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An Extension of Caputo's k-Fractional Derivative Operator and Its Applications

Aftab Ahmad[1], *Mukhtar Ahmad*[2,*], *Yuhani Binti Yusof*[3], *Muhammad Muawwaz*[1], *Muhammad Maaz*[1], *Ather Qayyum*[4,*]

[1] Department of Mathematics, Institute of Southern Punjab Multan, Pakistan
[2] Khwaja Fareed University of Engineering and Information Technology, RYK, Pakistan
[3] Department of Mathematics, Universiti Malaysia Pahang Al-Sultan Abdullah, Malaysia
[4] QFBA-Northumbria University Doha, Qatar;Department of Mathematics, Institute of Southern Punjab Multan, Pakistan

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Abstract: This research introduces a novel extension of the Caputo fractional derivative operator, characterized by a new parameter $k > 0$. We establish several properties of the Caputo k-fractional derivative operator and present a series of results related to its application. Additionally, we extend the concept of k-hypergeometric functions and derive their integral representations utilizing the k-fractional derivative operator. Our work further includes the development of linear and bilinear generating relations for the extended k-hypergeometric functions, as well as the Mellin transform of selected extended k-fractional derivatives.

Keywords: Caputo fractional derivative operator; k-fractional derivative operator; k-hypergeometric functions; linear generating relations; mellin transform; extended k-fractional derivatives.

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1 Introduction

Fractional Calculus is a branch of mathematical investigation that studies a lot of different possibilities of defining real number powers or complex figure powers of the differentiation operator D. The initial emergence of the idea of a fractional derivative is originate in a letter written to Guillaume de l'Hopital by Gottfried Wilhelm Leibniz in 1695. The Riemann-Liouville fractional derivative is calculated by means of Lagrange's rule for differential operators. For computing fractional derivative; the Caputo fractional derivative is also used, it was introduced by Caputo. As compared to the Riemann Lioville fractional derivative, when solving differential equations using Caputo's definition, it is not essential to describe the fractional order initial conditions. The knowledge of fractional calculus is very important in number of fields of pure and applied mathematics. The Caputo fractional derivative is defined as

$$D^{\nu} f(x) = \frac{1}{\Gamma(s-\nu)} \int_0^x (x-w)^{s-\nu-1} \frac{d^s}{dw^s} f(w) dw,$$

where $s-1 < Re(\nu) < s$.
Fractional derivatives are crucial in applied mathematics for modeling complex systems, particularly linear viscoelasticity and hereditary phenomena [1, 2]. Caputo's original definition of fractional differentiation was used in seismology lectures [3].

Fractional calculus is a mathematical theory that focuses on special functions and integral transforms. Previously abstract, it has gained application in various fields, including diffusion, advection, system management, finance, and economics. Current applications include fractional control of engineering systems, analytical and numerical tools, techniques, and improvements in Calculus of Variations and finest control. Special functions, such as gamma, beta, and hypergeometric functions, are important and given their own names and notation. They are widely used by physicists,

* Corresponding author e-mail: _itxmemuktar@gmail.com, dratherquyyum@um.edu.my

mathematicians, and engineers in physics, pure and applied mathematics, number theory, fluid dynamics, and quantum mechanics heat conduction [4-7].

Euler generalized the factorial function to the gamma function (1707-1983), which has valuable applications in physical and engineering problems [8]. Euler beta function and gamma function are closely related, with the extended Euler beta function being a related extension [9-11]. Generated generalized beta and gamma functions by Diaz and Teruel while Diaz and Pariguan developed Pochhammer symbols [12]. Fractional derivatives explain memory and hereditary properties [13].

Fractional derivative has applications in modeling mechanical and electrical properties of real materials and in many other fields such as mathematical physics, astrophysics, control theory, electric conductance of organic systems and engineering. The Caputo fractional derivatives are the forms of some existing fractional derivatives. Caputo k-fractional derivatives are the generalization of some fractional derivatives [14]. The concept of Riemann-Liouville k-fractional integral which is based on gamma k-function is introduce by Mubeen and Habibullah [15]. In 2013, a new fractional operator called k- Riemann-Liouville fractional derivative introduce by Romero et al. by using gamma k-function [16]. They have proved some properties of this defined functional operator and by using extension of Riemann-Liouville fractional derivative operator, the linear and bilinear generating relations for extended hypergeometric functions is obtained.Special functions, used in mathematical, physical, and engineering applications, include trigonometric, exponential, hyperbolic, gemma, beta, Caputo fractional derivative, Appell, and Lauricella functions.

The gemma k-function, part of special transcendental functions, is used in various fields like extended hypergeometric series, asymptotic series, Caputo fractional derivative, and number theory. It is related to factorial function.

$$\beta_k(i,j;m) = \frac{\Gamma_{m,k}(i)\Gamma_{m,k}(j)}{\Gamma_{m,k}(i+j)}, \quad Re(m) > 0. \tag{1}$$

The integral representation of gamma function is given by

$$\Gamma(i) = \int_0^\infty w^{i-1} e^{-w} dw, \quad Re(i) > 0. \tag{2}$$

The gamma functions in term of new parameter k are called gamma k-functions. The gamma k-functions are denoted by $\Gamma_k(i)$ and are defined as

$$\Gamma_k(i) = \lim_{r\to\infty} \frac{r! k^r (rk)^{\frac{i}{k}-1}}{(i)_{r,k}}. \tag{3}$$

Where $(\delta)_{r,k} = \delta(\delta+k)(\delta+2k)...(\delta+(r-1)k)$ is called the Pochhammer k-symbol, and is defined as,

$$(\delta)_{r,k} = \frac{\Gamma_k(\delta+rk)}{\Gamma_k(\delta)}. \tag{4}$$

The extension of Euler's beta k-function is a generalization of the classical Euler beta function, designed to introduce flexibility and accommodate additional parameters such as k, which scales and modifies the function. This extension is particularly useful in generalized calculus, fractional calculus, and applications where standard gamma and beta functions are inadequate.

1.1 Classical Euler Beta Function

Classical Euler Beta Function $B(x,y)$ is defined:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

where $Re(x) > 0$ and $Re(y) > 0$. It can also be expressed in terms of the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x,y)}$$

The beta function is symmetric:

$$B(x,y) = B(y,x)$$

, and arises naturally in problems involving integration of products of powers of t and $1-t$.

1.2 Extension of Euler Beta k-Function

The extension of Euler Beta k-Function often denoted $B(x,y)$ modifies the classical beta function by incorporating a parameter $k>0$, which scales and extends the function. It is defined as:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

where $Re(x)>0$ and $Re(y)>0$.

1.3 Applications and Utility

Replacement for Gamma Functions: he extended beta k-function is often used to simplify expressions involving combinations of extended gamma functions $\Gamma_k(x)$.
Generalized Special Functions: It appears in extended hypergeometric functions, Appell functions, and other special functions as a building block.
Fractional Calculus: The k-extension is particularly suited to fractional calculus, where the scaling parameter k aligns with generalized fractional integrals and derivatives.
Convergence and Flexibility: By adjusting k the k-beta function provides better control over convergence properties and parameter tuning.

1.4 Special Cases

When $k=1$ The extended beta k-function reduces to the classical beta function:

$$B(x,y)|_{k=1}=B(y,x)$$

. Symmetry in x and y: Just as in the classical case, the symmetry holds:

$$B(x,y)=B(y,x)$$

. It appears in definite integration, extended hypergeometric k-functions, and in Caputo k-fractional derivative etc. The extended beta k-function is defined as

$$\beta_{m,k}(i,j)=\frac{1}{k}\int_0^\infty w^{\frac{i}{k}-1}(1-w)^{\frac{j}{k}-1}e^{-(\frac{m^k}{kw(1-w)})}dw, \tag{5}$$

1.5 Extended hypergeometric k-function

$$F_{m,k}(\delta,\eta;\lambda;x)=\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}B_{m,k}(\eta+rk,\lambda-\eta)x^r}{B_k(\eta,\lambda-\eta)}, \tag{6}$$

This equation defines the extended hypergeometric k-function $F_{m,k}$ in terms of the extended k-beta function and a generalized k-Pochhammer symbol. Let's break it down:

1.5.1 Summation:

$$\sum_{r=0}^{\infty}$$

This indicates the function is expressed as a power series expansion in x, summing over r starting from 0.

1.5.2 Generalized k-Pochhammer Symbol $(\delta)_{r,k}$:

The term $(\delta)_{r,k}$ represents the k-Pochhammer symbol, which is a generalization of the classical Pochhammer symbol. It is typically defined recursively and incorporates k as a parameter.

1.5.3 Extnded k-Beta Function $B_{r,k}$:

$B_{r,k}(\eta + rk, \lambda - \eta)$ is an extension of the classical beta function, dependent on the parameters $\eta + rk, \lambda - \eta$ and m. This incorporates additional parameters k (scaling) and m (extension factor), introducing flexibility into the framework.

1.5.4 Denominator $B_k(\eta, \lambda - \eta)$:

The term $B_k(\eta, \lambda - \eta)$ normalizes the series, ensuring convergence and defining the extended k-beta function for given parameters.

1.5.5 Power Series x^r:

Each term of the series is weighted by x^r where x serves as the primary variable of the function.
Purpose of Equation (6): It extends classical hypergeometric functions by introducing parameters m and k, enabling applications to broader contexts, such as fractional calculus or generalized special functions.

1.6 Extended Riemann-Liouvill k-Fractional Derivative

$$D_{x,k}^{\delta,m} = \frac{d^s}{dx^s} D_{x,k}^{\delta-s} f(x) = \frac{d^s}{dx^s}\left(\frac{1}{\Gamma_k(-\delta+s)} \int_0^x (x-w)^{-\delta+s-k} e^{\frac{-mx^k}{w(x-w)}} f(w)dw\right). \tag{7}$$

This equation introduces an extended k-Fractional derivative operator $D_{x,k}^{\delta,m}$ Here's the step-by-step breakdown:

1.6.1 Outer Derivative:

$$\frac{d^s}{dx^s}$$

1.6.2 Fractional Integral Component:

$$D_{x,k}^{\delta-s} f(x)$$

Involves a fractional integral or derivative of order $\delta - s$ applied to $f(x)$. The order δ s a real parameter, satisfying $(s-1) < Re(\delta) < s$.

1.6.3 Normalization via $\Gamma_k(-\delta+s)$:

The term $\Gamma_k(-\delta+s)$ generalizes the gamma function, accounting for the $k$extension.

1.6.4 Integral Term:

$$\int_0^x (x-w)^{-\delta+s-k} e^{\frac{-mx^k}{w(x-w)}} f(w)dw$$

$(x-w)^{-\delta+s-k}$: A kernel incorporating fractional powers, k and δ.
$e^{\frac{-mx^k}{w(x-w)}}$: An exponential weight factor, dependent on m, k and w.
$f(w)$: The function under consideration.

1.6.5 Constraints on Parameters:

$Re(m) > 0$: Ensures convergence of the integral by controlling the exponential term.
$s-1 < Re(\delta) < s$: Restricts the order δ for a valid fractional derivative.
Purpose of Equation (7: This operator generalizes the Riemann-Liouville fractional derivative, incorporating k-extensions and an exponential weight. It's especially useful in fractional calculus and differential equations with complex boundary conditions.
The Caputo fractional derivative, along with Grunwald Letnikov and Riemann-Liouville ones, is commonly used for modeling applied difficulties, signal and image properties, and dispersion in permeable mediums, with the extended Gauss hypergeometric k-function defined.

$$_2F_{1,k}(u,v;f;x;m) = \sum_{r=0}^{\infty} \frac{(u)_{r,k}(v)_{r,k} B_{m,k}(v-sk+r, f-v+sk)x^r}{(v-sk)_{r,k} B_k(v-sk, f-v+sk) r!}. \quad (8)$$

for all $|x| < 1$ where $s < Re(v) < Re(f)$. The extended beta k-function is used to extend Appel hypergeometric functions as

$$F_{1,k}(u,v,f;g;i,j;m) = \sum_{r,t=0}^{\infty} \frac{(u)_{r+t,k}(v)_{r,k}(f)_{r,k} B_m(u-sk+r+t, g-u+sk) i^r j^t}{(u-sk)_{r+t,k} B(u-sk, g-u+sk) r! t!}, \quad (9)$$

$(u)_{r+t,k}(v)_{r,k}(f)_{r,k}(u-sk)_{r+t,k}$ k- pochhammer symbols:
$(a)_{r,k}$ is the k- pochhammer symbols, defined recursively as $(a)_{r,k} = a(a+k)(a+2k)....(a+(r-1)k)$ for $r > 0$ and $(a)_{0,k}$ = 1.
To ensure proper definition, the initial parameter $a(e.g., u, v, f, u-sk)$ must not take values where negative or undefined products occur.
For convergence, the k-Pochhammer terms should grow slower than the factorial terms in the denominator.

1.7 Summary of Requirements on Parameters

1.7.1 Series Expansion Variables:

$$|i| < 1 and |j| < 1.$$

1.7.2 Pochhammer Symbol Parameters:

- $u, v, f, u-sk$, must not lead to undefined terms in $(a)_{r,k}$,

1.7.3 Beta Function Parameters:

- $u-sk+r+t > 0$,
- $g-u+sk > 0$,
- $u-sk > 0$,
- $g-u+sk > 0$,

1.7.4 Real Part Constraint on u:

$$s < Re(u) < Re(g)$$

These constraints collectively ensure the proper definition and convergence of the extended Appell hypergeometric function $F_{1,k}$. Let me know if you need further clarification.

The Lauricella hypergeometric function has unique consequence for mathematical physics and applied mathematics due to elliptic integrals are hypergeometric functions of type F_D.

$$F^3_{D,m,k}(u,v,f,g;h;ix,jx,x;m)$$

$$= \sum_{r,l,t=0}^{\infty} \frac{(u)_{r+l+t,k}(v)_{r,k}(f)_{t,k}(g)_{l,k}B_{m,k}(u-sk+rk+tk+lk,sk-u+g)i^r j^t x^l}{(u-sk)_{r+l+t,k}B_k(u-sk,h-u+sk)} r!l!t!.$$

The extended Caputo k-fractional derivative is defined as

$$D^{v,m}_{x,k} f(x) = \frac{1}{\Gamma_k(sk-\nu)} \int_0^x (x-w)^{sk-\nu-k} e(\frac{-mx^2}{w(x-w)}) \frac{d^s}{dw^s} f(w)dw. \tag{10}$$

for all $Re(m) > 0$ and $s-1 < Re(\nu) < s$.

This project consists of paper, covering essential definitions, extended derivatives, linear and bilinear generating associations, Mellin transforms, and integral extended hypergeometric functions represented. The generalized fractional Caputo derivatives are used to calculate linear and bilinear generating associations, and the Mellin transforms are used to derive the integral representation of extended hypergeometric functions. The field of hypergeometric functions is crucial for ongoing research [17, 18].

2 Preliminaries

In this paper, we give some definitions and basic consequences that are used in later paper. It includes Pochhammer's symbol, extended gamma function, extended beta function,extended hypergeometric function, extended hypergeometric function.

Chaudhry and colleagues [13] derived the expansion of Euler's beta function in 1997. which is defined as;

$$\beta_m(i,j) = \int_0^1 w^{i-1}(1-w)^{j-1} e^{(\frac{-m}{w(1-w)})} dw, \tag{11}$$

where $Re(m) > 0, Re(i) > 0, Re(j) > 0$.

2.0.1 Pochhammer's Symbol:

The Pochhammer's symbol is denoted by $(\delta)_n$, δ is any complex number, as defined by German mathematician Leo Pochhammer.

$$(\delta)_r = \begin{cases} \delta(\delta+1)(\delta+2)...(\delta+r-1), & r \in \mathbb{N} \\ 1, & r = 0, \delta \neq 0. \end{cases} \tag{12}$$

It follows that $(1)_r = r!$ for $r \geq 1, a \neq 0$ and $(\delta)_0 = 1$.

Note that $r \in \mathbb{N}$ and also connection between the gamma function and Pochhammer's symbol is describe as

$$(\delta)_r = \frac{\Gamma(\delta+r)}{\Gamma(\delta)}. \tag{13}$$

The integral representation of gamma function is represented as

$$\Gamma(i) = \int_0^\infty w^{i-1} e^{-w} dw, \quad Re(i) > 0. \tag{14}$$

Chaudhry et al. [5] extended the hypergeometric function by taking $B_m(i,j)$ and the extended hypergeometric function is defined as

$$F_m(\delta,\eta;\lambda;x)=\sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}\frac{B_m(\eta+r,\lambda-\eta)}{B(\eta,\lambda-\eta)}x^r. \tag{15}$$

The linear and bilinear generating relations for extended hypergeometric functions by using the extension of the Riemann-Liouville fractional derivative operator was obtained by Özarslan and Özergin [21], define as

$$D_x^{\delta,m}f(x)=\frac{d^s}{dz^s}D_z^{\delta-s}f(x)=\frac{d^s}{dx^s}[\frac{1}{\Gamma(-\delta+s)}\int_0^x(x-w)^{\delta+s-1}e^{(\frac{-mx^2}{w(x-w)})}f(w)dw], \tag{16}$$

where $Re(m)>0$ and $s-1<Re(\delta)<s$, If we put $m=0$, these extension concide with original ones.

2.1 Results in $k>0$ form:

Some properties of gamma k-function, beta k-function and Pochhammer k-symbol that are useful in present work are introduced and proved by Diaz and Pariguan [13]

$$\beta_k(i,j)=\frac{\Gamma_k(i)\Gamma_k(j)}{\Gamma_k(i+j)}, \tag{17}$$

$Re(i)>0, Re(j)>0,\ k>0$

$$\Gamma_k(i+k)=i\Gamma_k(i), \tag{18}$$

$$(i)_{r,k}=\frac{\Gamma_k(i+rk)}{\Gamma_k(i)}. \tag{19}$$

The Pochhammer k-symbol is defined as

$$(\delta)_{r,k}=\delta(\delta+k)(\delta+2k)...(\delta+(r-1)k), \tag{20}$$

$r\in\mathbb{N}^+,\delta\in\mathbb{C}$.
Researchers also proved some other important results, that are given as

$$(i)_{s,k}(i+sk)_{r,k}=(i)_{s+r,k}, \tag{21}$$

$$(i)_{r-s,k}=\frac{(-1)^s(i)_{r,k}}{(k-i-rk)_{s,k}}, \tag{22}$$

and

$$\frac{(-1)^r}{(i)_{r,k}}=\frac{\Gamma_k(k-i-rk)}{\Gamma_k(k-i)}. \tag{23}$$

2.1.1 Extended hypergeometric k-functions:

The extended Gauss hypergeometric k-function is

$${}_2F_{1,k}(u,v;f;x;m)=\sum_{r=0}^{\infty}\frac{(u)_{r,k}(v)_{r,k}}{(v-sk)_{r,k}}\frac{B_{m,k}(v-sk+r,f-v+sk)}{B(v-sk,f-v+sk)}\frac{x^r}{r!}. \tag{24}$$

2.1.2 The extended Appell hypergeometric k-functions F_1:

The extended Appell hypergeometric k-functions F_1 is

$$F_{1,k}(u,v,f;g;i,j;m)=\sum_{r,t=0}^{\infty}\frac{(u)_{r+t,k}(v)_{r,k}(f)_{t,k}}{(u-sk)_{r+t,k}}\frac{B_{m,k}(u-sk+r+t,g-u+sk)}{B(u-sk,g-u+sk)}\frac{i^r}{r!}\frac{j^t}{t!}. \tag{25}$$

2.1.3 The extended Appell k-hypergeometric function F_2:

The extended Appell k-hypergeometric function F_2 is

$$F_{2,k}(u,v,f;g,h;i,j;m)=\sum_{r,t=0}^{\infty}\frac{(u)_{r+t,k}(b)_{r,k}(f)_{t,k}}{(v-sk)_{r,k}(h)_{t,k}}\frac{B_{m,k}(v-sk+r,g-v+sk)}{B(v-sk,g-v+sk)}\frac{i^r j^t}{r!t!}$$
$$=\sum_{r,t=0}^{\infty}\frac{(u)_{r+t,k}(v)_{r,k}(c)_{t,k}}{(g)_{r,k}(f-sk)_{t,k}}\frac{B_{m,k}(f-sk+r,h-f+sk)}{B_k(f-sk,h-f+sk)}\frac{i^r j^t}{r!t!}$$
$$=\sum_{r,t=0}^{\infty}\frac{(u)_{r+t,k}(v)_{r,k}(f)_{t,k}}{(v-sk)_{r,k}(f-sk)_{t,k}}\frac{B_{m,k}(v-sk+r,g-v+sk)}{B_k(v-sk,g-v+sk)}\frac{B_{m,k}(f-sk+r,h-f+sk)}{B(f-sk,h-f+sk)}\frac{i^r j^t}{r!t!}.$$

3 Summation Theorems

In this paper, we proof the theorems that give the generalization of the results.

3.1 Extended fractional derivatives of some elementary functions:

We calculate extended fractional derivative of some elementary functions, such as Caputo fractional derivative.

Remark.:

By setting $\psi=0,1,...,s-1$.
Then $D_x^{v,m}(z^{\psi})=0$

Now, an expression is given for the extended Caputo fractional derivative of an analytic function.

Theorem 1.*If $f(x)$ has a power series expansion and is an analytic function on the disk $|x|<m$ we can express $f(x)=\Sigma_{r=0}^{\infty}u_r x^r$, subsequently*

$D_x^{v,m}[x^{\psi-1}f(x)]=\Sigma_{r=0}^{\infty}u_r D_x^{v,m}[x^{r+\psi-1}]$.

*Proof.*If $f(x)$ has a power series expansion and is an analytic function on the disk $|x|<m$ we can express $f(x)=\Sigma_{r=0}^{\infty}u_r x^r$, subsequently

$D_x^{v,m}[x^{\psi-1}f(x)]=\Sigma_{r=0}^{\infty}u_r D_x^{v,m}[x^{r+\psi-1}],.$

where $s-1<Re(\psi)<s<Re(\psi)$.
Since $f(x)=\sum_{r=0}^{\infty}u_r x^r$, we consider:

$$x^{\psi-1}f(x)=x^{\psi-1}\sum_{r=0}^{\infty}u_r x^r=\sum_{r=0}^{\infty}u_r x^{r+\psi-1}$$

Now, apply the fractional derivative operator $D_x^{\nu,m}$ to both sides:

$$D_x^{\nu,m}[x^{\psi-1}f(x) = D_x^{\nu,m}[\sum_{r=0}^{\infty} u_r x^{r+\psi-1}]$$

Using the linearity of the operator $D_x^{\nu,m}$, the derivative of the sum is the sum of the derivative:

$$D_x^{\nu,m}[x^{\psi-1}f(x) = \sum_{r=0}^{\infty} u_r D_x^{\nu,m} x^{r+\psi-1}$$

.

This conforms the structure form the theorem. Next, evaluate $D_x^{\nu,m}x^{r+\psi-1}$.
The fractional derivative $D_x^{\nu,m}$ of $x^{r+\psi-1}$

$$\mathscr{D}_x^{\nu,m}[x^{\psi+r-1}] = \frac{\Gamma(\psi+r)B_m(\psi-s+r,s-\nu)x^{\psi+r-1-\nu}}{\Gamma(\psi-\nu+r)B(\psi-s+r,s-\nu)}.$$

Substituting this into the summation:

$$D_x^{\nu,m}(x^{\psi-1}\sum_{r=0}^{\infty} u_r x^r) = \sum_{r=0}^{\infty} u_r D_x^{\nu,m}(x^{\psi+r-1}) = \sum_{r=0}^{\infty} u_r \frac{\Gamma(\psi+r)B_m(\psi-s+r,s-\nu)x^{\psi+r-1-\nu}}{\Gamma(\psi-\nu+r)B(\psi-s+r,s-\nu)}.$$

Introduce the r-Pochhammer symbol:

$$(\delta)_r = \frac{\Gamma(\delta+r)}{\Gamma(\delta)}$$

Using this, rewrite $\Gamma(\psi+r)$ and $\Gamma(\psi-v+r)$:

$$\Gamma(\psi+r) = \Gamma(\psi)(\psi)_r, \Gamma(\psi-v+r) = \Gamma(\psi-v)(\psi-v)_r$$

Also, recall the definition of the classical beta function:

$$B(i,j) = \frac{\Gamma(i)\Gamma(j)}{\Gamma(i+j)}.$$

Substituting these into the summation:

$$D_x^{\nu,m}[(x^{\psi-1}f(x)] = \sum_{r=0}^{\infty} u_r \frac{\Gamma(\psi)x^{\psi-\nu-1+r}}{\Gamma(\psi-\nu)} \frac{(\psi)_r B_m(\psi-s+r,s-\nu)}{(\psi-\nu)_r B(\psi-s+r,s-\nu)}$$

Notice that $\Gamma(\psi)x^{\psi-\nu-1}$ and $\Gamma(\psi-\nu)$ are independent of r factor of terms out:

$$D_x^{\nu,m}[(x^{\psi-1}f(x)] = \frac{\Gamma(\psi)x^{\psi-\nu-1}}{\Gamma(\psi-\nu)} \sum_{r=0}^{\infty} u_r \frac{(\psi)_r B_m(\psi-s+r,s-\nu)}{(\psi-\nu)_r B(\psi-s+r,s-\nu)} x^r$$

After further simplification, we recognize the structure of the derivative applied to

$$D_x^{\nu,m}[(x^{\psi-1}f(x)] = \frac{\Gamma(\psi)x^{\psi-\nu-1}}{\Gamma(\psi-\nu)} \sum_{r=0}^{\infty} u_r \frac{(\psi)_r B_m(\psi-s+r,s-\nu)}{(\psi-s)_r B(\psi-s,s-\nu)} x^r,$$

For this expression to hold, the parameters must satisfy:

• $s-1 < Re(\nu) < s$,
• $s < Re(\psi)$,
• $Re(\psi) > 0$,
This ensures the convergence of the series and the proper definition of the beta and gamma functions.
Thus, the theorem is proven.

Theorem 2. *Let $s-1<Re(\psi-\nu)<s<Re(\psi)$, then*

$$D_x^{\psi-\nu,m}[x^{\psi-1}(1-x)^{-\delta}]=\frac{\Gamma(\psi)x^{\nu-1}}{\Gamma(\nu)}\sum_{r=0}^{\infty}\frac{(\delta)_r(\psi)_r}{(\psi-s)_r}\frac{B_m(\psi-s+r,\nu-\psi+s)x^r}{B(\psi-s,\nu-\psi+s)r!}$$
$$=\frac{\Gamma(\psi)x^{\nu-1}}{\Gamma(\nu)}{}_2F_1(\delta;\psi;\nu;x;m),$$

for $|x|<1$.

Proof.: We take the power series expansion of $(1-x)^{-\delta}$ and (8). The power series expansion of $(1-x)^{-\delta}$ is known as

$$(1-x)^{-\delta}=\sum_{r=0}^{\infty}\frac{(\delta)_r x^r}{r!}. \tag{26}$$

$$D_x^{\psi-\nu,m}[x^{\psi-1}(1-x)^{-\delta}]=D_x^{\delta-\nu,m}[x^{\psi-1}\Sigma_{r=0}^{\infty}(\delta)_r\frac{x^r}{r!}]$$
$$=\sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}D_x^{\psi-\nu,m}[x^{\psi+r-1}].$$

We know that

$$D_x^{\nu,m}[x^{\psi+r-1}]=\frac{\Gamma(\psi)x^{\psi-\nu-1}}{\Gamma(\psi-\nu)}\sum_{r=0}^{\infty}u_r\frac{(\psi)_rB_m(\psi-s+r,s-\nu)}{(\psi-s)_rB(\psi-s,s-\nu)}x^r.$$

Replace $\nu=\psi-\nu$, in the above equation we get

$$D_x^{\psi-\nu,m}[x^{\psi+r-1}]=\frac{\Gamma(\psi)x^{\nu-1}}{\Gamma(\nu)}\frac{(\psi)_r(B_m(\psi-s+r,s-\psi+\nu)}{(\psi-s)_rB(\psi-s,s-\psi+\nu)}x^r.$$

By putting this , we get

$$\sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}D_x^{\psi-\nu,m}[x^{\psi+r-1}]=\sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}\frac{\Gamma(\psi)x^{\nu-1}}{\Gamma(\nu)}\frac{(\psi)_r(B_m(\psi-s+r,s-\psi+\nu)}{(\psi-s)_rB(\psi-s,s-\psi+\nu)}x^r$$
$$=\frac{\Gamma(\psi)}{\Gamma(\nu)x^{\nu-1}}\sum_{r=0}^{\infty}\frac{(\delta)_r(\psi)_r(B_m(\psi-s+r,s-\psi+\nu)}{(\psi-s)_rB(\psi-s,s-\psi+\nu)}\frac{x^r}{r!}.$$

Now by using (8), we get

$$D_x^{\psi-\nu,m}[x^{\psi-1}(1-x)^{-\delta}]=\frac{\Gamma(\psi)x^{\nu-1}}{\Gamma(\nu)}{}_2F_{1(\delta,\psi;\nu;x;m)}.$$

Theorem 3. *The extended Caputo fractional derivative ${}_2F_1(u,v;f;x)$ is*

$$D_x^{\nu,m}[{}_2F_1(u,v;f;x)]=\frac{(u)_s(v)_s}{(f)_s}\frac{x^{s-\nu}}{\Gamma(1-\nu+s)}\sum_{r=0}^{\infty}\frac{(u+s)_r(v+s)_r}{(f+s)_r(1-\nu+s)_r}\frac{B_m(s-\nu,r+1)x^r}{B(s-\nu,r+1)}$$

for $|x|<1$.

Proof.: Using the power series expansion ${}_2F_1(u,v;f;x)$ and by calculation, the power series expansion ${}_2F_1(u,v;f;x)$ is given as

$${}_2F_1(u,v;f;x)=\sum_{r=0}^{\infty}\frac{(u)_r(v)_r}{(f)_r}\frac{x^r}{r!},$$

$$D_x^{\nu,m}[{}_2F_1(u,v;f;x)]=D_x^{\nu,m}[\sum_{r=0}^{\infty}\frac{(u)_r(v)_r}{(f)_r}\frac{x^r}{r!}]$$
$$=[\sum_{r=0}^{\infty}\frac{(u)_r(v)_r}{(f)_rr!}D_x^{\nu,m}[x^r].$$

And $D_x^{\nu,m}(x^r)$ is given as

$$D_x^{\nu,m}(x^r) = \frac{\Gamma(r+1)B_m(s-\nu,r-s+1)}{\Gamma(r-\nu+1)B(s-\nu,r-s+1)}x^{r-\nu},$$

$$D_x^{\nu,m}[{}_2F_1(u,v;f;x)] = [\sum_{r=s}^{\infty}\frac{(u)_r(v)_r}{(f)_r r!}\frac{\Gamma(r+1)B_m(s-\nu,r-s+1)}{\Gamma(r-\nu+1)B(s-\nu,r-s+1)}x^{r-\nu}].$$

By setting $r=r+s$, we get

$$D_x^{\nu,m}[{}_2F_1(u,v;f;x)] = [\sum_{r=0}^{\infty}\frac{(u)_{r+s}(v)_{r+s}}{(f)_{r+s}(r+s)!}\frac{\Gamma(r+s+1)B_m(s-\nu,r+1)}{\Gamma(r+s-\nu+1)B_c(s-\nu,r+1)}x^{r+s-\nu}].$$

By setting $(u)_{r+s}=(u)_s(u+s)_r$, $(v)_{r+s}=(v)_s(v+s)_r$ and $(f)_{r+s}=(f)_s(f+s)$, we get our result

$$D_x^{\nu,m}[{}_2F_1(u,v;f;x)] = \frac{(u)_s(v)_s}{(f)_s}\frac{x^{s-\nu}}{\Gamma(1-\nu+s)}\sum_{r=0}^{\infty}\frac{(u+s)_r(v+s)_r}{(f+s)_r(1-\nu+s)_r}\frac{B_m(s-\nu,r+1)x^r}{B(s-\nu,r+1)}.$$

Theorem 4. *Let $Re(w)>0$ and $|x|<1$, then*

$$M[D_x^{\nu,m}(x^{\psi}):w] = \frac{\Gamma(s)x^{s-\nu}}{\Gamma(s-\nu)}\sum_{r=0}^{\infty}\frac{B(s-\nu+w,r+w+1)}{\Gamma(r+1)}(\nu)_r x^r. \quad (27)$$

Proof.: Expanding the power series of $(1-x)^{-\delta}$ and considering ψ=r in previous theorem, we find

$$M[D_x^{\nu,m}((1-x)^{-\delta}):w] = M[D_x^{\nu,m}[\sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}x^r:w]$$

$$= \sum_{r=0}^{\infty}\frac{(\delta)_r}{r!}M[D_x^{\nu,m}(x^r):w]$$

$$= \frac{\Gamma(s)x^{-\nu}}{\Gamma(s-\nu)}\sum_{r=s}^{\infty}\frac{B(s-\nu+w,r-s+w+1)}{\Gamma(r-s+1)}(\delta)_r x^r.$$

Replace $r=r+s$ and after simplification, we get

$$M[D_x^{\nu,m}((1-x)^{-\delta}):w] = \frac{\Gamma(w)x^{s-\nu}}{\Gamma(s-\nu)}\sum_{r=0}^{\infty}B(s-\nu+w,r+w+1)\frac{(\delta)_{r+s}x^r}{r!}.$$

3.2 Summation Theorems in k-form

This paper introduces a new $k>0$-parameter fractional derivative, generalizes Caputo derivative, and finds extended k-fractional derivative of basic functions. It also presents k-hypergeometric extension, calculates generating relations, and derives Mellin transforms.

Theorem 5. *If $f(x)$ is an analytic function on the disk $|x|<m$ and has a power series expansion $f(x)=\Sigma_{r=0}^{\infty}u_{r,k}x^{\frac{r}{k}}$, then*

$\mathscr{D}_{x,k}^{\nu,m}f(x) = \Sigma_{r=0}^{\infty}u_{r,k}D_{x,k}^{\nu,m}x^{\frac{r}{k}}$

Proof.: The power series expansion of *f*, we have

$$\mathscr{D}_{x,k}^{\nu,m}f(x) = \frac{1}{k\Gamma_k(sk-\nu)}\int_{r=0}^{x}(x-w)^{\frac{-\nu}{k}+s-1}e^{\frac{-m^k x^2}{kw(x-w)}}\sum_{r=0}^{\infty}u_{r,k}\frac{d^s}{dw^s}w^{\frac{r}{k}}dw. \quad (28)$$

(29)

As the power series converges regularly and the integral converges completely, then the sort of the integration and the summation can be altered. So we have,

$$D_{x,k}^{\nu,m}f(x) = \sum_{r,k=0}^{\infty} u_{r,k}\frac{1}{k\Gamma_k(sk-\nu)}\int_0^x (x-w)^{\frac{-\nu}{k}+s-1}e^{\frac{-m^k x^2}{kw(x-w)}}\frac{d^s}{dw^s}w^{\frac{r}{k}}dw \tag{30}$$

$$= \sum_{r=0}^{\infty} u_{r,k}D_{x,k}^{\nu,m}x^{\frac{r}{k}}. \tag{31}$$

Theorem 6. *Let $sk-1 < Re(\psi-\nu) < sk < Re(\psi)$, then*

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\,{}_2F_{1,k}(\delta+r,\psi;\nu;m)w^r = (1-w)^{\frac{-\delta}{k}}\,{}_2F_{1,k}(\delta,\psi;\nu;\frac{xk}{1-w};m), \tag{32}$$

where $|xk| < min[1,|1-w|]$.

Proof.: Consider the identity

$$[(1-xk)-w]^{\frac{-\delta}{k}} = (1-w)^{\frac{-\delta}{k}}(1-\frac{xk}{1-w})^{\frac{-\delta}{k}}$$

we start by analyzing and expanding the left-hand side of the given equation step by step. The equation is:

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}(1-xk)^{\frac{-\delta}{k}}(\frac{x}{1-xk})^r = (1-w)^{\frac{-\delta}{k}}(1-\frac{xk}{1-w})^{\frac{-\delta}{k}}. \tag{33}$$

where $|w| < |1-xk|$.
Now, derive Equation (33) by incorporating the Caputo k-fractional derivative operator $D_{x,k}^{\psi-\nu,m}$ and using the series expansion properties.
If we multiply both sides $z^{\frac{\psi}{k}-1}$ and by using the extended Caputo k fractional derivative operator $D_{x,k}^{\psi-\nu,m}$, we get

$$D_{x,k}^{\psi-\nu,m}[\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}w^r}{r!}x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\delta}{k}-r}] = D_{x,k}^{\psi-\nu,m}[(1-w)^{\frac{-\delta}{k}}x^{\frac{\psi}{k}-1}(1-\frac{xk}{1-w})^{\frac{-\delta}{k}}]. \tag{34}$$

Since $|w| < |1-xk|$ and $Re(\psi) > Re(\nu) > 0$, it is feasible to change the order of the summation and the derivative as

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}D_{x,k}^{\psi-\nu,m}[x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\delta}{k}-r}]w^r = (1-w)^{\frac{-\delta}{k}}D_{x,k}^{\psi-\nu,m}[x^{\frac{\psi}{k}-1}(1-\frac{xk}{1-w})^{\frac{-\delta}{k}}].$$

So we get the result after using previous theorem on both sides.

Theorem 7. *Let $sk-1 < Re(\psi-\nu) < sk < Re(\psi)$, then*

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\,{}_2F_{1,k}(\eta-r,\psi;\nu;x;m)w^r = (1-w)_2^{\frac{-\delta}{k}}F_{1,k}(\eta,\delta,\psi;\nu;x;\frac{xkw}{1-w};m),$$

Additionally, x appears in the fractional argument of the hypergeometric function. To ensure the overall convergence of the series representation and the stability of the transformation, the constraint $|w| < \frac{1}{1+|x|}$ is imposed.
This restriction on w guarantees that the series converges. The transformed hypergeometric k-function ${}_2F_{1,k}$ is well-defined.

Proof. Let the identity

$$[1-(1-xk)w]^{\frac{-\delta}{k}} = (1-w)^{\frac{-\delta}{k}}(1+\frac{xkw}{1-w})^{\frac{-\delta}{k}}$$

expending the left hand side, we have

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}(1-xk)^r w^r = (1-w)^{\frac{-\delta}{k}}(1-\frac{-xkw}{1-w})^{\frac{-\delta}{k}}.$$

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

When $|w| < |1-xk|$. If we multiply the both side $x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\eta}{k}}$ and by using the extended Caputo k- fractional derivative operator $D_{x,k}^{\delta-\nu,m}$, we obtain

$$D_{x,k}^{\psi-\nu,m}[\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\eta}{k}}(1-xk)^r w^r] = D_{x,k}^{\psi-\nu,m}[(1-w)^{\frac{-\delta}{k}}x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\eta}{k}}(1-\frac{-xkw}{1-w})^{\frac{-\delta}{k}}].$$

Since $|w| < |1-xk|$ and $Re(\psi) > Re(\nu) > 0$, it is likely to change the place of the summation and the derivative as

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}D_{x,k}^{\psi-\nu,m}[x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\eta}{k}+r}]w^r = (1-w)^{\frac{-\delta}{k}}D_{x,k}^{\psi-\nu,m}[x^{\frac{\psi}{k}-1}(1-xk)^{\frac{-\eta}{k}}(1-\frac{xk}{1-w})^{\frac{-\delta}{k}}].$$

• w is an auxiliary variable introduced to expand and transform the hypergeometric k-function.
• The condition $|w| < |1-xk|$ ensures convergence of the series and the validity of the transformed hypergeometric function.
• w is linked to x in the argument of the transformed ${}_2F_{1,k}$ which incorporates $\frac{xkw}{1-w}$

Theorem 8. *Let $sk-1 < Re(\eta-\varphi) < sk < Re(\eta)$ and $sk < Re(\psi) < Re(\nu)$, then*

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\ {}_2F_{1,k}(\delta+r,\psi;\nu;x;m){}_2F_{1,k}(-r,\eta;\varphi;q;m) = \ {}_2F_{1,k}(\delta,\psi;\eta;\nu,\varphi;x,\frac{qw}{1-w};m)$$

Proof. Let consider $w \mapsto (1-qk)w$ in (4.1) and then multiply the both side with $q^{\frac{\eta}{k}-1}$, we obtain

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\ {}_2F_{1,k}(\delta+r,\psi;\nu;x;m)q^{\frac{\eta}{k}-1}(1-qk)^r w^r = q^{\frac{\eta}{k}-1}[1-(1-q)w]_2^{\frac{-\delta}{k}}F_{1,k}(\delta,\psi;\nu;\frac{x}{1-(1-q)w};m).$$

Using the fractional derivative $D_{\nu,k}^{\eta-\varphi}$ to both sides and changing the order, we get

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\ {}_2F_{1,k}(\delta+r,\psi;\nu;x;m)D_{\nu,k}^{\eta-\varphi}[q^{\frac{\eta}{k}-1}(1-qk)^r]w^r$$
$$= D_{\nu,k}^{\eta-\varphi}[q^{\frac{\eta}{k}-1}[1-(1-q)w]^{\frac{-\delta}{k}}\ {}_2F_{1,k}(\delta,\psi;\nu;\frac{x}{1-(1-q)w};m)].$$

when $|x| < 1, |\frac{1-q}{1-x}w| < 1$ and $|\frac{x}{1-w}|+|\frac{qw}{1-w}| < 1$. By taking the equality as

$$\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}\ {}_2F_{1,k}(\delta+r,\psi;\nu;x;m)D_{\nu,k}^{\eta-\varphi}[q^{\frac{\eta}{k}-1}(1-qk)^r]w^r$$
$$= D_{\nu,k}^{\eta-\varphi}[q^{\frac{\eta}{k}-1}[1-\frac{-qkw}{1-w}]^{\frac{-\delta}{k}}\ {}_2F_{1,k}(\delta,\psi;\nu;\frac{x}{1-\frac{-qw}{1-w}};m)].$$

and applying previous theorem we obtain the desired result.

3.3 Further results and observations in k- form

We use the extended Caputo fractional derivative operator to common functions e^2 and ${}_2F_{1,k}(u,v;f;x)$. Additionally, we compute the integral representations of various extended hypergeometric k-functions and obtain the Mellin transforms of some extended Caputo k-fractional derivatives.

Theorem 9. *Caputo's extended fractional derivative of $f(x)$=e^x is*

$$D_{x,k}^{\nu,m}e^x = \frac{x^{sk-\nu}}{\Gamma(sk-\nu)}\sum_{r=0}^{\infty}\frac{x^r}{r!}B_{m,k}(sk-\nu,r+k) \quad (35)$$

for all x.

*Proof.*Applying the power series expansion of e^x and previous theorem , we have

$$D_{x,k}^{\nu,m}[e^x]=\sum_{r=0}^{\infty}\frac{1}{r!}D_{x,k}^{\nu,m}[x^r]=\sum_{r=s}^{\infty}\frac{\Gamma_k(r+k)B_{m,k}((r-sk+k,sk-\nu)}{\Gamma_k(r-\nu+k)B_k(r-sk+k,sk-\nu)}\frac{x^{r-\nu}}{r!} \tag{36}$$

$$=\sum_{r=0}^{\infty}\frac{\Gamma_k(r+sk+k)B_{m,k}(r+k,sk-\nu)x^{r+nk-\nu}}{\Gamma_k(r+sk-\nu+k)B_k(r+k,sk-\nu)}\frac{x^{r+sk-\nu}}{(r+sk)!} \tag{37}$$

$$=\frac{x^{sk-\nu}}{\Gamma_k(sk-\nu)}\sum_{r=0}^{\infty}\frac{x^r}{r!}B_{m,k}(sk-\nu,r+k). \tag{38}$$

Theorem 10.*The extended Caputo k-fractional derivative* ${}_2F_{1,k}(u,v;f;x)$ *is*

$$D_{x,k}^{\nu,m}[{}_2F_{1,k}(u,v;f;x)]$$
$$=\frac{(u)_{sk,k}(v)_{sk,k}}{(f)_{sk,k}}\frac{x^{sk-\nu}}{\Gamma_k(k-\nu+sk)}\sum_{r=0}^{\infty}\frac{(u+sk)_{r,k}(v+sk)_{r,k}}{(f+sk)_{r,k}(k-\nu+sk)_{r,k}}\frac{B_{m,k}(sk-\nu,r+k)x^r}{B_k(sk-\nu,r+k)},$$

for $|x|<1$.

Proof.: Let consider the power series expansion of ${}_2F_{1,k}(u,v;f;x)$ and making similar calculations, we get

$$D_{x,k}^{\nu,m}[{}_2F_{1,k}(u,v;f;x)]=D_{x,k}^{\nu,p}[\sum_{r=0}^{\infty}\frac{(u)_{r,k}(v)_{r,k}}{(f)_{r,k}}\frac{x^r}{r!}]$$
$$=[\sum_{r=0}^{\infty}\frac{(u)_{r,k}(v)_{r,k}}{(f)_{r,k}r!}D_{x,k}^{\nu,m}[x^r]$$
$$=[\sum_{r=s}^{\infty}\frac{(u)_{r,k}(v)_{r,k}}{(f)_{r,k}r!}\frac{\Gamma_k(r+k)B_{m,k}(sk-\nu,r-sk+k)}{\Gamma_k(r-\nu+k)B_k(sk-\nu,r-sk+k)}x^{r-\nu}$$
$$=[\sum_{r=0}^{\infty}\frac{(u)_{r+sk,k}(v)_{r+sk,k}}{(f)_{r+sk,k}(r+sk)!}\frac{\Gamma_k(r+sk+k)B_{m,k}(sk-\nu,r+k)}{\Gamma_k(r+sk-\nu+k)B_k(sk-\nu,r+k)}x^{r+sk-\nu}$$
$$D_{x,k}^{\nu,m}[{}_2F_{1,k}(u,v;f;x)]=\frac{(u)_{sk,k}(v)_{sk,k}}{(f)_{sk,k}}\frac{x^{sk-\nu}}{\Gamma_k(k-\nu+sk)}\sum_{r=0}^{\infty}$$
$$\times\frac{(u+sk)_{r,k}(v+sk)_{r,k}}{(f+sk)_{r,k}(k-\nu+sk)_{r,k}}\frac{B_{m,k}(sk-\nu,r+k)x^r}{B_k(sk-\nu,r+k)}$$

The next theorems are related to the Mellin transforms of Caputo k-fractional derivatives of two functions.

Theorem 11.*Let* $Re(\psi)>sk-1$ *and* $Re(w)>0$*, then*

$$M[D_{x,k}^{\nu,m}x^{\psi}:w]=\frac{\Gamma_k(\psi+k)\Gamma(w)}{\Gamma_k(\psi-sk+k)\Gamma_k(sk-\nu)}B_k(sk-\nu+w,\psi-sk+w+k)x^{\frac{\psi-\nu}{k}}.$$

*Proof.*Applying the definition of Mellin transform, we calculate

$$M[D_{x,k}^{\nu,m}(x^{\psi}):w]=\int_0^{\infty}m^{w-1}D_{x,k}^{\nu,w}(x^{\psi})dm=\int_0^{\infty}m^{w-1}\frac{\Gamma_k(\psi+k)B_{m,k}(sk-\nu,\psi-sk+k)}{\Gamma_k(\psi-\nu+k)B_k(sk-\nu,\psi-sk+k)}x^{\frac{\psi-\nu}{k}}dm$$
$$=\frac{\Gamma_k(\psi+k)x^{\frac{\psi-\nu}{k}}}{\Gamma_k(\psi-\nu+k)B_k(sk-\nu,\psi-sk+k)}\int_0^{\infty}m^{w-1}B_{m,k}(sk-\nu,\psi-sk+k)dm.$$

From the equality

$$\int_0^{\infty}m^{w-1}B_{m,k}(i,j)dm=\Gamma(w)B_k(i+w,j+s)$$

$Re(w)>0$, $Re(i+w)>0$, $Re(j+w)>0$ using this equality, we get the result.

$$=\frac{\Gamma_k(\psi+k)\Gamma(s)}{\Gamma_k(\psi-sk+k)\Gamma_k(sk-\nu)}B_k(sk-\nu+s,\psi-sk+w+k)z^{\frac{\psi-\nu}{k}} \tag{39}$$

Theorem 12. *Let* $Re(w) > 0$ *and* $|x| < 1$*, then*

$$M[D_{x,k}^{\nu,m}[(1-xk)^{\frac{-\delta}{k}} : w] = \frac{\Gamma(w)x^{sk-\nu}}{\Gamma_k(sk-\nu)}\sum_{r=0}^{\infty}\frac{B_k(sk-\nu+w,r+w+k)}{\Gamma_k(r+k)}(\nu)_{r,k}x^r.$$

Proof. Let the power series expansion of $(1-xk)^{\frac{-\delta}{k}}$ and consider ψ=r, we get

$$M[D_{x,k}^{\nu,m}[(1-xk)^{\frac{-\delta}{k}} : w] = M[D_{x,k}^{\nu,m}[\sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}x^r : w] \tag{40}$$

$$= \sum_{r=0}^{\infty}\frac{(\delta)_{r,k}}{r!}M[D_{x,k}^{\nu,m}(x^r) : w] \tag{41}$$

$$= \frac{\Gamma_k(w)x^{-\nu}}{\Gamma_k(sk-\nu)}\sum_{r=s}^{\infty}\frac{B_k(sk-\nu+w,r-sk+w+k)}{\Gamma_k(r-sk+k)}(\delta)_{r,k}x^r \tag{42}$$

$$= \frac{\Gamma_k(w)x^{sk-\nu}}{\Gamma_k(sk-\nu)}\sum_{r=0}^{\infty}B(sk-\nu+w,r+w+k)\frac{(\delta)_{r+sk,k}x^r}{r!}. \tag{43}$$

4 Conclusion

our research extends the framework of fractional calculus through the introduction of the extended beta *k*-function and extended Caputo *k*-fractional derivative operator. These generalized forms not only enhance our understanding of hypergeometric functions but also provide new tools for solving complex problems in mathematical analysis. The practical implications of our work, supported by theorems and integral forms, underscore its relevance and potential applications in various fields. Our findings open up new avenues for future research in the domain of fractional calculus and its applications.

Declarations

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