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New Results of Itô's Formula Using q-Calculus

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Abstract: In this paper, we extend the classical Itô formula by employing the framework of quantum calculus to derive its q-analogue, applicable to both classical Brownian motion and q-Itô processes via the quantum Taylor expansion. We also determine the q-infinitesimal generators associated with stochastic q-differential equations. To illustrate the applicability of our results, we present the q-Black-Scholes equation as a concrete example. Throughout the study, we assume $0 < q < 1$, and we show that the classical results of Itô calculus are recovered in the limit as $q \to 1$.

Keywords: q-calculus; Itô calculus; semigroup; stochastic differential equations.

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1 Introduction

Itô calculus extends classical calculus to incorporate stochastic processes such as Brownian motion (see [22,24,34]) and plays a foundational role in mathematical finance and stochastic differential equations (see [5,9,14,26]). A central result in this theory is Itô's formula, which enables the computation of differentials for time-dependent functions of stochastic processes. Moreover, Itô's formula is instrumental in identifying the infinitesimal generators of stochastic models (see [7, 21]). Let us define the following space

$$\mathscr{C}_b^2 = \{\phi \in \mathscr{C}^2(\mathbb{R},\mathbb{R}) : \phi, \phi', \phi'' \text{ are bounded}\}.$$

For $\phi \in \mathscr{C}_b^2$, Francis and Meyre [17] stated Itô's lemma, which is expressed as

$$\phi(B(t)) = \phi(B(0)) + \int_0^t \phi'(B(s))dB(s) + \frac{1}{2}\int_0^t \phi''(B(s))ds, \ \forall t. \tag{1}$$

They considered the probabilistic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \tag{2}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(x)$ is a $d \times k$ matrix assumed to be Lipschitz. Here, $B(t)$ is a Brownian motion in $\mathbb{R}^k$. Let us define the space $\mathscr{C}_c^2(\mathbb{R}^d)$ as

$$\mathscr{C}_c^2(\mathbb{R}^d) = \{\phi \in \mathscr{C}^2(\mathbb{R}^d) \text{ with compact support}\}.$$

For $\phi \in \mathscr{C}_c^2(\mathbb{R}^d)$, they proved that the infinitesimal operator L of equation (2) is given by

$$L\phi(x) = \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\phi(x) + \sum_{1\le i\le d} b_i(x)\frac{\partial}{\partial x_i}\phi(x), \tag{3}$$

where $a = \sigma\sigma^*$, and σ^* denotes the adjoint of σ.

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Applications of Itô's formula have become increasingly diverse. In finance, it underpins models of option pricing, interest rate evolution, and portfolio optimization. More recently, Itô-type stochastic differential frameworks have also been applied to population dynamics, epidemiological modeling, and biological systems. For instance, the study in [1] employs stochastic modeling techniques to analyze competing strains in disease propagation. Similarly, delay-based nonlinear stochastic systems, such as those discussed in [2], use differential structures akin to Itô calculus to explore biological population control mechanisms.
On a different front, quantum calculus, also known as q-calculus, is a limit-free alternative to classical calculus. Unlike traditional derivatives, which require the existence of limits, q-derivatives circumvent this constraint. Introduced by Jackson in the early 1900s [20], quantum calculus encompasses q-differentiation and q-integration. Cheung and Kac [13] expanded on these ideas, highlighting their significance in various mathematical and physical contexts. Recent works (see [3,6,8,12,25,28]) further emphasize the utility of q-calculus, particularly in theoretical physics and analytic number theory. It also serves as a bridge connecting discrete mathematics and continuous systems (see [4,16,18,29]). Classical results in calculus often have elegant q-analogues, and as $q \to 1$, quantum calculus converges to the traditional form.
This work aligns with the scope of the **Jordan Journal of Mathematics and Statistics** by contributing to the ongoing development of analytical techniques in modern stochastic modeling. Specifically, we aim to construct a q-analogue of Itô's formula, thereby extending classical stochastic analysis to discrete-time and deformed-time frameworks. This not only enriches the theoretical foundations of q-calculus and offers novel tools for applied probability, finance, and biological modeling, fields that frequently involve non-classical time structures. Our work bridges gaps between continuous stochastic calculus, quantum deformations, and applied mathematical modeling, resonating with the journal's focus on original research in mathematical theory and statistical applications, as evidenced by recent studies on stochastic equations [30], numerical schemes for diffusion models with discontinuous coefficients [10], and applications of q-calculus in analytic function theory [33].
The need to generalize Itô calculus in the context of q-calculus arises naturally when modeling systems where time is deformed or discrete, as encountered in quantum mechanics and other discrete-scale phenomena. Classical Itô theory is limited to continuous-time dynamics, while q-calculus enables the formulation of stochastic behavior on a discretized or deformed time scale. This generalization can be applied in various fields, including finance, where it facilitates the construction of discrete-time models for asset prices and risk management under non-classical temporal structures. In growth dynamics, it offers a framework for systems with discontinuous time evolutions, paving the way for new research in quantum finance, discrete-time stochastic modeling, and statistical physics, and highlighting the broad relevance of q-calculus in modern applied mathematics.
Hudson and Parthasarathy [19], Kholevo [23], and Parthasarathy [32] developed quantum stochastic calculus within the framework of non-commutative probability and operator-valued processes. In this work, we propose a q-deformation approach based on q-calculus principles, aiming to construct a q-analogue of the classical Itô formula. This perspective allows the modeling of stochastic systems where time exhibits a discrete or deformed structure, providing a complementary tool to the operator-based quantum stochastic methods.

We begin by introducing foundational functional spaces. For all $p \geq 1$, $\mathbf{L}^p$-space is defined as

$$\mathbf{L}^p(\Omega) = \{X : \mathbb{E}(|X|^p) < \infty\},$$

and the $\mathbf{L}^p$-norm of the stochastic variable X is given by

$$\|X\|_p = (\mathbb{E}(|X|^p))^{\frac{1}{p}}.$$

The set of continuous functions on the real line is denoted by $\mathscr{C}(\mathbb{R})$. In addition, the space $\mathscr{C}^2_{b\,q}$ is defined as

$$\mathscr{C}^2_{b\,q} = \{\phi \in \mathscr{C}(\mathbb{R},\mathbb{R}) : \phi, D_q\phi, D_q^2\phi \text{ are bounded}\},$$

where D_q and D_q^2 denote the first and second q-derivatives, respectively.
The space $\mathscr{C}_c(\mathbb{R})$ refers to the set of continuous functions on $\mathbb{R}$ with compact support.

In this paper, we work with classical Brownian motion. We adopt the following hypothesis:

(H) Let $0 < q < 1$, and let B denote a standard Brownian motion. The random variables

$$(B(t_i) - B(t_{i-1}))_q^2 \sim \sqrt{t_i - t_{i-1}}\sqrt{t_i - q^2 t_{i-1}}(\mathscr{N}(0,1))^2$$

are independent for all $1 \leq i \leq n$, where the q-square of the increment $B(t_i) - B(t_{i-1})$ is defined by

$$(B(t_i) - B(t_{i-1}))_q^2 = (B(t_i) - B(t_{i-1}))(B(t_i) - qB(t_{i-1})).$$

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The main result of this paper is stated in the following theorem, which provides the quantum Itô formula for Brownian motion:

Theorem 1. *Let $0 < q < 1$, and suppose the standard Brownian motion B satisfies hypothesis* $(\mathbf{H})$. *Then, for every $\phi \in \mathscr{C}^2_{b\,q}$, we have*

$$\phi(B(t)) = \phi(B(0)) + \int_0^t D_q\phi(B(s))dB(s) + \frac{1}{[2]_q}\int_0^t D_q^2\phi(B(s))d_qs, \ \forall t. \tag{4}$$

This generalizes the classical Itô's formula (1). As $q \to 1$, (4) reduces to the classical case.
Theorem 1 will be proved in Section 3 using a q-adapted Taylor expansion. The q-stochastic integral

$$\int_0^t D_q\phi(B(s))dB(s)$$

generalizes the classical Itô integral $\int_0^t \phi'(B(s))dB(s)$, where the integrand is replaced by its q-analogue. It arises from q-Itô calculus, based on the q-Taylor expansion, where the first term corresponds to the q-stochastic integral and the second provides an additional correction. The parameter $0 < q < 1$ governs the deformation, and as $q \to 1$, the q-stochastic integral recovers the classical Itô integral. This framework offers greater flexibility for modeling q-analytic stochastic processes.
In the context of q-calculus, the integral $\int_0^t B(s)dB(s)$ should be interpreted as a q-stochastic integral, as defined in Theorem 1. This integral is a generalization of the classical stochastic integral and is based on the q-derivatives D_q and D_q^2, which modify the structure of the integral according to the value of $0 < q < 1$. Consider the example $\phi(x) = x^2$, a function commonly used to test the properties of stochastic integrals. The q-derivatives of x^2 are given by

$$D_qx^2 = (1+q)x, \text{ and } \ D_q^2x^2 = 1+q.$$

According to Theorem 1, the q-stochastic integral is expressed as

$$B^2(t) = (1+q)\int_0^t B(s)dB(s) + t.$$

In this case, the q-stochastic integral yields a relation modified by the parameter $0 < q < 1$. As $q \to 1$, the q-stochastic integral converges to the classical Itô stochastic integral. Specifically, the q-stochastic integral

$$\int_0^t B(s)dB(s) \text{ converges to } \frac{B^2(t)-t}{2} \text{ as } q \to 1.$$

This shows that while the q-stochastic integral introduces a deformation through q, this deformation disappears as $q \to 1$, and the integral reduces to its classical form. In the context of the q-stochastic integral, although the notation used for $\int_0^t B(s)dB(s)$ is identical to that of the classical stochastic integral, it is important to clarify that this integral refers to the q-modified version, as defined in Theorem 1, where $0 < q < 1$.
We now define the q-extension of equation (2) as

$$dX(t) = b(X(t))d_qt + \sigma(X(t))dB(t), \tag{5}$$

where B denotes a classical Brownian motion.
The q-analogue of the infinitesimal generator will now be presented.

Theorem 2. *Let $0 < q < 1$, and suppose B satisfies hypothesis* $(\mathbf{H})$. *The infinitesimal generator L associated with the diffusion process governed by the stochastic q-differential equation (5) is given by*

$$\mathscr{C}_c(\mathbb{R}^d) \subset Dom(L),$$

and for any function $\phi \in \mathscr{C}_c(\mathbb{R}^d)$, we have

$$L\phi(x) = \frac{1}{[2]_q}\sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial_q^2}{\partial_q x_i \partial_q x_j}\phi(x) + \sum_{1\le i\le d} b_i(x)\frac{\partial_q}{\partial_q x_i}\phi(x), \tag{6}$$

where $a = \sigma\sigma^*$, and σ^* denotes the adjoint of the matrix σ.

The next section provides a brief overview of some fundamental concepts that form the foundation of our work. Section 4 presents the proof of Theorem 2. In Section 5, we introduce a practical example, the q-Black-Scholes equation, to illustrate our results. Section 6 concludes with remarks and perspectives.

2 Basic concepts of q-calculus

We introduce the following concepts (see [11,13,15,31]).
Let $0 < q < 1$. The q-differential of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$d_q f(x) = f(qx) - f(x).$$

Using this, we define the q-derivative as

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \qquad x \neq 0.$$

If $f'(0)$ exists, we set

$$D_q f(0) = f'(0).$$

It is clear that if f is differentiable, then

$$\lim_{q \to 1} D_q f(x) = f'(x).$$

Note that a function continuous on an interval not containing 0 is also continuously q-differentiable.
The higher-order q-derivatives are defined recursively by

$$D_q^n f(x) = \begin{cases} f(x) & \text{if}\, n = 0, \\ D_q(D_q^{n-1} f)(x), & \text{if}\, n = 1, 2, \ldots. \end{cases}$$

For a real-valued function f of n variables, the q-partial derivative with respect to x_i, is defined as

$$\frac{\partial_q f(x)}{\partial_q x_i} = \frac{f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n) - f(x)}{(q-1)x_i}, \qquad x_i \neq 0,$$

$$\left.\frac{\partial_q f(x)}{\partial_q x_i}\right|_{x_i=0} = \lim_{x_i \to 0} \frac{\partial_q f(x)}{\partial_q x_i}, \ \text{where}\, x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

For real numbers a, b, and a function $f : \mathbb{R} \to \mathbb{R}$, the Jackson integral is defined by

$$\int_0^b f(x) d_q x = (1-q) b \sum_{i=0}^{\infty} q^i f(q^i b),$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

provided that the infinite series is absolutely convergent. If f is continuous on $[a,b]$, then

$$\lim_{q \to 1} \int_a^b f(x) d_q x = \int_a^b f(x) dx.$$

If f is continuous at $x = 0$, then

$$\int_0^b D_q f(x) d_q x = f(b) - f(0).$$

The q-antiderivative of $\dfrac{1}{x}$ is given by

$$\int \frac{d_q x}{x} = \frac{q-1}{\log q} \log x.$$

Let f and g be functions with classical derivatives defined in a neighborhood of $x = 0$, and continuous at $x = 0$. The q-integration by parts formula is

$$\int_a^b g(x) D_q f(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b f(qx) D_q g(x) d_q x.$$

We now introduce the following notation. Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space. We define $M^1_{q,loc}$ as the set of progressively measurable functions φ defined over $\mathbb{R}_+ \times \Omega$ such that

$$\int_0^T |\varphi(t,\omega)| d_q t < \infty \text{ a.s. for all T}.$$

Similarly, $M^2_{q,loc}$ denotes the set of progressively measurable functions φ for which

$$\int_0^T \varphi^2(t,\omega) d_q t < \infty \text{ a.s. for all T}.$$

3 Generalization of Itô's lemma for Brownian motion

The central objective of this section is to prove Theorem 1. To that end, we rely on the q-Taylor formula [27], which is presented in the following proposition:

Proposition 1. *Let $\phi(x)$ be a function continuous on the interval $[a,b]$, and let $z, c \in]a,b[$. Then, there exists a point $\hat{q} \in]0,1[$ such that for all $q \in]\hat{q},1[$, there exists a point $\xi \in]a,b[$, located between c and z, for which the following formula holds*

$$\phi(z) = \sum_{k=0}^{n-1} \frac{D_q^k \phi(c)}{[k]_q!}(z-c)_q^k + \frac{D_q^n \phi(\xi)}{[n]_q!}(z-c)_q^n,$$

where

$$[n]_q = \frac{q^n - 1}{q-1}, \qquad \text{for any positive integer } n,$$

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q \times [n-1]_q \times \ldots \times [1]_q & \text{if } n = 1,2,\ldots, \end{cases}$$

and

$$(z-c)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (z-c)(z-qc)\ldots(z-q^{n-1}c) & \text{if } n \geq 1. \end{cases}$$

We now prove Theorem 1.

Proof. We consider $t_i = \dfrac{2it}{n(1+\sqrt{(1-q^2)n+q^2})}$.
Applying the second-order q-Taylor expansion to ϕ, we obtain

$$\phi(B(t)) = \phi(B(0)) + \sum_{i=1}^n [\phi(B(t_i)) - \phi(B(t_{i-1}))]$$

$$= \phi(B(0)) + \sum_{i=1}^n D_q\phi(B(t_{i-1}))(B(t_i) - B(t_{i-1})) + \frac{1}{[2]_q}\sum_{i=1}^n D_q^2\phi(B(\theta_i))(B(t_i) - B(t_{i-1}))_q^2,$$

where $\theta_i \in]t_{i-1}, t_i[$. According to Remark 3.1.4 in [17], we have

$$\lim_{n\to+\infty} \sum_{i=1}^n D_q\phi(B(t_{i-1}))(B(t_i) - B(t_{i-1})) = \int_0^t D_q\phi(B(s))dB(s).$$

Therefore, we aim to show that

$$\lim_{n\to+\infty} \sum_{i=1}^n D_q^2\phi(B(\theta_i))(B(t_i) - B(t_{i-1}))_q^2 = \int_0^t D_q^2\phi(B(s))d_q s, \text{ in } \mathbf{L}^2.$$

From this perspective, we assume

$$U_n = \sum_{i=1}^n D_q^2\phi(B(\theta_i))(B(t_i) - B(t_{i-1}))_q^2.$$

We now define

$$V_n=\sum_{i=1}^{n}D_q^2\phi(B(q^{i-1}t))(B(t_i)-B(t_{i-1}))_q^2,\ \text{ and }\ W_n=\sum_{i=1}^{n}D_q^2\phi(B(q^{i-1}t))(1-q)q^{i-1}t.$$

Hence, we have, by Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}|U_n-V_n| &\le \mathbb{E}\left(\sup_{1\le i\le n}|D_q^2\phi(B(\theta_i))-D_q^2\phi(B(q^{i-1}t))|\times\sum_{i=1}^{n}|(B(t_i)-B(t_{i-1}))_q^2|\right)\\ &\le \left[\mathbb{E}\left(\sup_{1\le i\le n}|D_q^2\phi(B(\theta_i))-D_q^2\phi(B(q^{i-1}t))|^2\right)\right]^{\frac12}\times\left[\mathbb{E}\left(\left(\sum_{i=1}^{n}|(B(t_i)-B(t_{i-1}))_q^2|\right)^2\right)\right]^{\frac12}.\end{aligned}$$

By Lebesgue's theorem and using (**H**), we get

$$\lim_{n\to+\infty}\mathbb{E}|U_n-V_n|=[0\times t^2]^{\frac12}=0.$$

Additionally, we have

$$\begin{aligned}\mathbb{E}|V_n-W_n|^2 &= \mathbb{E}\left(\left|\sum_{i=1}^{n}D_q^2\phi(B(q^{i-1}t))((B(t_i)-B(t_{i-1}))_q^2-(1-q)q^{i-1}t)\right|^2\right)\\ &=\sum_{i=1}^{n}\mathbb{E}\left(|D_q^2\phi(B(q^{i-1}t))((B(t_i)-B(t_{i-1}))_q^2-(1-q)q^{i-1}t)|^2\right)\\ &\le \sup(D_q^2\phi)^2\times\sum_{i=1}^{n}\mathbb{E}\left(|(B(t_i)-B(t_{i-1}))_q^2-(1-q)q^{i-1}t|^2\right).\end{aligned}$$

Using assumption (**H**), we obtain

$$\lim_{n\to+\infty}\|V_n-W_n\|_2\le \sup(D_q^2\phi)^2\times\frac{1-q}{1+q}t^2<\infty.$$

Finally, using the definition of the Jackson integral, since $D_q^2\phi$ is bounded, it follows that

$$\lim_{n\to+\infty}\left\|W_n-\int_0^t D_q^2\phi(B(s))d_qs\right\|_1=0.$$

Then, we have the $\mathbf{L}^1$ convergence of

$$\phi(B(t))=\phi(B(0))+\sum_{i=1}^{n}D_q\phi(B(t_{i-1}))(B(t_i)-B(t_{i-1}))+\frac{1}{[2]_q}\sum_{i=1}^{n}D_q^2\phi(B(\theta_i))(B(t_i)-B(t_{i-1}))_q^2$$

to

$$\phi(B(t))=\phi(B(0))+\int_0^t D_q\phi(B(s))dB(s)+\frac{1}{[2]_q}\int_0^t D_q^2\phi(B(s))d_qs.$$

This completes the proof.

4 Infinitesimal generator in quantum calculus

Before proceeding, we introduce the q-Itô's formula for q-Itô's process, which is formalized in Theorem 3.

4.1 Generalization of Itô's lemma for quantum Itô's process

Theorem 3. *Let $0 < q < 1$. Suppose $X(t)$ is a q-Itô's process of the form*

$$dX(t) = \varphi(t)dB(t) + \psi(t)d_qt,$$

where $\varphi \in M^2_{q,loc}$, $\psi \in M^1_{q,loc}$ and $B(t)$ denotes standard Brownian motion.
If ϕ is continuous and B satisfies condition ($\mathbf{H}$), then almost surely, for all $t \geq 0$, the q-Itô's formula holds

$$\phi(X(t)) = \phi(X(0)) + \int_0^t D_q\phi(X(s))\varphi(s)dB(s) + \int_0^t D_q\phi(X(s))\psi(s)d_qs + \frac{1}{[2]_q}\int_0^t D_q^2\phi(X(s))\varphi(s)^2 d_qs.$$

Proof. Similar to the approach used in Theorem 1, let $t_i = \dfrac{2it}{n(1+\sqrt{(1-q^2)n+q^2})}$.
Using the q-Taylor expansion, we analyze the behavior of the terms as $n \to \infty$.

$$\begin{aligned}\phi(X(t)) &= \phi(X(0)) + \sum_{i=1}^n [\phi(X(t_i)) - \phi(X(t_{i-1}))] \\ &= \phi(X(0)) + \sum_{i=1}^n D_q\phi(X(t_{i-1}))(X(t_i) - X(t_{i-1})) + \frac{1}{[2]_q}\sum_{i=1}^n D_q^2\phi(X(\theta_i))(X(t_i) - X(t_{i-1}))_q^2,\end{aligned}$$

where $\theta_i \in]t_{i-1}, t_i[$. Obviously, we can obtain

$$\lim_{n\to+\infty}\sum_{i=1}^n D_q\phi(X(t_{i-1}))(X(t_i) - X(t_{i-1})) = \int_0^t D_q\phi(X(s))dX(s), \ \text{ in } \mathbf{L}^2.$$

Next, we show that

$$\lim_{n\to+\infty}\sum_{i=1}^n D_q^2\phi(X(\theta_i))(X(t_i) - X(t_{i-1}))_q^2 = \int_0^t D_q^2\phi(X(s))\varphi(s)^2 d_qs, \ \text{ in } \mathbf{L}^2.$$

We thus have

$$\begin{aligned}\sum_{i=1}^n D_q^2\phi(X(\theta_i))(X(t_i) - X(t_{i-1}))_q^2 &= \sum_{i=1}^n D_q^2\phi(X(\theta_i))\varphi(t_{i-1})^2(B(t_i) - B(t_{i-1}))(B(t_i) - qB(t_{i-1})) \\ &+ \sum_{i=1}^n D_q^2\phi(X(\theta_i))\varphi(t_{i-1})\psi(t_{i-1})(t_i - qt_{i-1})(B(t_i) - B(t_{i-1})) \\ &+ \sum_{i=1}^n D_q^2\phi(X(\theta_i))\psi(t_{i-1})\varphi(t_{i-1})(t_i - t_{i-1})(B(t_i) - qB(t_{i-1})) \\ &+ \sum_{i=1}^n D_q^2\phi(X(\theta_i))\psi(t_{i-1})^2(t_i - t_{i-1})(t_i - qt_{i-1}).\end{aligned}$$

According to Theorem 1, the first sum converges in $\mathbf{L}^2$ to the q-integral

$$\int_0^t D_q^2\phi(X(s))\varphi(s)^2 d_qs.$$

We square the second sum and evaluate its expectation

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{i=1}^n D_q^2\phi(X(\theta_i))\varphi(t_{i-1})\psi(t_{i-1})(t_i - qt_{i-1})(B(t_i) - B(t_{i-1}))\right)^2\right] \\ &\leq \sup(D_q^2\phi)^2 \times \sup(\varphi)^2 \times \sup(\psi)^2 \sum_{i=1}^n (t_i - qt_{i-1})^2(t_i - t_{i-1}),\end{aligned}$$

which converges to zero. A similar calculation yields that the third sum converges in $\mathbf{L}^2$ to zero.
It remains to analyze the last sum. Since

$$\sum_{i=1}^{n} D_q^2\phi(X(\theta_i))\psi(t_{i-1})^2(t_i-t_{i-1})(t_i-qt_{i-1}) \leq \sup(D_q^2\phi)\times\sup(\psi)^2\times\frac{2t^2}{(1+\sqrt{(1-q^2)n+q^2})^2}(-q+1+\frac{(1+q)}{n}),$$

this term converges to zero as $n\to+\infty$. Finally, we conclude that

$$\lim_{n\to+\infty}\sum_{i=1}^{n} D_q^2\phi(X(\theta_i))(X(t_i)-X(t_{i-1}))_q^2 = \int_0^t D_q^2\phi(X(s))\varphi(s)^2 d_qs, \text{ in } \mathbf{L}^2.$$

Therefore, we obtain convergence in $\mathbf{L}^1$ of

$$\phi(X(t)) = \phi(X(0)) + \sum_{i=1}^{n} D_q\phi(X(t_{i-1}))(X(t_i)-X(t_{i-1})) + \frac{1}{[2]_q}\sum_{i=1}^{n} D_q^2\phi(X(\theta_i))(X(t_i)-X(t_{i-1}))_q^2$$

to

$$\phi(X(t)) = \phi(X(0)) + \int_0^t D_q\phi(X(s))\varphi(s)dB(s) + \int_0^t D_q\phi(X(s))\psi(s)d_qs + \frac{1}{[2]_q}\int_0^t D_q^2\phi(X(s))\varphi(s)^2 d_qs.$$

The proof is complete.

4.2 Generalization of Infinitesimal generator

To proceed with the proof of Theorem 2, we define the semigroup P_t by

$$(P_t\phi)(x) = \mathbb{E}_x\phi(X(t)),$$

where $\mathbb{E}_x$ denotes the expectation with respect to the process starting at x, and $\phi:\mathbb{R}^d\to\mathbb{R}$ is a bounded Borel function.
We define the domain of the generator L as

$$Dom(L) = \{\phi\in\mathscr{C}_0 : \lim_{t\to 0}\frac{1}{t}(P_t\phi-\phi) \text{ exists}\},$$

where $\mathscr{C}_0 = \mathscr{C}_0(\mathbb{R}^d,\mathbb{R})$ is the space of continuous functions vanishing at infinity.
For $\phi\in Dom(L)$, we define

$$L\phi = \lim_{t\to 0}\frac{1}{t}(P_t\phi-\phi).$$

We now present the proof of Theorem 2.

*Proof.*Let $\mathfrak{L}$ be the differential operator described in equation (6).
By the q-Itô's formula, for $\phi\in\mathscr{C}_c(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
\phi(X(t))-\phi(x) &= \int_0^t\sum_{i=1}^{d} b_i(X(s))\frac{\partial_q}{\partial_q x_i}\phi(X(s))d_qs + \int_0^t\sum_{i=1}^{d}\frac{\partial_q}{\partial_q x_i}\phi(X(s))\sum_{l=1}^{k}\sigma_{il}(X(s))dB_l(s)\\
&+\frac{1}{[2]_q}\int_0^t\sum_{1\leq i,j\leq d}\frac{\partial_q^2}{\partial_q x_i\partial_q x_j}\phi(X(s))\sum_{l=1}^{k}\sigma_{il}(X(s))\sigma_{jl}(X(s))d_qs\\
&= \int_0^t\sum_{1\leq i\leq d} b_i(X(s))\frac{\partial_q}{\partial_q x_i}\phi(X(s))d_qs + \int_0^t\sum_{i=1}^{d}\frac{\partial_q}{\partial_q x_i}\phi(X(s))\sum_{l=1}^{k}\sigma_{il}(X(s))dB_l(s)\\
&+\frac{1}{[2]_q}\int_0^t\sum_{1\leq i,j\leq d} a_{ij}(X(s))\frac{\partial_q^2}{\partial_q x_i\partial_q x_j}\phi(X(s))d_qs\\
&= \int_0^t\mathfrak{L}\phi(X(s))d_qs + \int_0^t\sum_{i=1}^{d}\frac{\partial_q}{\partial_q x_i}\phi(X(s))\sum_{l=1}^{k}\sigma_{il}(X(s))dB_l(s).
\end{aligned}$$

Since ϕ has a compact support, the stochastic integral is centered, and we obtain

$$\frac{1}{t}(P_t\phi(x)-\phi(x)) = \mathbb{E}_x\frac{1}{t}\int_0^t \mathfrak{L}\phi(X(s))d_qs.$$

As $\mathfrak{L}\phi$ is continuous on $\mathbb{R}$ and $X(t)$ is almost surely continuous on $\mathbb{R}_+$, we thus obtain

$$\frac{1}{t}\int_0^t \mathfrak{L}\phi(X(s))d_qs \underset{t\to 0}{\longrightarrow} \mathfrak{L}\phi(x).$$

Moreover, as $\mathfrak{L}\phi$ is bounded, it follows from the dominated convergence theorem that

$$\lim_{t\to 0}\frac{1}{t}(P_t\phi(x)-\phi(x)) = \mathfrak{L}\phi(x),$$

which confirms that $\mathscr{C}_c(\mathbb{R}^d)\subset Dom(L)$ and $L=\mathfrak{L}$ on this set.

5 Application

In this section, we apply the q-Black-Scholes equation to support our theoretical findings. We determine the infinitesimal generator of this equation and solve it using advanced tools from q-calculus, stochastic calculus, and q-differential equations, contributing to our understanding of option pricing and financial mathematics. We consider the q-Black-Scholes equation

$$dX(t) = \mu X(t)d_qt + \sigma X(t)dB(t),$$

where σ and μ are constants with $\sigma > 0$, $B(t)$ is a classical Brownian motion satisfying condition $(\mathbf{H})$.
The associated infinitesimal generator is then defined as

$$L\phi(x) = \frac{\sigma^2x^2}{[2]_q}D_q^2\phi(x) + \mu x D_q\phi(x).$$

The solution $X(t)$ can be obtained by applying q-Itô's formula to $\log X(t)$. We therefore get

$$\log(X(t)) = \log(X(0)) + \int_0^t D_q\log(X(s))dX(s) + \frac{1}{[2]_q}\int_0^t D_q^2\log(X(s))\sigma^2X^2(s)d_qs.$$

This yields

$$\log(X(t)) = \log(X(0)) + \frac{\log q}{q-1}\left[\left(\mu-\frac{\sigma^2}{[2]_q}\right)t+\sigma B(t)\right].$$

Taking the exponential of both sides, we obtain

$$X(t) = X(0)\exp\left(\frac{\log q}{q-1}\left[\left(\mu-\frac{\sigma^2}{[2]_q}\right)t+\sigma B(t)\right]\right).$$

We conclude by conducting a simulation study to illustrate this example, as shown in Fig 1.

Fig 1 presents simulation results for various values of $q\in]0,1[$ with $\mu=\sigma=X(0)=1$. We observe that increasing q leads to higher values of $X(t)$. Therefore, choosing an appropriate value of q is crucial to achieving a good fit with real data.

6 Concluding Remarks and Perspectives

In this paper, we have established a q-analogue of the classical Itô formula by employing the quantum Taylor expansion within the framework of quantum calculus. This construction provides a consistent extension of stochastic analysis to q-Itô processes, encompassing both standard Brownian motion and q-deformed stochastic differential equations. We have also derived the associated q-infinitesimal generators and illustrated the theoretical developments through the q-Black-Scholes equation. As $q\to 1$, our results naturally converge to those of classical Itô calculus, ensuring consistency with the standard theory. Future research could focus on developing a rigorous q-stochastic integration theory grounded in martingale techniques and q-measure theory, and on extending the formalism to multidimensional settings. Furthermore, establishing connections between q-stochastic calculus and quantum probability may yield new insights into mathematical physics and the modeling of complex systems with time-scale deformation.

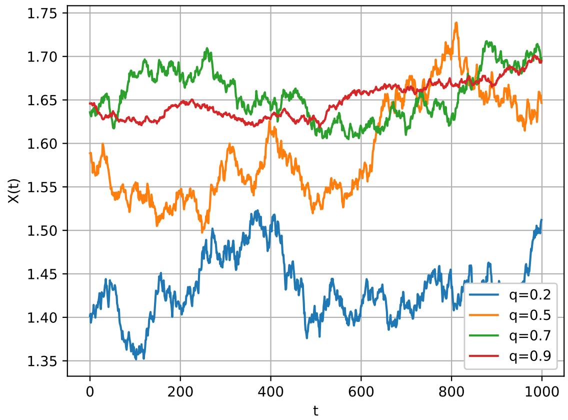

Fig. 1: Solutions of q-Black-Scholes with $\mu = \sigma = X(0) = 1$.

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