G) = \mu_{\chi_i^+}(G) = \frac{2(t+1)}{2}$ and $\sigma^2_{\chi_i}(G) = \sigma^2_{\chi_i^+}(G) = \frac{t(t+1)}{3}$.*

3 Conclusion

In this article, the statistical parameters such as the chromatic mean and chromatic variance with respect to the improper injective coloring of graphs are introduced and these parameters are determined for certain classes of graphs. As the approach of graph coloring in terms of statistics is a novel area of research, and we are extending it to the improper coloring of graphs, it offers a broad scope for future study, wherein other measures of central tendency and dispersion can be defined in similar lines and studied.

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