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New Extension of Weibull Distribution: Theory, Properties, and Applications of Water Runoff Data

H. M. Kashif Rasheed[1], *Sajid Hussain*[1,*], *Shafqat Ali*[2], *Shazia Bibi Nadia*[1], *Muhammad Asghar*[1], *Abid Khan*[1]

[1] Dept. of Statistics, The Islamia University of Bahawalpur, Bahawapur, Pakistan
[2] Dept. of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan

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Abstract: Probability distribution theory offers flexibility in modeling for different categories of data sets. That flexibility of the models makes it suitable for quantifying risks and uncertainties in extreme events. This ia also valuable for decision-making. This study proposed new model as extension of Weibull model, known as exponentiated power alpha index generalized Weibull (EPAIGW). Graphical study of densities and reliability functions are given for EPAIGW model. Several statistical properties of the new model are derived. For parameters estimation, maximum likelihood method is used. Simulation study is done for EPAIGW model. EPAIGW model is compared with other competitive models to check model performance using water runoff data. This leads to more precise risk assessment, and more solid and economical decisions. It also offers a methodical approach to estimate the probability of unfavourable occurrences and bolstering plans to lesson their effects in fields of epidemiology, climate science, economics and reliability engineering.

Keywords: Weibull; exponentiated; hazard; alpha; simulation; likelihood; runoff; extreme; empirical.

2010 Mathematics Subject Classification. 60E05; 62E15.

1 Introduction

Statistical probability distribution (PDn) theory related to hydrological extreme events (EXEs) play a fundamental role in risk assessment, designing infrastructure, response in emergency, climate adaptation, protection of environment, etc. It provides a quantitative and structured approach to understanding the occurrence and impact of EXEs, enabling informed decision-making and enhancing societal resilience. In reliability analysis, Weibull model [1] is a very popular lifetime model. It has advantage over the exponential model because of its increasing and decreasing trends of hazard rate function (HRFn) based on shape parameter. Normally, it is used for analysis of hydrological, medical, biological, reliability and survival, and failure time of equipments data sets. It gives an inappropriate fit for several data sets, especially, when shapes of HRFn are bathtub, upside down bathtub, or bimodal. To handle such situations, recently, several researchers have attempted various generalizations and extensions of Weibull distribution (WD) to improve model performance by injecting more parameters. Among these, the most useful distributions are: the exponentiated Weibull (EW) that allow bathtub-shaped HRFn by [2,3], the additive Weibull [4], modified Weibull (MW) [5], EW with flood applications [6], extended Weibull [7], flexible MW [8], generalized MW [9], Weibull-G family [10], exponential-Weibull [11], exponentiated Weibull [12], beta Sarhan-Zaindin MW [13], Alpha power Weibull [14], modified generalized WD [15], inverse extended WD for insurance loss data [16], bivariate generalized WD [17], new extended WD [18], power generalized WD [19], generalization of the inverse generalized WD [20], alpha power transformed generalized WD [21], extended Rayleigh Weibull [22], exponentiated generalized Weibull exponential [23], extended odd Weibull Lindly [24], modified exponentiated inverted WD [25]. For readers, we suggest some of the latest extensions: [26,27,28,29,30,31,32,33,34,35,36,37,38].
For utilizing the non-symmetrical behavior of the parent distributions, [39] proposed a technique, Alpha Power Transformation (APT). Let a continuous random variable (r.v.) X with cumulative distribution function (CDF), the CDF

* Corresponding author e-mail: sajidhussain060917@gmail.com

of APT is as.

$$F^{APT}(x) = \begin{cases} \frac{\alpha^{G(x)}-1}{\alpha-1} & \text{if } \alpha \neq 1, \alpha > 0, x > 0, \\ G(x) & \text{if } \alpha = 1. \end{cases}$$

The associated probability density function (PDF) is as.

$$f^{APT}(x) = \begin{cases} \frac{\log(\alpha)}{\alpha-1}\alpha^{G(x)}g(x) & \text{if } \alpha \neq 1, \alpha > 0, x > 0, \\ g(x) & \text{if } \alpha = 1. \end{cases}$$

A family of exponentiated distributions developed by [40] has CDF and PDF with an additional shape parameter β are as follows.

$$F(x) = [G(x)]^{\beta},$$

and

$$f(x) = \beta[G(x)]^{\beta-1}g(x), \beta > 0, x > 0.$$

[41] proposed the T-X family of distributions by CDF and PDF as.

$$F(x) = \int_{\lambda_1}^{L[G(x)]} \pi(t)\mathrm{d}t = \Pi\{L[G(x)]\},$$

and

$$f(x) = \pi\{L[G(x)]\}\frac{d}{dx}L[G(x)], x > 0.$$

Where, $\pi(t)$ is the PDF of a r.v. $T \in [\lambda_1, \lambda_2]$ for $-\infty < \lambda_1 < \lambda_2 < \infty$ and $F(x)$ is the CDF of a r.v. X such that the link function $L(\cdot) : [0,1] \to [\lambda_1, \lambda_2]$ satisfies the following conditions.

(i)$L(\cdot)$ is differentiable and monotonically non-decreasing.
(ii)

$$L(\cdot) \to \begin{cases} \lambda_1 & \text{when } \lim_{x\to-\infty} G(x) = 0 \\ \lambda_2 & \text{when } \lim_{x\to+\infty} G(x) = 1. \end{cases}$$

If $[\lambda_1, \lambda_2]$ is open or half open interval, we replace $L(0)$ with $\lim_{t\to 0^+} L(t) \to \lambda_1$ and/or $L(1)$ with $\lim_{t\to 1^-} L(t) \to \lambda_2$.
[42] proposed a family named as Exponentiated Power Alpha Index Generalized (EPAIG) family having CDF and PDF as.

$$F(x) = \left[\frac{\alpha^{\frac{-\log\overline{G}(x)}{1-\log\overline{G}(x)}} - 1}{\alpha - 1}\right]^{\delta}, \tag{1}$$

and

$$f(x) = \frac{\delta\log(\alpha)}{\alpha-1}\frac{g(x)}{\overline{G}(x)\left[1-\log\overline{G}(x)\right]^2}\left[\frac{\alpha^{\frac{-\log\overline{G}(x)}{1-\log\overline{G}(x)}} - 1}{\alpha - 1}\right]^{\delta-1}\alpha^{\frac{-\log\overline{G}(x)}{1-\log\overline{G}(x)}}. \tag{2}$$

where α, δ and x are real positive, and $\alpha \neq 1$.
A key part of hydrology and water resources mangement is data modeling, which often calls for flexible PDn's that can capture complex data behavior such as skewness, heavy or light tails, and varying hazard rate structures. Although classical WD and its general forms have been widely applied in hydrology, environment science, and reliability because of their mathematical robustness and ability to represent the increase or decrease in failure rates. Despite this, they still have significant flaws. In particular, WD is not adaptive enough to handle diverse data patterns, especially when the data exhibit multi model trends or non-monotonic hazard rates shapes. Furthermore, EXEs and tail behaviours that are often observed in hydrological phenomena such as high runoff or floods, are not adequately captured by many traditional models. The "Exponentiated Power Alpha Index Generalized Weibull (EPAIGW) distribution" is suggested as to overcome these flaws. EPAIGW improves the shape elasticity by incorporating a combination of the alpha index parameter, and exponentiated transformed power parameter into the traditional Weibull frame work. This enables it to describe the different types of data types and tail behaviors more accurately. This three-parameter distribution can simulate complex skewness patterns, fit light- and heavy-tailed data and accommodate different forms of hazard rates such as constant, increasing, decreasing, and non-monotonic forms. When applied to runoff data (ROfD), The EPAIGW distribution performed superior to existing models, as confirmed by goodness-of-fit (GFt) criteria. The empirical results

demonstrate the practical relevance of the EPAIGW model as a reliable and adaptable tool for statistical modeling of environment and geophysical data, validating its use in hydrology. Therefore, the development of the distribution has been motivated by the need to create a more comprehensive and flexible statistical framework that truly captures the complex behaviours of the hydrological processes more accurately as compared to Weibull or other traditional distributions.

EPAIGW distribution develops a systematic mathematical framework for modeling the uncertainty present in man made, artificial or natural processes, that significantly improves risk assessment or decision making in hydrology and various other fields. Hydrological variables such as precipitation, stream flow, runoff, and flood levels are naturally random and affected by a number of unpredictable environmental factors. By fitting EPAIGW distribution to these variables, researchers can estimate the probability of EXEs such as droughts, heavy floods, or peak discharge which is necessary to design reliable water infrastructure, manage reservoirs, create floodplain zoning, and install early warning systems. Engineers can estimate return periods (such as floods that occur every 10 or 20 years) using the EPAIGW distribution, and this information directly influence the design specifications of drainage systems, levees, and dams. It can also be used in water resource planning to predict low flow scenarios that could affect agriculture, municipal water supplies, and power generation. In general, it helps in failure risk prediction, safety assessment, and reliability analysis of systems and structures under uncertain environmental stresses.

The main objective of this manuscript is to propose and study a new three parameters lifetime model known as "Exponentiated Power Alpha Index Generalized Weibull" distribution based on the EPAIG family [42]. The basic purpose of the EPAIGW model is that it can give several desirable properties and more flexibility in the shapes of HRFn: constant, decreasing, increasing, bathtub, upside-down bathtub, J, reversed-J and S.

The rest of this manuscript is arranged as follows: In Section 2, sub-models of EPAIG family are proposed. Graphical presentations of PDF and CDF for three parameters EPAIGW model are given in Section 3. Reliability measures and statistical properties of the new EPAIGW model are derived in Sections 4 and 5, respectively. Simulation and bootstrapping are performed in Section 6 and 7. Estimation method is provided in Section 8. Section 9 described empirical illustration using water runoff data. Lastly, conclusion is drawn in Section 10.

2 New Sub-Model

Let X be a Weibull r.v., then the CDF and PDF of the one-parameter baseline WD are given as.

$$G(x) = 1 - \exp\left(-x^{\gamma}\right), \tag{3}$$

and

$$g(x) = \gamma x^{\gamma-1} \exp\left(-x^{\gamma}\right), x > 0. \tag{4}$$

where, $\gamma > 0$ is shape parameter. Substituting, Equations (3) and (4) in Equations (1) and (2), we obtain new sub model of the EPAIG family [42] called as three parameters EPAIGW distribution with CDF and PDF as.

$$F(x) = \left[\frac{\alpha^{\frac{x^{\gamma}}{1+x^{\gamma}}} - 1}{\alpha - 1}\right]^{\delta}, \tag{5}$$

and

$$f(x) = \frac{\delta\gamma\log(\alpha)}{\alpha-1}\frac{x^{\gamma-1}}{\left[1+x^{\gamma}\right]^{2}}\left[\frac{\alpha^{\frac{x^{\gamma}}{1+x^{\gamma}}} - 1}{\alpha - 1}\right]^{\delta-1}\alpha^{\frac{x^{\gamma}}{1+x^{\gamma}}}; \quad \alpha, \gamma, \delta, x > 0. \tag{6}$$

Setting $\delta = 1$ in Equations (5) and (6), the sub model of the EPAIGW distribution is two parameters Alpha Index Generalized Weibull (AIGW) distribution with CDF and PDF as.

$$F(x) = \frac{\alpha^{\frac{x^{\gamma}}{1+x^{\gamma}}} - 1}{\alpha - 1}, \tag{7}$$

and

$$f(x) = \frac{\gamma\log(\alpha)\, x^{\gamma-1}}{\alpha-1}\frac{\alpha^{\frac{x^{\gamma}}{1+x^{\gamma}}}}{\left[1+x^{\gamma}\right]^{2}}; \quad \alpha, \gamma, x > 0. \tag{8}$$

In the following sections, three parameters EPAIGW model will be studied in details.

3 Graphical Representation of the new Three Parameters Model

Graphs for PDF and CDF of the three parameters EPAIGW distribution for different values of parameters are given in Figures 1-2.

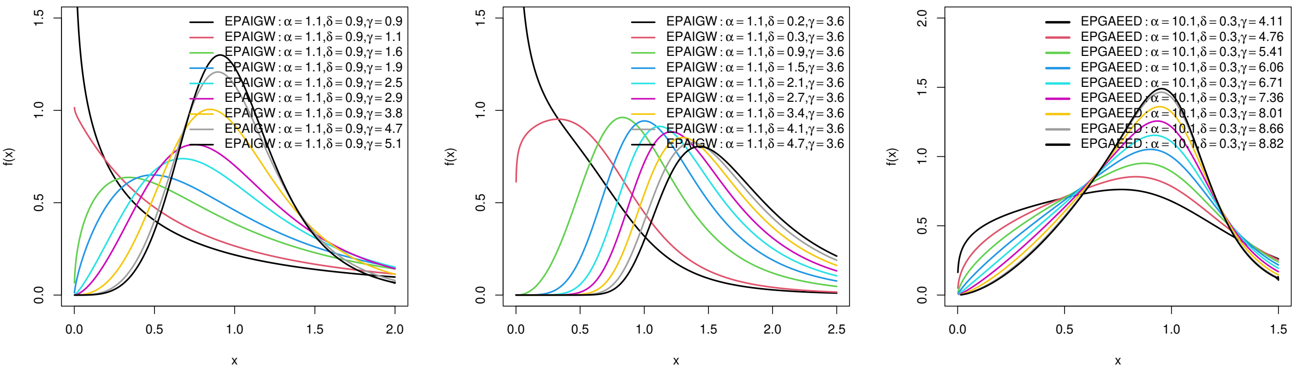

Fig. 1: PDF graphs for three parameters EPAIGW distribution.

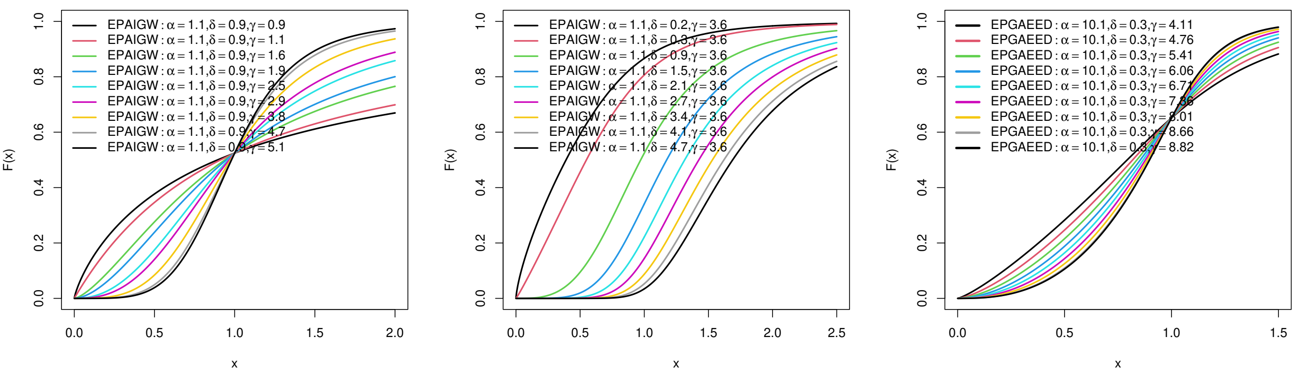

Fig. 2: CDF graphs for three parameters EPAIGW distribution.

4 Reliability Measures

In distribution theory, reliability measures (RMs) are very important, especially in fields such as engineering, hydrology, and environment science. To understand these measures, imagine yourself trying to predict the likelihood of significant rainfall in your area. If you really want to know the probability of a major rainfall event that could result in flooding, not just normal rain. RMs are useful in these situations. They help you understand the likelihood (or risk) that an unusual event could occur. The 'return period' that indicates how often an event such as flood occurs, for example, once every ten years, is a RMs. 'Probability of Exceedance', which indicates the likelihood that a particular value (such as a runoff or river flow) will be exceeded is another important measure. To provide a clear picture of the probabilities of different events, these measures are used under several PDn's. This information is extremely useful when planning safe buildings, bridges, and drainage systems. It also helps in emergency preparedness for community and governments. To put it briefly, RMs turn raw data into useful information that can be used to inform safe and intelligent decisions based on the long term

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

behaviour of nature.
In aspect, PDn's give mathematical tools to modelled behavior of component or system over time, and RMs quantify the performance and likelihood of success based on those models. Here, EPAIGW distribution is used as mathematical model to describe the patterns of failure or success of the components or systems.
A life time r.v. t is said to have EPAIGW(α, δ, γ) distribution, if its has CDF and PDF as.

$$F(t) = \left[\frac{\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1}{\alpha - 1}\right]^\delta, \tag{9}$$

and

$$f(t) = \frac{\delta\gamma\log\alpha}{\alpha - 1}\frac{t^{\gamma-1}}{\left[1+t^\gamma\right]^2}\left[\frac{\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1}{\alpha - 1}\right]^{\delta-1}\alpha^{\frac{t^\gamma}{1+t^\gamma}}, \quad t, \alpha, \gamma, \delta > 0. \tag{10}$$

The well-known RMs are survival function (SFn), $S(t)$; HRFn, $h(t)$; cumulative HRFn (CHRFn), $H(t)$; reversed HRFn (RHRFn), $r(t)$; mean residual life (MRLe), $M(t)$; and mean waiting time (MWTe), $\overline{M}(t)$.
In hydrology, SFn, describes the probability that a component or system (such as a water reservoir or drainage system) will operate without failure for a specific period of time. For runoff, SFn is related to the probability that a specified flow rate will not be exceeded or that a specified storage capacity will not be exhausted within a given time. A reservoir is supposed to be designed to supply water to a city for 60 years. SFn determines the probability that the reservoir will not experience any failures (such as overtopping or water depletion) within 60 years. At $t = 0$, SFn, S(0) is 1 (or 100%), because the reservoir has not experienced any failures. Over time, the likelihood of failure increases due to several factors such as unexpected water usage or weather EXEs. SFn, S(60), will define the probability that the reservoir operates without failure for the entire 60 year period. Factors affecting the flow and reliability of water management system are rainfall patterns (i.e., variability in rainfall duration and intensity), catchment characteristics (i.e., drainage area, soil type, and land cover), and system design (i.e., reservoir capacity, spill way design). In essence, SFn is often based on statistical analysis of historical data or the use of probabilistic models that account for various factors affecting runoff and system performance. The survival function for EPAIGW distribution is given as.

$$S(t) = 1 - \left[\frac{\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1}{\alpha - 1}\right]^\delta, \tag{11}$$

The HRFn (or failure rate function) expresses the instantaneous probability that an event will occur at a particular time if it has not already occurred. This is a way to calculate the probability that an event will occur soon, taking into account the time that has passed since the last event. It is calculated as the ratio of the PDF of the time to the event $f(t)$ to the SFn $S(t)$, which represents the probability that the event will not occur up to time t. Runoff event is more likely to occur when the hazard rate is high. Consider a mountainous area where snow freezes in winter. In the spring, snowfall begins to melt, and the resulting melt water increases runoff. The probability of a runoff event occurring at a particular moment during the melt period can be modeled using the HRFn. The risk rate may be very low in the early stages of snow melt. This is a result of the slow melt rate and the small snow pack. The probability of a major runoff event occurring is minimal. The rate of melt increases in conjunction with increasing temperatures and deeper snowpack. Additionally, the hazard rate increases, suggesting a greater likelihood of runoff event such as flash floods or increased river flow. The rate of melting eventually slows as the snow pack gets smaller. As the probability of significant runoff events decreases, the hazard rate may begin to decrease. In essence, the HRFn primarily helps hydrologists better predict floods and manage water resources by highlighting how runoff risk varies overtime during a snow melt event. The hazard rate function for EPAIGW distribution is given as.

$$h(t) = \frac{\gamma\delta\log\alpha\, t^{\gamma-1}\left(\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1\right)^{\delta-1}\alpha^{\frac{t^\gamma}{1+t^\gamma}}}{(1+t^\gamma)^2\left[(\alpha-1)^\delta - (\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1)^\delta\right]}, \tag{12}$$

The CHRFn represents the accumulated risk of experiencing an event (e.g. runoff, flood) over a specified time. It is derived from HRFn, which defines the immediate risk at a given time. Essentially, CHRFn shows how the total risk of an event occurring increases overtime. An analysis of flood events in a river shows that the hazard rate may be higher during the rainy season, indicating a higher risk of flooding at any given time. It can also be higher in area with impervious surface and steep slopes (high runoff) than in area with pervious surface and gentle slopes (low runoff). During this time, the cumulative hazard will increases more rapidly, indicating a higher overall flood risk. If there is a long period with out sufficient rainfall, the overall risk may increases more slowly, suggesting a lower risk through that period. The overall risk will be greater in locations with a high runoff coefficient than in location with a low runoff coefficient. The cumulative hazard rate function for EPAIGW distribution is given as.

$$H(t) = -\log\left\{1 - \left(\frac{\alpha^{\frac{t^\gamma}{1+t^\gamma}} - 1}{\alpha - 1}\right)^{\delta}\right\}, \tag{13}$$

The RHRFn at time t describes the conditional probability of a past event occurring within a short period before a specific time $(t - t_0, t]$, given that it occurred before that time t, as t_0 approaches 0. This is a concept used to analyzed the time of events, especially when dealing with incomplete or censored data. For r.v. t representing the time of an event, with CDF $F(t)$ and PDF $f(t)$, the RHRFn $r(t)$ is the ratio of $f(t)$ to $F(t)$. This is especially helpful in hydrology when working with flood data that may be censored or incomplete, such as when the exact time of a flood is unknown but its occurrence is known before a specific date. Let's look at a hypothtical situation where we are examining when significant floods occur in a river. We know in which years there were floods exceeding a certain threshold. We know what year each flood occurred, but we don't have exact time stamps for each one. Consider year of flood events are 1996, 2001, 2006, 2011, 2016, 2021. Assume, a PDF where each event has an equal probability 1/6, 1/6, 1/6, 1/6,1/6, 1/6. So, we can estimate CDF based on the number of flood events up to given year as 1/6, 2/6, 3/6, 4/6, 5/6, 6/6=1. Now, we can compute the reverse hazard rate for each year as 1, 1/2, 1/3, 1/4, 1/5, 1/6. The reversed hazard rate function for EPAIGW distribution is given as.

$$r(t) = \delta\gamma\log\alpha \frac{t^{\gamma-1}}{(1+t^\gamma)^2\left[1 - \alpha^{-\frac{t^\gamma}{1+t^\gamma}}\right]}, \tag{14}$$

Graphs for SFn, HRFn, CHRFn, and RHRFn of three parameters EPAIGW distribution for different values of parameters are given in Figures 3-6.

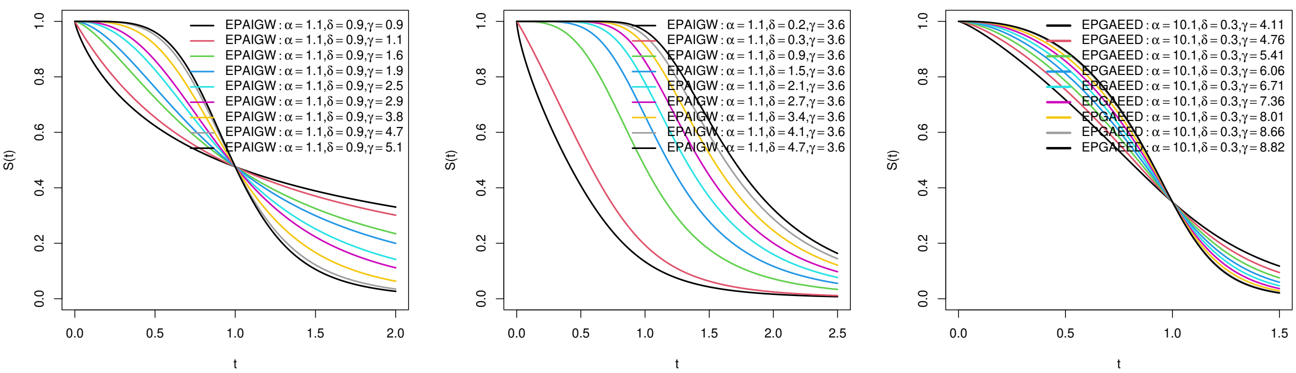

Fig. 3: SFn graphs for three parameters EPAIGW distribution.

4.1 Mean Residual Life

The mean residual life (MRLe) function calculates how long an event is likely to last if it continues or exceeds a predetermined threshold. To help urban flood managers assess the continued risk of road or infrastructures flooding, MRLe can calculate the remaining length of a runoff event that has already lasted ten hours. In catchment modeling, the MRLe function can assist in sizing a culvert, spillway, or warning system to prevent the expected average excess flow above a critical threshold, such as 600 m^3/s, if measured runoff exceeds that level. The MRLe, $M(t)$, for three parameters EPAIGW distribution is

$$M(t) = \frac{1}{S(t)}\int_t^\infty t f(t) d\text{t} - t, \tag{15}$$

or

$$M(t) = \frac{1}{S(t)}\left[E(t) - \int_0^t t f(t) dt\right] - t. \tag{16}$$

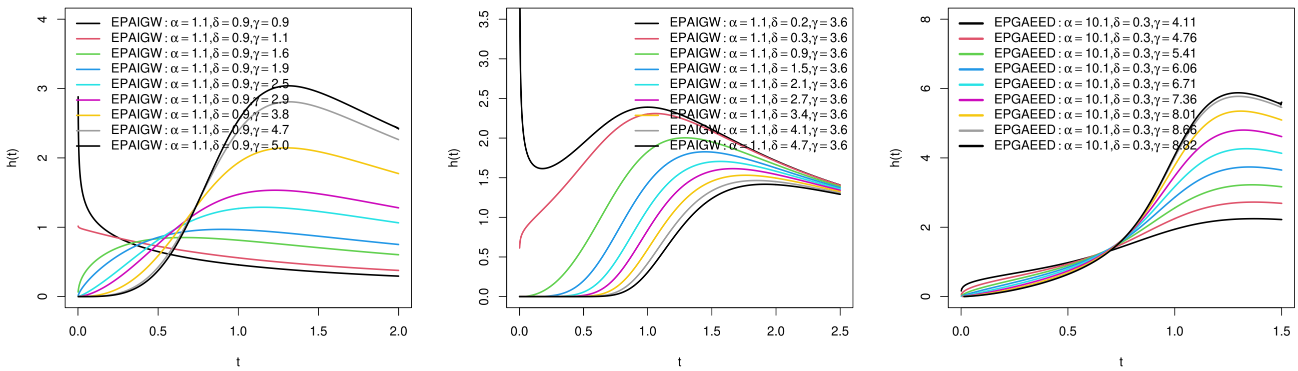

Fig. 4: HRFn graphs for three parameters EPAIGW distribution.

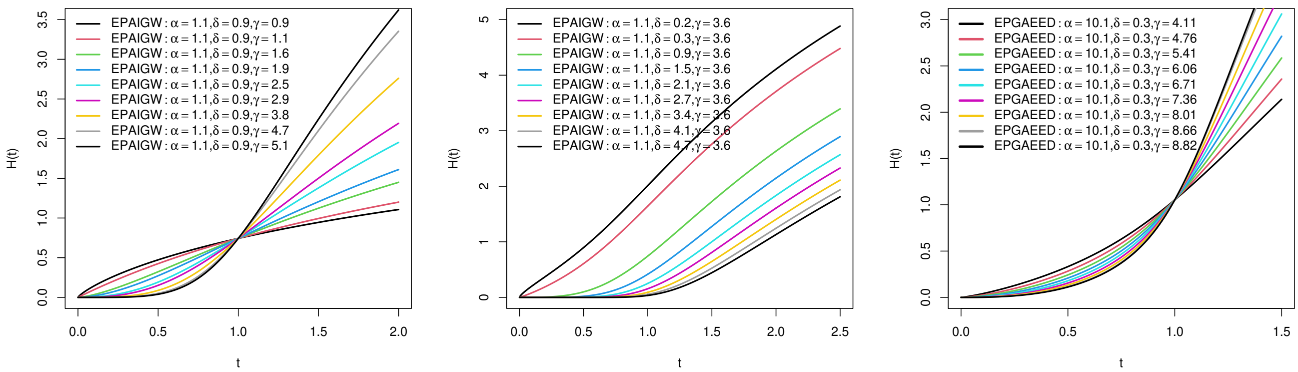

Fig. 5: CHRFn graphs for three parameters EPAIGW distribution.

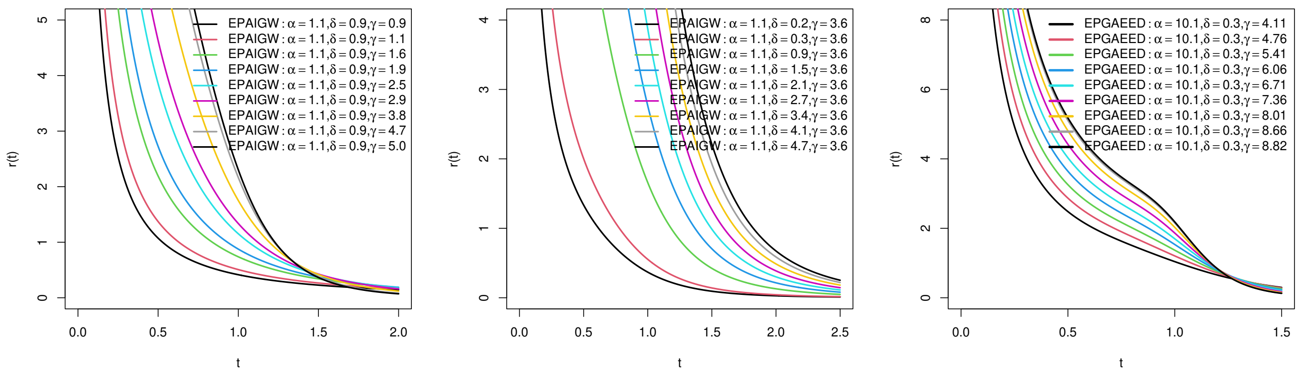

Fig. 6: RHRFn graphs for three parameters EPAIGW distribution.

The binomial expansion is

$$(1-Z)^b = \sum_{k=0}^{\infty} (-1)^k \binom{b}{k} Z^k. \tag{17}$$

The Maclaurin series is

$$\alpha^{kt} = \sum_{j=0}^{\infty} \frac{(\log(\alpha))^j (kt)^j}{j!}. \tag{18}$$

Using Equations (10), (17) and (18), we have

$$\int_0^t t f(t) dt = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+\delta-1}\delta(k+1)^j(\log(\alpha))^{(1+j)}}{j!(\alpha-1)^{\delta}} \binom{\delta-1}{k} B_u\big((j+\frac{1}{\gamma}+1),(1-\frac{1}{\gamma})\big). \tag{19}$$

and

$$E(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{j+\delta-1}(\log(\alpha))^{(1+k)}\delta(j+1)^k}{k!(\alpha-1)^{\delta}} \binom{\delta-1}{j} B\big((k+\frac{1}{\gamma}+1),(1-\frac{1}{\gamma})\big). \tag{20}$$

Using Equations (11), (19), and (20) in Equation (16), we obtain $M(t)$ as

$$\begin{aligned} M(t) = {} & \frac{\delta}{(\alpha-1)^{\delta} - (\alpha^{\frac{t^{\gamma}}{1+t^{\gamma}}}-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \Bigg\{ \frac{(-1)^{j+\delta-1}(\log(\alpha))^{(1+k)}(j+1)^k}{k!} \binom{\delta-1}{j} \\ & \times B\big((k+\frac{1}{\gamma}+1),(1-\frac{1}{\gamma})\big) - \frac{(-1)^{k+\delta-1}(k+1)^j(\log(\alpha))^{(1+j)}}{j!}\binom{\delta-1}{k} \\ & \times B_u\big((j+\frac{1}{\gamma}+1),(1-\frac{1}{\gamma})\big)\Bigg\} - t. \end{aligned} \tag{21}$$

where $u = t^{\gamma}$, $B(c_1,c_2) = \int_0^{\infty} \frac{u^{c_1-1}}{(1+u)^{c_1+c_2}} du$ beta function (BF) and $B_u(c_1,c_2) = \int_0^{u} \frac{u^{c_1-1}}{(1+u)^{c_1+c_2}} du$ is incomplete BF.

4.2 Mean Waiting Time

The expected average time before a particular event occurs is known as the mean waiting time (MWTe) and is calculated using PDn's that model the timing of events such as high flow peak, extreme runoff volume, and threshold excedance due to rainfall. For example, the MWTe untill the next high runoff event (e.g. runoff exceeded 700 m^3/s) occurs is 5 seasons if the event occurs with a probability 0.2 during each monsoon season. This indicates that runoff of this size is expected to occur once every five seasons on average. Understanding the frequency and timing of excessive flows, the MWTe is important for planning and preparedness in area such as drainage system design, reservoir operation, and flood risk management. The MWTe, $\overline{M}(t)$, for three parameters EPAIGW distribution is

$$\overline{M}(t) = t - \left[\frac{1}{F(t)} \int_0^t t f(t) dt\right]. \tag{22}$$

Using Equations (9) and (19) in Equation (22), we obtain $\overline{M}(t)$ as

$$\begin{aligned} \overline{M}(t) = t - \Bigg[& \frac{\delta}{(\alpha^{\frac{t^{\gamma}}{1+t^{\gamma}}}-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \binom{\delta-1}{k} \frac{(k+1)^j(-1)^{k+\delta-1}(\log(\alpha))^{(1+j)}}{j!} \\ & \times B_u\big((j+\frac{1}{\gamma}+1),(1-\frac{1}{\gamma})\big)\Bigg]. \end{aligned} \tag{23}$$

5 Statistical properties

In this section, we derived some statistical properties of the three parameters EPAIGW distribution.

5.1 Random number generation through quantile function

When generating random numbers that follows EPAIGW distribution, the quantile function (QFn), $Q(p) = F^{-1}(p) = x_p$ where $p \in (0,1)$ is important. This helps in the simulation of synthetic flow data in hydrology which replicate the real behavior of a watershed. Using the Equation (5), the pth quantile of the three parameters EPAIGW distribution can be obtained as.

$$\left[\frac{\alpha^{\frac{x_p^\gamma}{1+x_p^\gamma}}-1}{\alpha-1}\right]^\delta = p,$$

For solving x_p, we obtain QFn as.

$$x_p = \left\{\frac{\log[1+(\alpha-1)p^{\frac{1}{\delta}}]}{\log\alpha-\log[1+(\alpha-1)p^{\frac{1}{\delta}}]}\right\}^{\frac{1}{\gamma}}; 0<p<1. \tag{24}$$

When $p = 0.50, 0.25, \& \ 0.75$ in Equation (24), we get median $x_{0.50}$, lower quartile $x_{0.25}$ and upper quartile $x_{0.75}$. Using inversion method in Equation (5), the random numbers can be obtained as.

$$x = \left\{\frac{\log[1+(\alpha-1)u^{\frac{1}{\delta}}]}{\log\alpha-\log[1+(\alpha-1)u^{\frac{1}{\delta}}]}\right\}^{\frac{1}{\gamma}}, \quad u \sim U(0,1). \tag{25}$$

Consider, we draw three random numbers for $u = 0.25, 0.50, 0.95$, the runoff values obtained by applying the QFn of the EPAIGW distribution are $x = 156\ m^3/s, 310\ m^3/s, 940\ m^3/s$. This indicates there is a 25 percent chance that runoff remaining below 156 m^3/s with average (median) runoff rate of 310 m^3/s, and an extreme upper tail scenario of 940 m^3/s. This enables to plan for flood risks by simulating future flow scenarios.
Table1 shows the percentage value points, as well as skewness and kurtosis using QFn of the three parameters EPAIGW distribution for some fixed values of the parameters. These can be obtained, to provide a reliable and intuitive understanding of the shape of the data distribution. In hydrology, they support policy decisions in case of uncertainty, informs reservoir designs, and helps hydrologists to determine the probabilities and impacts of EXEs. Furthermore, it confirms that EPAIGW is more realistic and flexible than traditional models in simulating asymmetric and heavy tailed behavior.

Table 1: Percentage value points, skewness and kurtosis for different values of α, δ and γ for three parameters EPAIGW distribution.

		$\alpha = 1.1$					$\alpha = 10.1$				
δ	γ	25%	50%	75%	Skewness	Kurtosis	25%	50%	75%	Skewness	Kurtosis
0.9	0.9	0.2739	1.2513	3.5267	4.5334	19.0357	0.9445	3.7704	10.0384	4.5309	19.0201
	1.6	0.4827	1.1344	2.0319	3.6710	13.6364	1.1210	1.5280	2.5700	1.5319	1.3579
	2.5	0.6274	1.0841	1.5742	2.6677	7.8997	1.0933	1.6964	2.3409	1.1714	1.5991
	3.8	0.7359	1.0545	1.3478	1.9075	4.2392	0.9866	1.3693	1.7268	1.9601	4.4739
	5.1	0.7957	1.0404	1.2491	1.5311	2.7255	0.9900	1.2640	1.5020	1.5943	2.9700
2.1	0.9	1.2147	3.9607	9.9457	4.5298	19.0130	5.7287	10.4141	22.4298	1.1103	0.5784
	1.6	1.1156	2.1689	3.6406	3.6992	13.8061	2.3160	2.9600	4.6480	1.5648	1.4522
	2.5	1.0725	1.6413	2.2864	2.7794	8.4657	1.7330	2.4980	3.3610	1.3044	1.9935
	3.8	1.0471	1.3854	1.7230	2.0969	5.0126	1.3665	1.6416	2.0479	1.9947	2.9559
	5.1	1.0349	1.2749	1.4999	1.7608	3.5276	1.2341	1.4857	1.6096	2.0038	5.0107
4.1	0.9	3.0720	8.9710	21.6450	4.5287	19.0068	10.9210	16.0750	30.9220	1.7383	1.7766
	1.6	1.8800	3.4350	5.6380	3.7133	13.8898	3.7600	7.1110	12.8810	0.9910	0.0525
	2.5	1.6180	2.3020	3.0830	1.3350	2.0856	2.1790	2.9360	4.2880	1.2429	0.0055
	3.8	1.3050	1.6810	2.0710	2.1730	5.3339	1.7310	1.9040	2.1510	1.6515	2.4292
	5.1	1.2590	1.3550	1.5550	0.9577	0.0730	1.3750	1.5730	1.8200	1.2552	1.1206

5.2 Measures of skewness and kurtosis based on quantile function

In hydrology, ROfD sometimes shows extreme and asymmetric values due to irregular precipitation, land cover, and watershed characteristics. To accurately understand the shape parameters effects and simulate their behaviors using three parameters EPAIGW distribution, two important shape indicators are used. First is skewness or asymmetry measured by

Bowley's skewness [43] using QFn that indicates whether runoff events are small or large and how the data is distributed around the mean. A positive skew indicates that most flow values are small, but that there are occasional large floods which are common in arid and semi-arid areas. Negative skewness indicates that there are many high flows and few small values that are less common in ROfD, and zero skew indicates symmetric behavior around the mean. Second is kurtosis measured by Moor's kurtosis [44] that tells how peaked the distribution is and how light or heavy the tails. It's value grater than 3 indicates sharp peak, frequent EXEs (flash floods) while value less than 3 indicates flatter curve, less frequent extreme. The value equal to 3 describes curve is normal. For three parameter EPAIGW distribution, these indicators using QFn can be obtained as.

$$S_{BS} = \frac{Q_{\frac{3}{4}} + Q_{\frac{1}{4}} - 2Q_{\frac{2}{4}}}{Q_{\frac{3}{4}} - Q_{\frac{1}{4}}},$$

and

$$K_{MK} = \frac{Q_{\frac{3}{8}} + Q_{\frac{1}{8}} + Q_{\frac{7}{8}} - Q_{\frac{5}{8}}}{Q_{\frac{6}{8}} - Q_{\frac{2}{8}}}.$$

Where $Q_{(.)}$ is the QFn. Using above formulas, we can estimate the shape of the runoff distribution more robustly than with classical moment based measures. The zero skewness indicates a symmetric runoff distribution with large and small runoff events are equally likely around the median while 0.41 shows moderate positively skewed distribution meaning that frequently small runoff, rare large events. If the skewness is 0.99, it indicates that the runoff distribution is skewed to the right, with more frequent moderate flows and rare severe floods. The kurtosis of 5.2 suggests that the EPAIGW distribution has heavy tails, indicating a higher probability of very large runoff events.
Graphs for skewness and kurtosis of three parameters EPAIGW distribution for specific values of parameters are given in Figures 7-8.

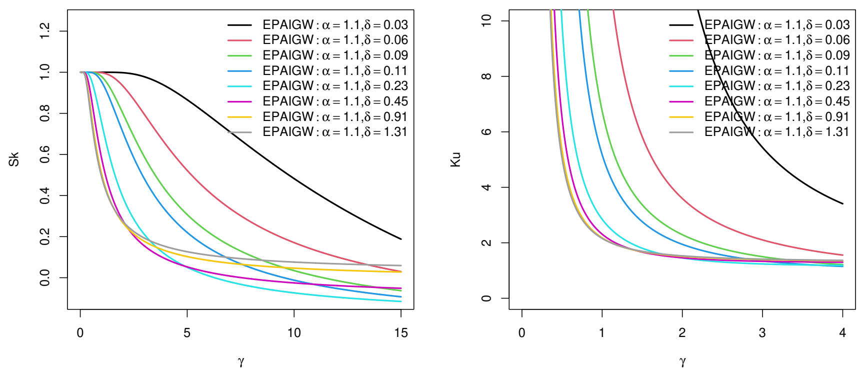

Fig. 7: Plots of skewness and kurtosis for EPAIGW (α, δ, γ) as function of γ for specific α, δ.

5.3 Mode

Unlike the mean, or median, the mode indicates where the data most often cluster. In skewed hydrological data, which is common in rainfall runoff modeling, the mode is particularly important because it reflects the most probable flow rate, not necessarily the mean. The mode is the value of the r.v. (here, runoff) at which the PDF reaches its maximum. Using Equation (6), we take natural logarithm of $f(x)$ and equate to zero the differentiation of $\log[f(x)]$ i.e. $\frac{d}{dx}\log[f(x)] = 0$. After simplification, we obtain it as.

$$\frac{\gamma - 1}{x} - 2\gamma\frac{x^{\gamma-1}}{1+x^{\gamma}} + \gamma\log\alpha\frac{x^{\gamma-1}}{(1+x^{\gamma})^2}\left[1 + \frac{(\delta-1)}{1-\alpha^{-\frac{x^{\gamma}}{1+x^{\gamma}}}}\right] = 0. \tag{26}$$

Clearly, Equation (26) is nonlinear equation (NLE) and with respect to x its analytical solution (ASn) is not possible; so it has numerical solution (NSn) if must $\frac{d^2}{dx^2}\{\log[f(x_0)]\} < 0$ then x_0 is root of Equation (26). The most typical or

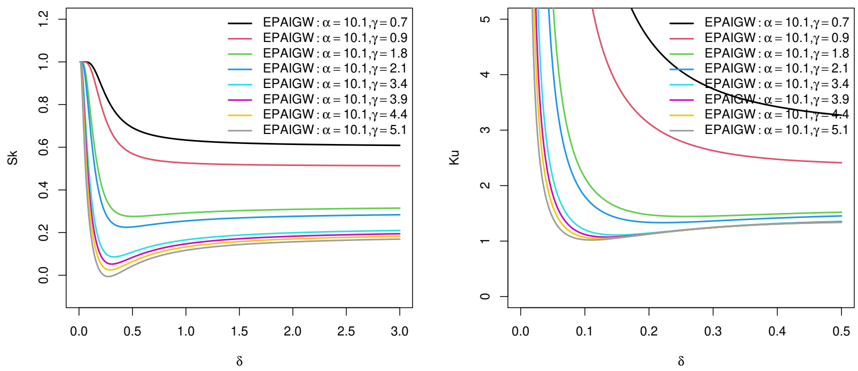

Fig. 8: Plots of skewness and kurtosis for EPAIGW (α, δ, γ) as function of δ for specific α, γ.

frequent behavior of a watershed depends largely on the mode of the PDn. For EPAIGW distribution, the mode interpretation enables, to determine the runoff value that occurs with the highest probability, which is especially helpful for flood forecasting, resource management, and design structure. The EPAIGW distribution is fitted to ROfD and obtained maximum likelihood estimates (MLEs), the PDF of the EPAIGW distribution peaks at a 270 m^3/s runoff value. This means the most probable value of runoff is 270 m^3/s. The other central measures such as mean (340 m^3/s), median (290 m^3/s), and skewness (1.2) suggest mode $<$ median $<$ mean. This indicates distribution is positively skewed, a common trend in flow data, where low to moderate flow occurs frequently, but major floods are occasional.

5.4 Incomplete moments

Because runoff events often need to be assessed under specific conditions (low flow or flood), incomplete moments (IMs) are essential tools for quantifying the partial behavior of a r.v., that is, behavior below or above a given threshold. Because they consider only a portion of the distribution range, IMs provide a more focused picture than full moments (such as mean or variance). If r.v. X follows PDF specified in Equation (6), then nth IMs of three parameters EPAIGW distribution is.

$$M_n(x) = \int_{-\infty}^{x} x^n f(x) \mathrm{d}x, \tag{27}$$

By combining Equations (6), (17), (18) and (27), we obtain.

$$M_n(x) = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{\delta(\alpha-1)^{-\delta}(\log(\alpha))^{1+l}(-1)^{\delta-1+i}\binom{\delta-1}{i}}{l!(i+1)^{-l}} B_t\big((\frac{n}{\gamma}+l+1),(1-\frac{n}{\gamma})\big). \tag{28}$$

where

$$\int_0^t \frac{t^{\frac{n}{\gamma}+l}}{(1+t)^{2+l}}\mathrm{d}t = B_t\big((\frac{n}{\gamma}+l+1),(1-\frac{n}{\gamma})\big) \text{ is incomplete BF and } t = x^{\gamma}.$$

For $n=1$, Equation (28) provides first IM as.

$$M_1(x) = \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{\delta(\alpha-1)^{-\delta}(\log\alpha)^{1+l}(-1)^{\delta-1+i}\binom{\delta-1}{i}}{l!(i+1)^{-l}} B_t\big((\frac{1}{\gamma}+l+1),(1-\frac{1}{\gamma})\big). \tag{29}$$

The EPAIGW distribution fitted to ROfD and obtained PDF, $f(x)$. To estimate low runoff trend below a threshold $x = 190$ m^3/s, we computed lower IMs $M_1(190) = 110$ m^3/s. This suggests average runoff below 190 m^3/s is 110 m^3/s. If 190 m^3/s is threshold for moderate flow. Then 110 m^3/s denotes typical low flow. Water managers can use IMs to determine what percentage of flow is below or above a threshold for urban, industrial use, or ecological balance. Lower IMs help estimate expected reductions under low flows, which inform decisions about crop irrigation, ground water recharge planning, and water conversation. Upper IMs characterize the expected magnitude of extreme flows above a critical level, which aids in flood early warning system, spillway design, and emergency response planning.

5.5 Moments

Moments are essential statistical characteristics that characterize the scale, shape, and behavior of a PDn. These help in verifying the model's flexibility, realism, and practicality in real world applications. In hydrology, when analyzing the central tendency, spread, asymmetry, and peakedness of runoff events by ROfD, the first four moments are particularly crucial. With respect to origin, the qth raw moment, when X follows three parameters EPAIGW distribution having PDF defined in Equation (6) is given as.

$$\mu_q'(x) = \int_{-\infty}^{\infty} x^q f(x)\mathrm{d}x, \tag{30}$$

Substituting Equation (6) in Equation (30), and applying Equations (17) and (18), we get.

$$\mu_q'(x) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{\delta(j+1)^m(-1)^{j+\delta-1}(\log\alpha)^{m+1}}{m!(\alpha-1)^{\delta}}\binom{\delta-1}{j}B\big((\frac{q}{\gamma}+m+1),(1-\frac{q}{\gamma})\big). \tag{31}$$

where

$$t = x^{\gamma} \text{ and } \int_0^{\infty} \frac{t^{\frac{q}{\gamma}+m}}{(t+1)^{2+m}}\mathrm{d}t = B\big((\frac{q}{\gamma}+m+1),(1-\frac{q}{\gamma})\big) \text{ is BF.}$$

By setting $q = 1,2,3$ and 4, we obtain first four raw moments i.e. $\mu_1'(x), \mu_2'(x), \mu_3'(x)$ and $\mu_4'(x)$. The first moment (mean) provides expected average runoff value over time. The EPAIGW distribution is fitted to ROfD and find $\mu_1'(x) = 319\, m^3/s$. According to obtained value, the runoff is roughly 319 m^3/s on average. This is useful for water supply system design, hydropower yield estimation, and baseline water availability analysis. The degree to which runoff values deviate from the mean is indicated by the second central moment (variance, μ_2) i.e., $\sigma^2 = \mu_2'(x) - \{\mu_1'(x)\}^2 = 11200\ (m^3/s)^2$. This suggests that there may be unstable flows in the watershed because of the considerable fluctuation in runoff, which could be caused by seasonal or extreme weather influences. The standard deviation $\sigma = 105.83\, m^3/s$. This helps in evaluating reservoir buffering capacity and design safety margins. The asymmetry is measured by the third standardized moment, or skewness $= [\mu_3'(x) - 3\mu_1'(x)\mu_2'(x) + 2\{\mu_1'(x)\}^3]/\sigma^3 = 1.2$. A positive skewness of 1.2 indicates a right-skewed distribution, which is common in hydrological data where long tail are caused by sporadic high runoff occurrences (such as flash floods) but low to moderate flows predominate. This makes EPAIGW model flexible and superior to less skewed or symmetric models. The peakedness or heaviness of the tails is reflected in the fourth standardized moment, or kurtosis $= [\mu_4'(x) - 4\mu_3'(x)\mu_1'(x) + 6\mu_2'(x)\{\mu_1'(x)\}^2 - 3\{\mu_1'(x)\}^4]/\sigma^4 = 5.1$. The distribution is more peaked than the normal distribution (ND) and has heavy tails, as indicated by the high kurtosis. In hydrology, this is crucial because it ensures that the model appropriately captures the infrequent but severe runoff episodes that can cause flooding.

5.6 Moment generating function

An effective tool for understanding the underlying structure of data is the moment generating function (MGFn) which summarizes all moments in compact form. This enables accurate estimation of expected runoff variability and behaviour in hydrology. Using Equation (6), the MGFn of X that follows three parameters EPAIGW distribution is.

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\mathrm{d}x, \tag{32}$$

Using the series $e^{tx} = \sum_{l=0}^{\infty}\frac{(tx)^l}{l!}$, Equations (17) and (18) in Equation (32), we obtain.

$$M_X(t) = \sum_{l,m,n=0}^{\infty}\binom{\delta-1}{m}\frac{t^l\ \delta(m+1)^n(\log\alpha)^{n+1}(-1)^{\delta-1+m}}{l!n!(\alpha-1)^{\delta}}B\big((\frac{l}{\gamma}+n+1),(1-\frac{l}{\gamma})\big). \tag{33}$$

where

$$u = x^{\gamma} \text{ and } \int_0^{\infty}\frac{u^{\frac{l}{\gamma}+n}}{(1+u)^{n+2}}\mathrm{d}u = B\big((\frac{l}{\gamma}+n+1),(1-\frac{l}{\gamma})\big) \text{ is BF.}$$

5.7 Mean deviation

The average amount by which values vary from a central point, usually the mean or median is statistically measured by the mean deviation (MDn). Because it offers a reliable measure of variability that is less sensitive to extreme values than the standard deviations, this measure is particularly important in hydrology. With respect to mean and median, the MDn of X for three parameters EPAIGW distribution specified by CDF in Equation (5) and PDF in Equation (6), can be defined as.

$$\mathrm{md}_1(x) = 2\mu_1' F(\mu_1') - 2M_1(\mu_1'), \tag{34}$$

and

$$\mathrm{md}_2(x) = \mu_1' - 2M_1(M). \tag{35}$$

The results of the Equations (34) and (35) easily can be obtained by using μ_1' from Equation (31) for $q = 1$, $F(x)$ evaluated at μ_1' from Equation (5) and $M_1(x)$ evaluated at μ_1' and M from Equation (24) for $p = 0.50$. If EPAIGW distribution fitted to ROfD and it gives MDn $\mathrm{md}_1(x) = 94\ m^3/s$ about mean runoff $\mu_1' = 319\ m^3/s$. This indicates moderate flow variability, with daily flows deviating from the mean by an average of 94 m^3/s. The MDn provides a clear picture of the variation than the variance, which squares and exaggerates the deviations. The MDn about the median is more robust in skewed runoff distribution. The MDn $\mathrm{md}_2(x)$ about median runoff $M = 289\ m^3/s$ is 74 m^3/s, which is smaller. It describes that distribution is skewed to right, common in catchment affected by seasonal floods.

5.8 Probability weighted moments

The probability weighted moments (PWMs) are an alternative to traditional moments. In hydrology, these are useful, particularly for modeling EXEs (here, runoff) and fitting asymmetric or heavy tailed distributions. The PWMs of X for three parameters EPAIGW distribution can be obtained by expression as.

$$\tau_{s_1,s_2} = E[X^{s_1} F(x)^{s_2}] = \int_{-\infty}^{\infty} x^{s_1} [F(x)]^{s_2} f(x) \mathrm{d}x, \tag{36}$$

Using Equations (5), (6), (17) and (18) in Equation (36), we have.

$$\begin{aligned}\tau_{s_1,s_2} = &\sum_{s_4=0}^{\infty}\sum_{s_3=0}^{\infty} \frac{(-1)^{\delta(s_2+1)+s_3-1}(\log\alpha)^{1+s_4}(s_3+1)^{s_4}\delta}{s_4!(\alpha-1)^{\delta(s_2+1)}} \\ &\times \binom{\delta(s_2+1)-1}{s_3} B((\frac{s_1}{\gamma}+s_4+1),(1-\frac{s_1}{\gamma})).\end{aligned} \tag{37}$$

where, $v = x^{\gamma}$ and BF is as.

$$B((\frac{s_1}{\gamma}+s_4+1),(1-\frac{s_1}{\gamma})) = \int_0^{\infty} \frac{v^{\frac{s_1}{\gamma}+s_4}}{(1+v)^{2+s_4}} \mathrm{d}v.$$

If first three PWMs are 310, 275, and 250 (in m^3/s) by fitting EPAIGW distribution to ROfD. The decreasing trend indicates a right skewed distribution, which is common in basins that are prone to flooding.

5.9 Rényi entropy

The randomness or uncertainty of PDn is measured by entropy. To measure uncertainty variation, [45] developed entropy called Rényi entropy (REy), which is generalization of shannon entropy as.

$$R^e(x) = \frac{1}{1-e} \log \int_{-\infty}^{\infty} [f(x)]^e \mathrm{d}x, \quad e \neq 1, e > 0, \tag{38}$$

For r.v. X that follows three parameters EPAIGW distribution, then the REy using Equations (6), (17) and (18) in Equation (38) is.

$$\begin{aligned}R^e(x) = &\frac{1}{1-e} \log \Bigg[\sum_{g_2=0}^{\infty}\sum_{g_1=0}^{\infty} \frac{\delta^e \gamma^{e-1} (\log\alpha)^{g_2+e}(-1)^{g_1+e(\delta-1)}}{g_2!(\alpha-1)^{e\delta}(g_1+e)^{-g_2}} \\ &\times \binom{e(\delta-1)}{g_1} B((e+(1-e)\frac{1}{\gamma}+g_2-1),(1+e-(1-e)\frac{1}{\gamma}))\Bigg].\end{aligned} \tag{39}$$

Here $w = x^\gamma$ and the BF used in Equation (39) is.

$$\int_0^\infty \frac{w^{e+(1-e)\frac{1}{\gamma}+g_2-1}}{(1+w)^{2e+g_2}} \mathrm{d}w = B((e+(1-e)\frac{1}{\gamma}+g_2-1),(1+e-(1-e)\frac{1}{\gamma})).$$

To measure the degree of dispersion or concentration of the runoff distribution. The EPAIGW distribution is fitted to ROfD. For $e = 2$, $R^2 = 4.56$, and $e = 0.5$, $R^2 = 5.12$. Higher entropy implies higher uncertainty, which is common in basins with highly variable rainfall or rugged topography.

5.10 Order statistic

To understand severe occurrences and their probabilities, in hydrology, order statistic (OSc) analyze ordered sequences of hydrological data such as precipitation or runoff. To compute statistics such as extremes, median, or quantiles of any distribution, which are utilized for flood frequency analysis, drought analysis, and other hydrological designs, the data must first be sorted from smallest to largest. Supposed $X_{1:s} \le X_{2:s} \le \cdots \le X_{s:s}$ are OSc of random sample $X_1 < X_2 < \cdots < X_s$ taken from three parameter EPAIGW distribution have CDF $F_X(x)$ in Equation (5) and PDF $f_X(x)$ in Equation (6), then PDF of $X_{\tau:s}$; $\tau = 1,2,\cdots,s$ is defined as.

$$f_{X_{\tau:s}}(x) = \frac{s!}{(\tau-1)!(s-\tau)!}[F_X(x)]^{\tau-1}[1-F_X(x)]^{s-\tau} f_X(x), \tag{40}$$

Substituting Equations (5) and (6) in Equation (40), and applying Equations (17) and (18), we obtain the PDF $f_{X_{\tau:s}}(x)$ of τth OSc for three parameters EPAIGW distribution as.

$$\begin{aligned} f_{X_{\tau:s}}(x) = &\sum_{i=0}^{s-\tau}\sum_{j=0}^{\delta(i+\tau)-1}\sum_{k=0}^{\infty} \frac{s!(j+1)^k\delta\gamma(\log\alpha)^{1+k}(-1)^{i+j+\delta(i+\tau)-1}}{(\tau-1)!(s-\tau)!k!(\alpha-1)^{\delta(i+\tau)}} \\ &\times \binom{s-\tau}{i}\binom{\delta(i+\tau)-1}{j}\frac{x^{\gamma(k+1)-1}}{(1+x^\gamma)^{2+k}}. \end{aligned} \tag{41}$$

5.11 Moment of order statistic

Moment of OSc are more flexible and less sensitive to outliers because they are linear combinations of sorted data values than to traditional moments. The nth moment of τth OSc for three parameters EPAIGW distribution is defined as.

$$E(X^n_{\tau:s}) = \int_{-\infty}^{\infty} x^n_{\tau:s} f_{X_{\tau:s}}(x)\mathrm{d}x_{\tau:s}, \tag{42}$$

Using Equation (41) in Equation (42) and by solving the integral using BF with substitution $y = x^\gamma_{\tau:s}$, we have.

$$\begin{aligned} E(X^n_{\tau:s}) = &\sum_{i=0}^{s-\tau}\sum_{j=0}^{\delta(i+\tau)-1}\sum_{k=0}^{\infty} \frac{s!(j+1)^k(-1)^{i+j+\delta(i+\tau)-1}\delta(\log\alpha)^{1+k}}{(\tau-1)!k!(s-\tau)!(\alpha-1)^{\delta(i+\tau)}} \\ &\times \binom{\delta(i+\tau)-1}{j}\binom{s-\tau}{i} B((\frac{n}{\gamma}+k+1),(1-\frac{n}{\gamma})). \end{aligned} \tag{43}$$

6 Simulation Study

Here, simulation study is performed to check the performance of the maximum likelihood estimates of the three parameters EPAIGW distribution. In this scenario, $N = 1{,}000$ samples are generated from EPAIGW (α,δ,γ) distribution with different sets of values for (α,δ,γ) using sample sizes $n = 25,50,\cdots,250$, and obtain the average estimates (AEs) with their corresponding absolute average biases (ABs) and mean square errors (MSEs). The simulation results are given in Table 2-3.

As expected, from Table 2-3, we can see that the AEs tend to the actual values as n increases. Also, when n increases, we observe that the absolute ABs and MSEs approaches to zero. These trends assessed that estimation method provides the accurate and reliable estimates for the EPAIGW distribution. This numerical work is also presented graphically by the Figures 9-10. Through this simulation study $N = 1,000$ samples of sizes $n = 25, 50, \cdots, 250$ are generated from three parameters EPAIGW distribution with actual parameter values $\alpha = 1.11, \delta = 0.79, \gamma = 2.78$; $\alpha = 1.04, \delta = 0.89, \gamma = 3.59$; $\alpha = 1.09, \delta = 0.83, \gamma = 2.99$; and $\alpha = 1.16, \delta = 0.85, \gamma = 3.29$; and ploted the curves of the absolute ABs in Figure 9 and the curves of the MSEs in Figure 10 for the estimates. The Figures 9 and 10 explore that the absolute ABs and MSEs curves approach quickly to the x-axis.

Table 2: AEs along with absolute ABs and MSEs of the three parameters EPAIGW distribution.

		Set I			Set II		
		$\alpha = 1.11, \delta = 0.79, \gamma = 2.78$			$\alpha = 1.04, \delta = 0.89, \gamma = 3.59$		
n	Parameter	AEs	abs. ABs	MSEs	AEs	abs. ABs	MSEs
25	$\hat{\alpha}$	1.555529	0.445529	1.559446	1.382346	0.342346	1.168178
	$\hat{\delta}$	0.763747	0.026253	0.014399	0.889109	0.000891	0.013696
	$\hat{\gamma}$	2.889052	0.109052	0.111988	3.658388	0.068388	0.069566
50	$\hat{\alpha}$	1.257429	0.147429	0.478105	1.181492	0.141492	0.445380
	$\hat{\delta}$	0.776813	0.013187	0.005091	0.884470	0.005530	0.004634
	$\hat{\gamma}$	2.826470	0.046470	0.034023	3.623529	0.033529	0.026239
75	$\hat{\alpha}$	1.140375	0.030375	0.069438	1.071067	0.031067	0.087668
	$\hat{\delta}$	0.785537	0.004462	0.001383	0.887666	0.002334	0.001083
	$\hat{\gamma}$	2.796183	0.016183	0.009006	3.597007	0.007007	0.005156
100	$\hat{\alpha}$	1.135290	0.025290	0.057121	1.053793	0.013793	0.029526
	$\hat{\delta}$	0.787062	0.002938	0.000939	0.888998	0.001002	0.000511
	$\hat{\gamma}$	2.789172	0.009172	0.004080	3.593302	0.003302	0.001700
125	$\hat{\alpha}$	1.124317	0.014317	0.042203	1.045916	0.005916	0.017591
	$\hat{\delta}$	0.787512	0.002488	0.000452	0.889689	0.000311	0.000192
	$\hat{\gamma}$	2.784348	0.004348	0.001695	3.591147	0.001147	0.000594
150	$\hat{\alpha}$	1.112194	0.002194	0.003092	1.039818	0.000181	0.000008
	$\hat{\delta}$	0.789142	0.000858	0.000126	0.890022	0.000021	0.000023
	$\hat{\gamma}$	2.782166	0.002166	0.000575	3.590206	0.000206	0.000184
175	$\hat{\alpha}$	1.109780	0.000220	0.000061	1.040017	0.000017	0.000016
	$\hat{\delta}$	0.789920	0.000080	0.000030	0.890025	0.000025	0.000011
	$\hat{\gamma}$	2.781456	0.001456	0.000371	3.590836	0.000837	0.000900
200	$\hat{\alpha}$	1.110438	0.000438	0.000108	1.040443	0.000443	0.000142
	$\hat{\delta}$	0.790244	0.000244	0.000024	0.890096	0.000096	0.000021
	$\hat{\gamma}$	2.780219	0.000219	0.000017	3.590061	0.000061	0.000050
225	$\hat{\alpha}$	1.109944	0.000056	0.000014	1.040473	0.000473	0.000224
	$\hat{\delta}$	0.789871	0.000129	0.000008	0.889895	0.000105	0.000011
	$\hat{\gamma}$	2.780442	0.000441	0.000179	3.590004	0.000004	0.000000
250	$\hat{\alpha}$	1.109861	0.000139	0.000009	1.040043	0.000043	0.000002
	$\hat{\delta}$	0.789907	0.000093	0.000005	0.890010	0.000010	0.000000
	$\hat{\gamma}$	2.780013	0.000013	0.000039	3.590092	0.000092	0.000008

7 Bootstrapping

In this section, we adopted method of percentile bootstrap (MPB) proposed by [46], for constructing bootstrap confidence intervals (CIs) through parametric bootstrapping. The MPB follows the following steps.

(a) Compute, MLEs of the parameters $\nu = (\alpha, \delta, \gamma)^T$ for equations (5) and (6) through maximum likelihood method (MLMd) using the real sample $\underline{x} = (x_1, x_2, \cdots, x_n)$.

Table 3: AEs along with absolute ABs and MSEs of the three parameters EPAIGW distribution.

		Set III			Set IV		
		$\alpha = 1.09, \delta = 0.83, \gamma = 2.99$			$\alpha = 1.16, \delta = 0.85, \gamma = 3.29$		
n	Parameter	AEs	abs. ABs	MSEs	AEs	abs. ABs	MSEs
25	$\hat{\alpha}$	1.527306	0.437306	1.559321	1.602782	0.442782	1.554406
	$\hat{\delta}$	0.812460	0.017540	0.013664	0.818407	0.031593	0.016959
	$\hat{\gamma}$	3.079829	0.089829	0.101932	3.408292	0.118292	0.126214
50	$\hat{\alpha}$	1.191996	0.101996	0.335676	1.238004	0.078004	0.246791
	$\hat{\delta}$	0.820218	0.009782	0.004188	0.840750	0.009250	0.003736
	$\hat{\gamma}$	3.022827	0.032827	0.028074	3.331005	0.041005	0.031388
75	$\hat{\alpha}$	1.129945	0.039945	0.109626	1.193175	0.033175	0.112955
	$\hat{\delta}$	0.824237	0.005763	0.001470	0.844567	0.005433	0.001660
	$\hat{\gamma}$	3.003878	0.013878	0.010357	3.303998	0.013998	0.007832
100	$\hat{\alpha}$	1.102886	0.012886	0.036214	1.166355	0.006355	0.013421
	$\hat{\delta}$	0.827978	0.002022	0.000612	0.847537	0.002463	0.000570
	$\hat{\gamma}$	2.995876	0.005876	0.003477	3.294903	0.004903	0.002586
125	$\hat{\alpha}$	1.096032	0.006032	0.008724	1.160922	0.000922	0.002489
	$\hat{\delta}$	0.830112	0.000112	0.000189	0.850301	0.000301	0.000182
	$\hat{\gamma}$	2.991698	0.001698	0.000884	3.292248	0.002248	0.000756
150	$\hat{\alpha}$	1.090960	0.000960	0.000571	1.166299	0.006299	0.017883
	$\hat{\delta}$	0.829749	0.000251	0.000056	0.849030	0.000970	0.000250
	$\hat{\gamma}$	2.990875	0.000875	0.000211	3.291975	0.001975	0.001278
175	$\hat{\alpha}$	1.089910	0.000090	0.000006	1.161362	0.001362	0.000981
	$\hat{\delta}$	0.829944	0.000056	0.000004	0.850050	0.000050	0.000085
	$\hat{\gamma}$	2.990223	0.000223	0.000026	3.290137	0.000137	0.000149
200	$\hat{\alpha}$	1.090159	0.000159	0.000049	1.162391	0.002391	0.005003
	$\hat{\delta}$	0.829924	0.000076	0.000019	0.849702	0.000298	0.000046
	$\hat{\gamma}$	2.990086	0.000086	0.000008	3.290583	0.000583	0.000334
225	$\hat{\alpha}$	1.090000	0.000000	0.000000	1.159871	0.000129	0.000009
	$\hat{\delta}$	0.830000	0.000000	0.000000	0.849988	0.000012	0.000017
	$\hat{\gamma}$	2.990000	0.000000	0.000000	3.290032	0.000032	0.000011
250	$\hat{\alpha}$	1.090032	0.000032	0.000001	1.160018	0.000018	0.000000
	$\hat{\delta}$	0.829922	0.000078	0.000006	0.850061	0.000061	0.000004
	$\hat{\gamma}$	2.990123	0.000123	0.000015	3.290077	0.000077	0.000006

(b) Using resampling technique of with replacement, obtain bootstrap sample $\underline{x}' = (x_1', x_2', \cdots, x_n')$ for required sample size.
(c) As in step (a), based on $\underline{x}'$ obtain MLEs of $\nu = (\alpha, \delta, \gamma)^T$ for equations (5) and (6) for bootstrap sample, say $\hat{\nu}' = (\hat{\alpha}', \hat{\delta}', \hat{\gamma}')^T$.
(d) Repeat steps (b) and (c) N times for booting and compute $\hat{\nu}_1' = (\hat{\alpha}_1', \hat{\delta}_1', \hat{\gamma}_1')^T$; $\hat{\nu}_2' = (\hat{\alpha}_2', \hat{\delta}_2', \hat{\gamma}_2')^T$; $\cdots$ $\hat{\nu}_{Nboot}' = (\hat{\alpha}_{Nboot}', \hat{\delta}_{Nboot}'), \hat{\gamma}_{Nboot}')^T$; where $\hat{\nu}_i' = (\hat{\alpha}_i', \hat{\delta}_i', \hat{\gamma}_i')^T$; $i = 1, 2, \cdots, Nboot$.
(e) Develop ascending order for $\hat{\nu}_i'; i = 1, 2, \cdots, Nboot$, to obtain bootstrap sample $(\hat{\nu}_1', \hat{\nu}_2', \cdots, \hat{\nu}_{Nboot}')$.
(f) Let $H(x) = P(\nu_i \leq x)$ be CDF of ν_i. Define $\nu_{i\ boot} = H^{-1}(x)$ provided $0 < x < 1$. The approximate bootstrap $100(1 - \beta^*)\%$ confidence interval of ν_i given by $\left\{\nu_{i\ MPB}\left(\frac{\beta^*}{2}\right), \nu_{i\ MPB}\left(1 - \frac{\beta^*}{2}\right)\right\}$.

For different sample size values, we obtained MLEs of parameters with their corresponding standard errors (SEs), biases, lower limit (L-L), upper limit (U-L) and length of percentile bootstrap CIs for EPAIGWD based on 1000 bootstrap samples. The results of bootstrapping are given in Table 4 and 5. The histogram, density curve, CIs plots, trace plots, Q-Q plots and box-plots for bootstrap distribution of α, δ and γ are given in Figures 11- 12. The results obtained for EPAIGWD after bootstrapping suggested that the estimated parameters or their variability's reliable and robust.

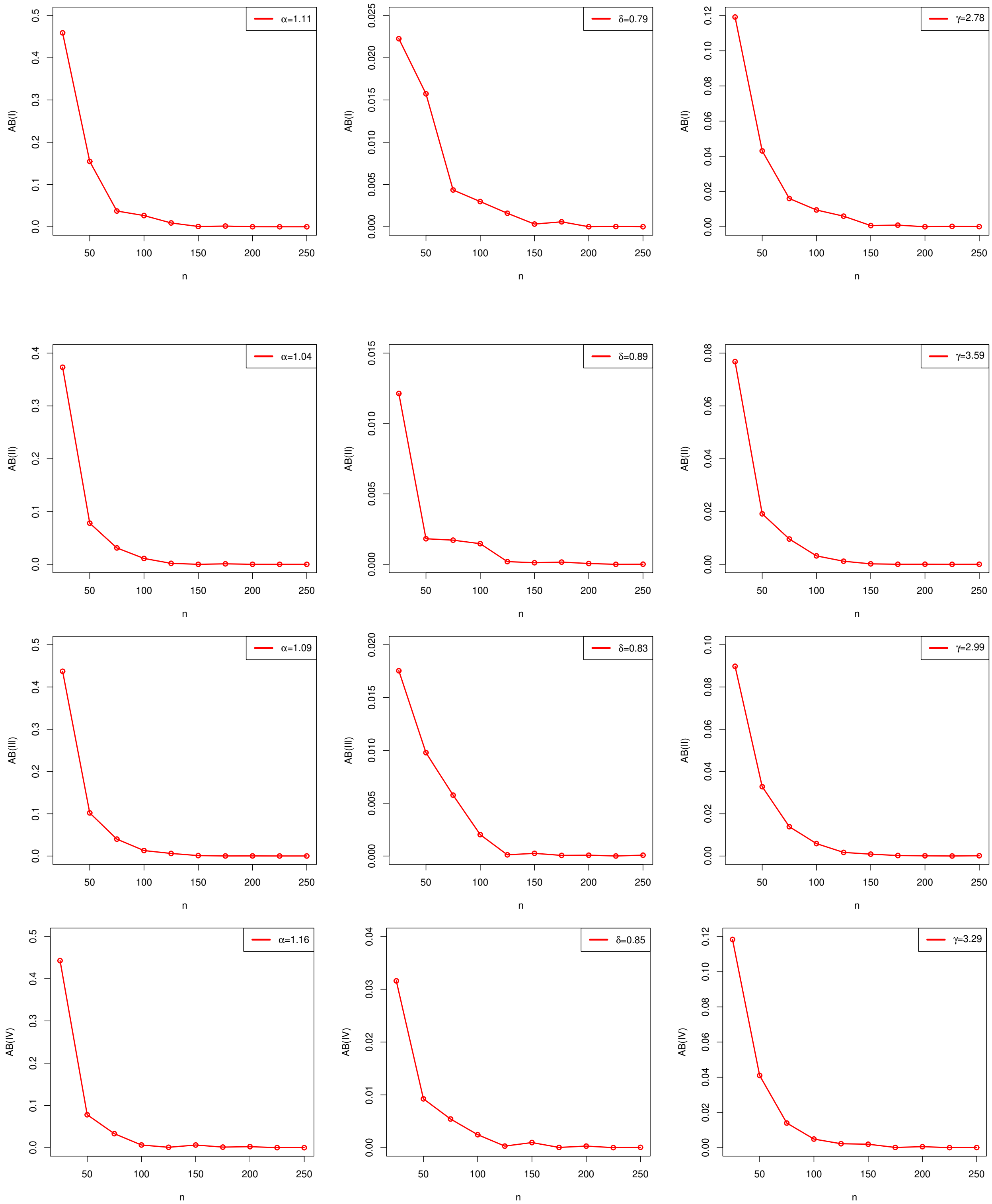

Fig. 9: Plots of absolute ABs
for EPAIGW (α, δ, γ) distribution of sets I (1.11, 0.79, 2.78), II (1.04, 0.89, 3.59), III (1.09, 0.83, 2.99) and IV (1.16, 0.85, 3.29).

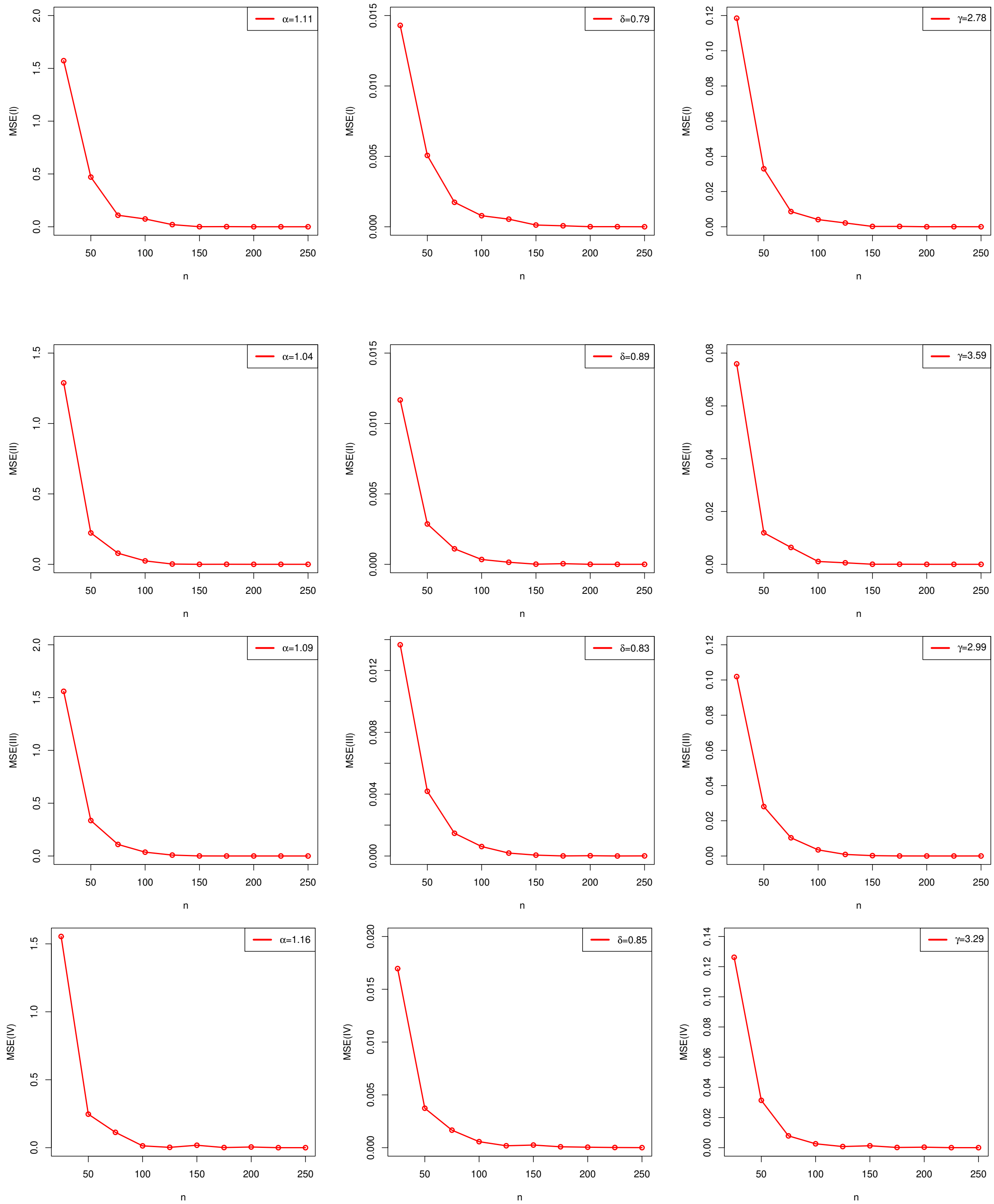

Fig. 10: Plots of MSEs
for EPAIGW (α, δ, γ) distribution of sets I (1.11, 0.79, 2.78), II (1.04, 0.89, 3.59), III (1.09, 0.83, 2.99) and IV (1.16, 0.85, 3.29).

8 Estimation through maximum likelihood method

Supposed sample $x_1, x_2, \cdots, x_n$ be taken from three parameters EPAIGW distribution with PDF described in Equation (6) for parameter vector $\nu = (\alpha, \delta, \gamma)^T$. To obtain MLEs using MLMd, we develop an expression of log-likelihood function

Table 4: Estimation of α, δ and γ, SEs, biases, 95% MPB CIs (L-L, U-L and length).

Set I							
$\alpha = 3.30, \delta = 0.39, \gamma = 0.02$							
n	Parameters	MLEs	SEs	Biases	L-L	U-L	Length
250	$\hat{\alpha}$	3.331251	0.051119	0.021753	3.325341	3.482073	0.156732
	$\hat{\delta}$	1.005034	0.420815	0.218059	0.943132	2.310918	1.367786
	$\hat{\gamma}$	0.655674	0.014984	-0.005824	0.616530	0.669018	0.052487
400	$\hat{\alpha}$	3.325991	0.043893	0.019582	3.325042	3.464463	0.139421
	$\hat{\delta}$	0.944732	0.361796	0.190716	0.937597	2.025651	1.088054
	$\hat{\gamma}$	0.624569	0.015395	0.017146	0.614057	0.668574	0.054517
450	$\hat{\alpha}$	3.326969	0.047912	0.024341	3.325300	3.486923	0.161622
	$\hat{\delta}$	0.957019	0.410805	0.218070	0.941055	2.097577	1.156522
	$\hat{\gamma}$	0.636471	0.013713	0.012116	0.622475	0.668972	0.046497
500	$\hat{\alpha}$	3.327082	0.088632	0.024882	3.326423	3.490980	0.164557
	$\hat{\delta}$	0.953688	0.389114	0.242526	0.949617	2.099719	1.150102
	$\hat{\gamma}$	0.622196	0.015516	0.019510	0.615827	0.668829	0.053002
550	$\hat{\alpha}$	3.329882	0.063922	0.023764	3.326099	3.483705	0.157605
	$\hat{\delta}$	0.996099	0.368578	0.196196	0.946311	2.013001	1.066690
	$\hat{\gamma}$	0.647590	0.014695	-0.002660	0.617601	0.668614	0.051013
600	$\hat{\alpha}$	3.357897	0.125668	-0.003406	3.326781	3.497141	0.170361
	$\hat{\delta}$	1.214248	0.405421	0.022372	0.952082	2.122731	1.170649
	$\hat{\gamma}$	0.650527	0.014827	-0.007311	0.615474	0.668679	0.053206
650	$\hat{\alpha}$	3.329318	0.108386	0.013364	3.325178	3.464371	0.139193
	$\hat{\delta}$	0.994134	0.356590	0.186676	0.943698	1.946338	1.002639
	$\hat{\gamma}$	0.655203	0.013885	-0.006947	0.620840	0.668940	0.048100
700	$\hat{\alpha}$	3.327974	0.123103	0.022293	3.326574	3.485339	0.158765
	$\hat{\delta}$	0.965463	0.395164	0.263748	0.951897	2.064418	1.112521
	$\hat{\gamma}$	0.630294	0.013757	0.016207	0.619453	0.668884	0.049431
750	$\hat{\alpha}$	3.405667	0.148122	-0.059175	3.325740	3.467883	0.142142
	$\hat{\delta}$	1.613268	0.385522	-0.373980	0.944705	1.988916	1.044211
	$\hat{\gamma}$	0.668681	0.014323	-0.023416	0.617453	0.668763	0.051311
850	$\hat{\alpha}$	3.469157	0.101241	-0.112443	3.326466	3.479378	0.152912
	$\hat{\delta}$	1.938313	0.408376	-0.675807	0.949271	2.060054	1.110783
	$\hat{\gamma}$	0.643309	0.014773	0.001569	0.616805	0.668763	0.051958
950	$\hat{\alpha}$	3.327185	0.141849	0.022843	3.326458	3.476648	0.150191
	$\hat{\delta}$	0.956624	0.431562	0.310124	0.950975	2.147015	1.196040
	$\hat{\gamma}$	0.625547	0.014373	0.021354	0.619858	0.668809	0.048951
1000	$\hat{\alpha}$	3.485422	0.066702	-0.128068	3.326781	3.492306	0.165524
	$\hat{\delta}$	1.930851	0.464887	-0.663539	0.954921	2.744780	1.789859
	$\hat{\gamma}$	0.658557	0.013885	-0.008743	0.622079	0.669015	0.046936

(LLFn) for ν as.

$$\begin{aligned} \ell = \ell(\nu) &= n\log(\delta) + n\log(\gamma) + n\log(\log(\alpha)) - n\delta\log(\alpha-1) + (\gamma-1)\sum_{i=1}^{n}\log(x_i) \\ &- 2\sum_{i=1}^{n}\log[1+x_i^{\gamma}] + (\delta-1)\sum_{i=1}^{n}\log[\alpha^{\frac{x_i^{\gamma}}{1+x_i^{\gamma}}}-1] + \log(\alpha)\sum_{i=1}^{n}\frac{x_i^{\gamma}}{1+x_i^{\gamma}}. \end{aligned} \tag{44}$$

We maximized the Equation (44) with respect to ν as.

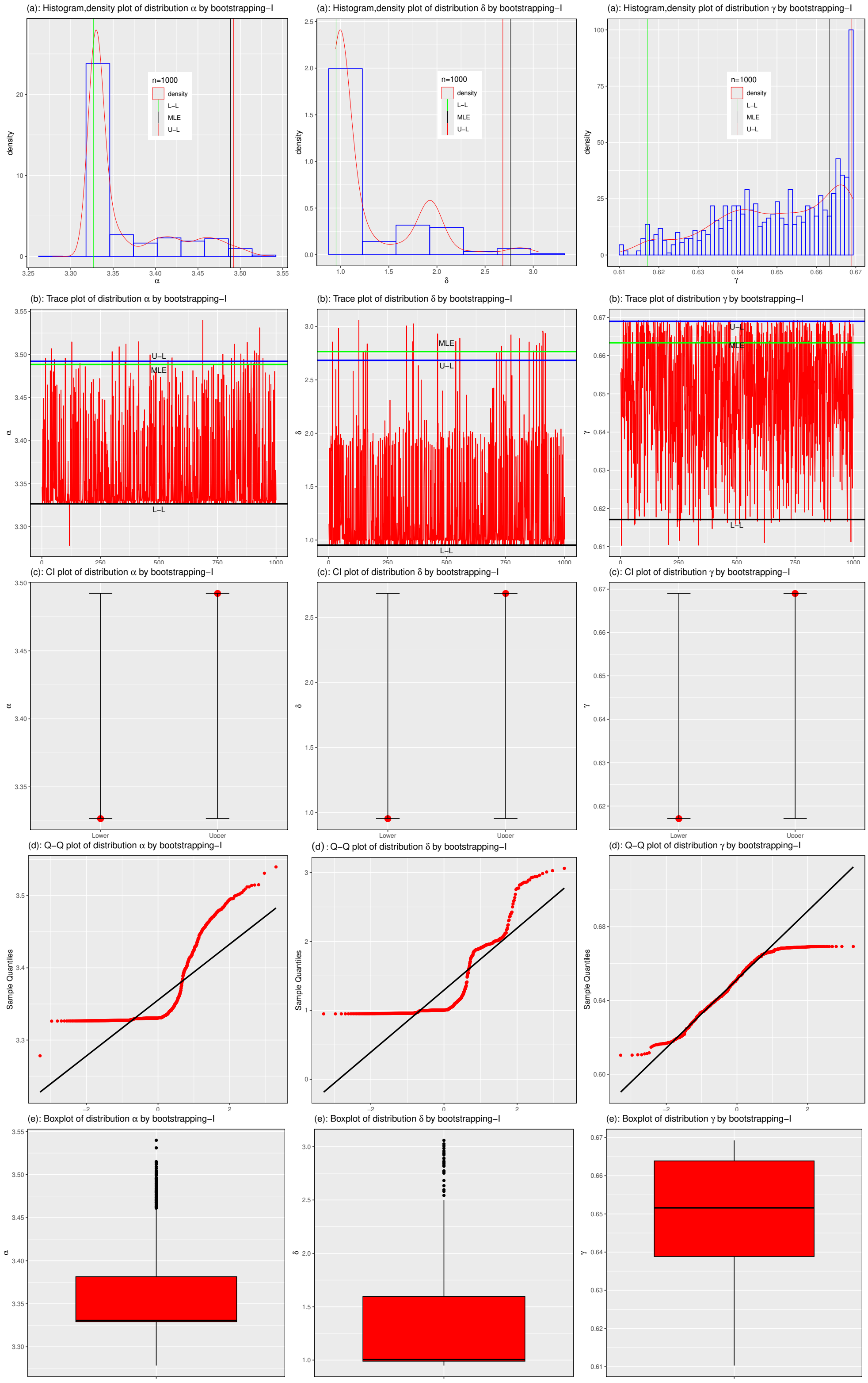

Fig. 11: Histogram, density plot, trace plot, CI plot, Q-Q plot and box plot for EPAIGW (α, δ, γ) distribution of sets I by bootstrapping.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha \log(\alpha)} - \frac{n\delta}{\alpha - 1} + (\delta - 1)\sum_{i=1}^{n} \frac{x_i^{\gamma}}{1+x_i^{\gamma}} \left[\alpha - \alpha^{\frac{1}{1+x_i^{\gamma}}}\right]^{-1} + \frac{1}{\alpha}\sum_{i=1}^{n} \frac{x_i^{\gamma}}{1+x_i^{\gamma}}, \quad (45)$$

Table 5: Estimation of α, δ and γ, SEs, biases, 95% MPB CIs (L-L, U-L and length).

Set II							
$\alpha = 2.83, \delta = 0.79, \gamma = 0.12$							
n	Parameters	MLEs	SEs	Biases	L-L	U-L	Length
250	$\hat{\alpha}$	2.849467	0.088160	0.044809	2.848461	3.055502	0.207040
	$\hat{\delta}$	1.034090	0.503273	0.366290	1.027574	2.425048	1.397474
	$\hat{\gamma}$	0.627990	0.016799	0.017454	0.612998	0.668975	0.055977
400	$\hat{\alpha}$	2.847858	0.074912	0.012864	2.846932	2.995458	0.148526
	$\hat{\delta}$	1.022595	0.322428	0.126600	1.017152	2.062900	1.045748
	$\hat{\gamma}$	0.623273	0.015572	0.006946	0.607422	0.666410	0.058988
450	$\hat{\alpha}$	2.850994	0.086118	0.008363	2.847632	3.009942	0.162310
	$\hat{\delta}$	1.051627	0.316235	0.096477	1.023128	2.096550	1.073422
	$\hat{\gamma}$	0.642936	0.013876	-0.008330	0.612891	0.665747	0.052857
500	$\hat{\alpha}$	2.954608	0.052572	-0.087873	2.848010	3.020086	0.172076
	$\hat{\delta}$	1.989647	0.331054	-0.830572	1.024336	2.154965	1.130629
	$\hat{\gamma}$	0.668571	0.014172	-0.036546	0.611681	0.665883	0.054202
550	$\hat{\alpha}$	2.855643	0.084383	0.030791	2.847911	3.031202	0.183291
	$\hat{\delta}$	1.090511	0.433975	0.229819	1.022176	2.195520	1.173344
	$\hat{\gamma}$	0.630367	0.019502	0.005454	0.608807	0.668653	0.059846
600	$\hat{\alpha}$	2.848379	0.142952	0.018042	2.847625	3.013238	0.165613
	$\hat{\delta}$	1.025754	0.412396	0.252305	1.021891	2.161821	1.139929
	$\hat{\gamma}$	0.622149	0.016775	0.015107	0.612807	0.668742	0.055935
650	$\hat{\alpha}$	2.848602	0.091033	0.011445	2.847865	2.995399	0.147534
	$\hat{\delta}$	1.029918	0.315396	0.133543	1.025416	2.081586	1.056170
	$\hat{\gamma}$	0.635905	0.012287	0.005845	0.621124	0.668504	0.047380
700	$\hat{\alpha}$	2.906115	0.093884	-0.016180	2.848184	3.035774	0.187590
	$\hat{\delta}$	1.392487	0.435596	-0.023285	1.024580	2.203868	1.179288
	$\hat{\gamma}$	0.653266	0.018482	-0.011168	0.611458	0.668825	0.057368
750	$\hat{\alpha}$	2.848552	0.082486	0.018394	2.847932	3.000794	0.152861
	$\hat{\delta}$	1.029380	0.355076	0.186370	1.025754	2.088791	1.063037
	$\hat{\gamma}$	0.633511	0.013154	0.010868	0.622735	0.668861	0.046126
850	$\hat{\alpha}$	2.945443	0.111268	-0.054387	2.848327	3.029585	0.181258
	$\hat{\delta}$	1.627807	0.457536	-0.223545	1.024348	2.197143	1.172795
	$\hat{\gamma}$	0.654262	0.018919	-0.010994	0.609856	0.668868	0.059012
950	$\hat{\alpha}$	2.997989	0.095832	-0.103183	2.848271	3.025506	0.177236
	$\hat{\delta}$	2.032690	0.465169	-0.613565	1.025111	2.239902	1.214791
	$\hat{\gamma}$	0.621509	0.018623	0.023791	0.608191	0.668846	0.060654
1000	$\hat{\alpha}$	2.949295	0.115662	-0.083724	2.848118	2.998823	0.150706
	$\hat{\delta}$	2.005852	0.395975	-0.753473	1.028629	2.281524	1.252894
	$\hat{\gamma}$	0.666312	0.012083	-0.015870	0.630383	0.669022	0.038638

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - n\log(\alpha-1) + \sum_{i=1}^{n} \log\left\{\alpha^{\frac{x_i^\gamma}{1+x_i^\gamma}} - 1\right\}, \tag{46}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n}\log(x_i) - 2\sum_{i=1}^{n}\frac{x_i^\gamma}{1+x_i^\gamma}\left[1-\frac{\log(\alpha)}{2[1+x_i^\gamma]}\right]\log(x_i) + (\delta-1)\log(\alpha)\\ &\times \sum_{i=1}^{n}\left[1-\alpha^{-\frac{x_i^\gamma}{1+x_i^\gamma}}\right]^{-1}\frac{1}{[1+x_i^\gamma]^2}x_i^\gamma\log(x_i).\end{aligned} \tag{47}$$

Fixing $\frac{\partial \ell}{\partial \alpha} = 0$, $\frac{\partial \ell}{\partial \delta} = 0$, and $\frac{\partial \ell}{\partial \gamma} = 0$ from Equations (45)–(47) gives the normal equations (NEns) which provides the MLEs $\hat{v} = (\hat{\alpha}, \hat{\delta}, \hat{\gamma})^T$. Simultaneous analytical solution of these NEns is tedious work, so, we obtained the numerical

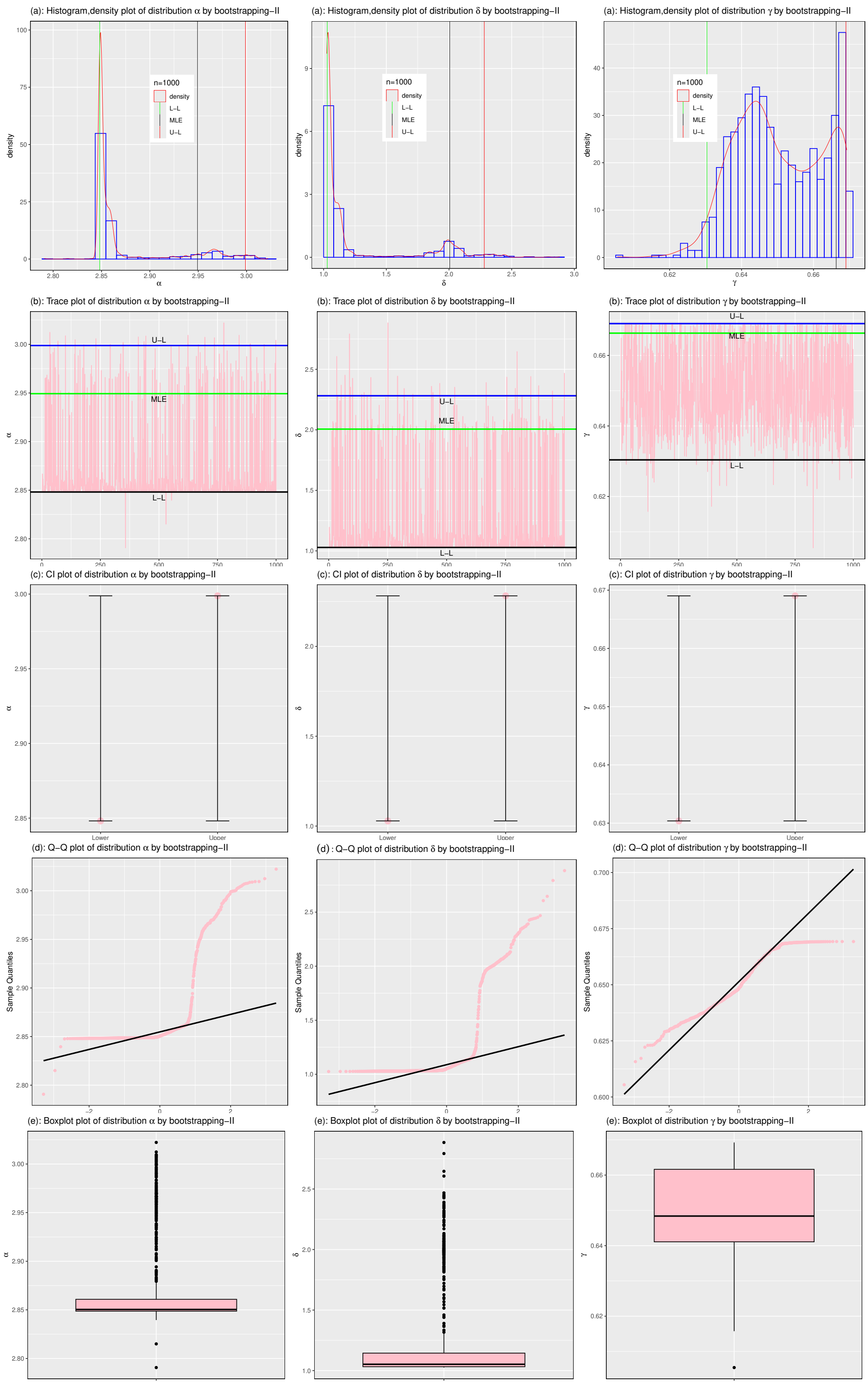

Fig. 12: Histogram, density plot, trace plot, CI plot, Q-Q plot and box plot for EPAIGW (α, δ, γ) distribution of sets II by bootstrapping.

solution through iterative method using R language for three parameters EPAIGW distribution.

To construct asymptotic CIs for parameters of the EPAIGW distribution, the observed information matrix (IMx) $J(\hat{\nu})$ is used as given below due to complications in expected IMx.

$$J(\hat{\nu}) = -\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\delta} & \frac{\partial^2\ell}{\partial\alpha\partial\gamma} \\ & \frac{\partial^2\ell}{\partial\delta^2} & \frac{\partial^2\ell}{\partial\delta\partial\gamma} \\ & & \frac{\partial^2\ell}{\partial\gamma^2} \end{pmatrix}$$

where $\nu = (\alpha, \delta, \gamma)T$. The explicit expression for elements of observed IMx are provided as.

$$\begin{aligned} \frac{\partial^2\ell}{\partial\alpha^2} = & -\frac{n[1+\log(\alpha)]}{[\alpha\log(\alpha)]^2} + \frac{n\delta}{(\alpha-1)^2} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{x_i^\gamma}{1+x_i^\gamma} + (1-\delta)\sum_{i=1}^{n}\frac{x_i^\gamma}{1+x_i^\gamma} \\ & \times\left[1-\frac{\alpha^{-\frac{x_i^\gamma}{1+x_i^\gamma}}}{1+x_i^\gamma}\right]\left[\alpha-\alpha^{\frac{1}{1+x_i^\gamma}}\right]^{-2}, \end{aligned} \tag{48}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\delta} = -\frac{n}{\alpha-1} + \sum_{i=1}^{n}\left[\alpha-\alpha^{\frac{1}{1+x_i^\gamma}}\right]^{-1}\frac{x_i^\gamma}{1+x_i^\gamma}, \tag{49}$$

$$\begin{aligned} \frac{\partial^2\ell}{\partial\alpha\partial\gamma} = & (\delta-1)\sum_{i=1}^{n}\frac{x_i^\gamma\log(x_i)}{1+x_i^\gamma}\left[\alpha-\alpha^{\frac{1}{1+x_i^\gamma}}\right]^{-1}\left\{1-\frac{x_i^\gamma}{1+x_i^\gamma}-\frac{x_i^\gamma\log(\alpha)\alpha^{\frac{1}{1+x_i^\gamma}}}{(1+x_i^\gamma)^2[\alpha-\alpha^{\frac{1}{1+x_i^\gamma}}]}\right\} \\ & +\frac{1}{\alpha}\sum_{i=1}^{n}\frac{x_i^\gamma\log(x_i)}{(1+x_i^\gamma)^2}, \end{aligned} \tag{50}$$

$$\frac{\partial^2\ell}{\partial\delta^2} = -\frac{n}{\delta^2}, \tag{51}$$

$$\frac{\partial^2\ell}{\partial\delta\partial\gamma} = \log(\alpha)\sum_{i=1}^{n}\frac{x_i^\gamma\log(x_i)}{(1+x_i^\gamma)^2}\left[\alpha^{\frac{x_i^\gamma}{1+x_i^\gamma}}-1\right]^{-1}\alpha^{\frac{x_i^\gamma}{1+x_i^\gamma}}, \tag{52}$$

$$\begin{aligned} \frac{\partial^2\ell}{\partial\gamma^2} = & -\frac{n}{\gamma^2} - 2\sum_{i=1}^{n}\frac{x_i^\gamma\{\log(x_i)\}^2}{(1+x_i^\gamma)^2} + (\delta-1)\log(\alpha)\sum_{i=1}^{n}\frac{\{\log(x_i)\}^2 x_i^\gamma}{(1+x_i^\gamma)^2} \\ & \times\left[1-\alpha^{-\frac{x_i^\gamma}{1+x_i^\gamma}}\right]^{-1}\left\{\frac{1-x_i^\gamma}{1+x_i^\gamma}-\left[1-\alpha^{-\frac{x_i^\gamma}{1+x_i^\gamma}}\right]^{-1}\frac{x_i^\gamma\log(\alpha)}{(1+x_i^\gamma)^2}\alpha^{-\frac{x_i^\gamma}{1+x_i^\gamma}}\right\} \\ & +\log(\alpha)\sum_{i=1}^{n}\frac{1-x_i^\gamma}{(1+x_i^\gamma)^3}x_i^\gamma\{\log(x_i)\}^2. \end{aligned} \tag{53}$$

When, parameters lying within interior of parameter space but not on boundary and follow usual regularity conditions then.

$$\sqrt{n}(\hat{\nu}-\nu) \simeq N_3(0, I^{-1}(\nu)),$$

where $I(\nu)$ is the expected IMx. Asymptotically its behaviour is valid if $I(\nu)$ is replaced by the $J(\hat{\nu})$ specified at $\hat{\nu}$. Using asymptotic multivariate ND, $N_3(0, J^{-1}(\hat{\nu}))$, the large sample $100(1-\beta)\%$ CIs of $\hat{\nu}$ for three parameters EPAIGW distribution are as.

$$\hat{\nu} \pm Z_{\frac{\beta}{2}}\sqrt{J^{-1}(\hat{\nu})},$$

where β is the significance level and $Z_{\frac{\beta}{2}}$ is upper $\frac{\beta}{2}$th percentile regarding standard ND.

9 Empirical Illustration

In statistical applications, the interest is focused on estimates of parameters and results of goodness-of-fit of the model to observe data on hand. Here, we check the effectiveness of the three parameter EPAIGW distribution by fitting to different datasets. We evaluate the GFt of the proposed model and make comparisons with other models in same sense. To compare competitive models (CMls), we computed measures of GFt statistics such as negative LLFn value ($-\ell$), Cramér-von Mises (W^*), Anderson-Darling (A^*), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogrov-Smirnov (KS) statistic with p-value. In general, the smaller the values of these statistics with high p-value, the better fit to the data. For more detail, see [47]. We used likelihood ratio test (LRT) to check whether EPAIGW distribution is superior to sub model, AIGW distribution, at given data set. Here, $H_0 : \delta = 1$ against $H_1 : H_0$ is not true, is comparable for EPAIGW distribution and AIGW distribution. The LRT statistic is evolved as $\omega = 2\{\ell(\hat{\alpha},\hat{\delta},\hat{\gamma}) - \ell(\widetilde{\alpha},1,\widetilde{\gamma})\}$, where $\hat{\alpha},\hat{\delta},\hat{\gamma}$ are MLEs under H_1 and $\widetilde{\alpha},\widetilde{\gamma}$ are estimates under H_0. The usual computations are done in the R language.

9.1 Application of the three parameter EPAIGW distribution on water runoff (in mm)

The 34 storm events were recorded from one watershed. According to data published by [48], one of the hydrological features studied since 1996 from the mall watershed in Korea (west of Suwon city) is storm water runoff (mm). The dataset is as: 0.80, 1.60, 8.10, 0.80, 1.80, 0.90, 59.30, 30.70, 3.40, 2.50, 216.20, 78.20, 0.60, 2.0, 24.10, 181.70, 7.30, 17.0, 16.80, 2.0, 146.80, 1.10, 6.10, 5.10, 2.0, 0.80, 2.90, 6.0, 89.10, 13.30, 20.50, 33.50, 7.20, 75.90.
The descriptive statistics of the water runoff dataset are: n=34, Min.=0.60, 1st Qu.=2.00, Median=6.65, 3rd Qu.=29.05, Mean=31.36, Max.=216.20, Standard deviation=53.75, Skewness=2.22, and Kurtosis=4.07; and TTT and Box plots are given in Figure 13.

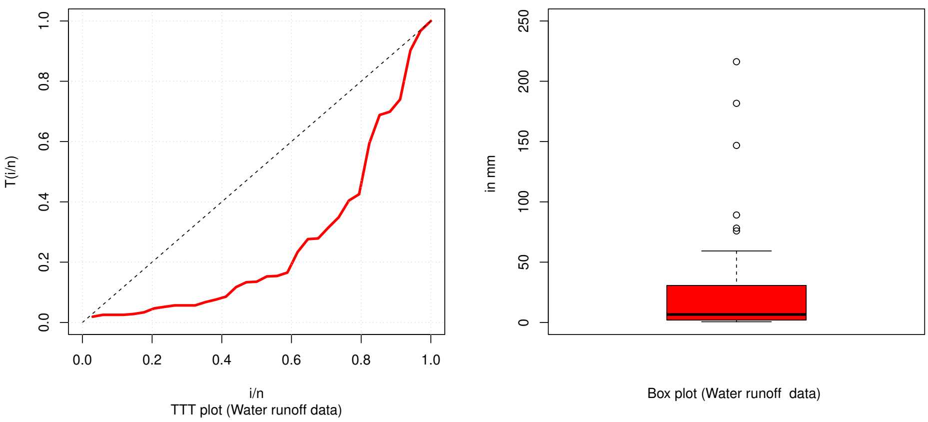

Fig. 13: TTT plot and Box plot for water runoff data.

9.2 Application of the three parameter EPAIGW distribution on fictitious water runoff data (in mm)

This dataset is generated through sample with replacement using real dataset [48]. The data points are as. 59.3, 0.9, 75.9, 0.8, 6.1, 8.1, 6.0, 2.0, 2.0, 3.4, 30.7, 1.8, 17.0, 2.0, 7.3, 75.9, 7.3, 17.0, 6.0, 2.0, 78.2, 3.4, 2.0, 20.5, 2.0, 0.8, 78.2, 7.3, 78.2, 78.2, 2.9, 7.2, 6.1, 0.6, 8.1, 0.8, 6.1, 0.9, 78.2, 0.8, 78.2, 1.1, 216.2, 2.5, 7.3, 7.3, 0.8, 2.0, 24.1, 2.5, 0.8, 24.1, 20.5, 6.0, 2.0, 89.1, 216.2, 1.8, 0.8, 1.6, 13.3, 33.5, 7.2, 24.1, 75.9, 3.4, 0.9, 13.3, 17.0, 24.1, 2.9, 0.8, 2.9, 20.5, 0.8, 6.0, 1.6, 13.3, 0.9, 24.1, 6.1, 0.8, 78.2, 20.5, 16.8, 0.8, 30.7, 181.7, 30.7, 2.0, 2.9, 5.1, 1.6, 0.6, 0.9.
The descriptive statistics of the fictitious water runoff dataset are: n=95, Min.=0.60, 1st Qu.=1.90, Median=6.10, 3rd Qu.= 22.30, Mean=22.77, Max.=216.20, Standard deviation=41.25, Skewness=3.05, and Kurtosis=10.21; and TTT and Box plots are given in Figure 14.
For the above discussed datasets, we compared the fits of the EPAIGW distribution with the Alpha index generalized Weibull (AIGW), Alpha power exponential (APE) [49], Kumaraswamy exponential (KmE) [50], Exponentiated exponential (EE) [51], Weibull [1] and Exponential (Expl) distributions.
The PDF of the competitive models are.

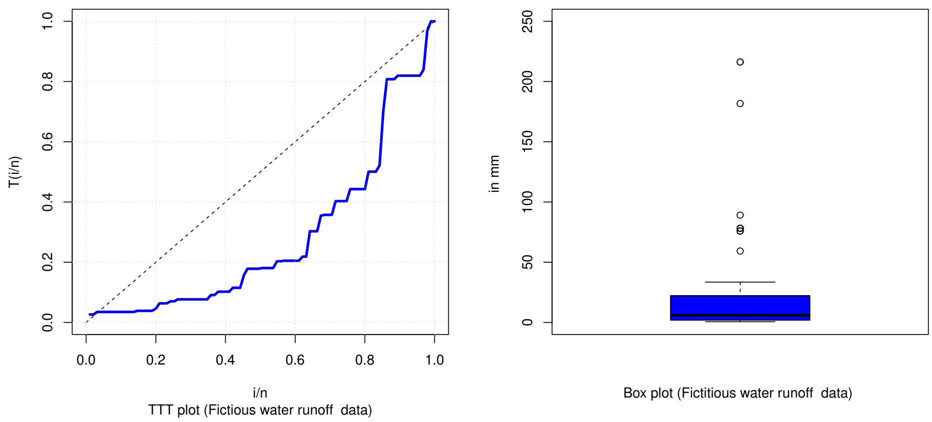

Fig. 14: TTT plot and Box plot for fictitious water runoff data.

The Alpha index generalized Weibull.

$$f(x) = \frac{\gamma \log(\alpha)\, x^{\gamma-1}}{\alpha - 1} \frac{\alpha^{\frac{x^\gamma}{1+x^\gamma}}}{\left[1+x^\gamma\right]^2}; \quad \alpha, \gamma, x > 0.$$

The Alpha power exponential.

$$f(x) = \frac{b \log(\alpha)}{\alpha - 1} \exp(-bx) \alpha^{1-\exp(-bx)}; \quad \alpha > 0, \alpha \neq 1, b, x > 0.$$

The Kumaraswamy exponential.

$$f(x) = \delta ab \exp(-\delta x)(1-\exp(-\delta x))^{a-1}(1-(1-\exp(-\delta x))^a)^{b-1}; \quad \delta, a, b, x > 0.$$

The Exponentiated exponential.

$$f(x) = ab \exp(-bx)(1-\exp(-bx))^{a-1} \quad a, b, x > 0.$$

The Weibull.

$$f(x) = \gamma \beta x^{\gamma-1} \exp(-\beta x^\gamma); \quad \gamma, \beta, x > 0.$$

The exponential.

$$f(x) = b \exp(-bx); \quad b, x > 0.$$

The results of MLEs with standard errors are given in Table 6-7 and GFt statistics values are provided in Table 8-9 of all fitted models for both cases of water runoff data. From Table 8 and 9, it could be observed that the EPAIGW model has the lowest GFt statistics values and highest p-value against all fitted CMls. The likelihood ratio test statistic to test the hypotheses H_0 against H_1 values are given in Table 10. According to these values, we reject the null hypothesis in both cases in favour of the EPAIGW distribution at the significance level 5%. Furthermore, Figure 15 and 16, also support these results for water runoff data sets.

Table 6: Parameter estimates and their standards errors for water runoff data of the EPAIGW model with other CMIs.

Model	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{a}$	$\hat{b}$
EPAIGW	1.00787	2.98842	–	0.66906	–	–
	(5.40319)	(6.10058)		(0.00244)		
AIGW	19.47574	–	–	0.66387	–	–
	(11.48519)			(0.00615)		
Weibull	–	–	0.17313	0.59149	–	–
			(0.05783)	(0.07608)		
KmE	–	0.00356	–	–	0.54371	3.62201
		(0.00208)			(0.08953)	(1.78116)
EE	–	–	–	–	0.46004	0.01801
					(0.09274)	(0.00513)
APE	0.04116	–	–	–	–	0.01805
	(0.04537)					(0.00578)
Expl	–	–	–	–	–	0.03189
						(0.00546)

Table 7: Parameter estimates and their standards errors for fictitious water runoff data of the EPAIGW model with other CMIs.

Model	$\hat{\alpha}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{a}$	$\hat{b}$
EPAIGW	7.04425	1.36774	–	0.66837	–	–
	(8.10213)	(0.45319)		(0.00174)		
AIGW	18.93196	–	–	0.66880	–	–
	(7.28605)			(0.00173)		
Weibull	–	–	0.62247	14.88656	–	–
			(0.04726)	(2.60551)		
KmE	–	0.00398	–	–	0.58079	4.48667
		(0.00150)			(0.05475)	(1.41407)
EE	–	–	–	–	0.49618	0.02616
					(0.06078)	(0.00439)
APE	0.03283	–	–	–	–	0.02265
	(0.02466)					(0.00483)
Expl	–	–	–	–	–	0.04392
						(0.00450)

Table 8: The measures $-\ell$, W^*, A^*, AIC, BIC, KS, and p-Value for water runoff data of the EPAIGW model with other CMIs.

Model	$-\ell$	W^*	A^*	AIC	BIC	KS	p-Value
EPAIGW	137.9472	0.04782	0.42643	281.8944	286.4735	0.09190	0.9363
AIGW	140.2786	0.06692	0.67948	284.5571	287.6099	0.13202	0.5939
Weibull	140.3388	0.15024	0.97064	284.6777	287.7304	0.13888	0.5283
KmE	140.9855	0.17509	1.07583	287.9711	292.5502	0.15692	0.3724
EE	142.2647	0.27183	1.48596	288.5293	291.5821	0.18921	0.1752
APE	144.2117	0.53074	3.67070	292.4233	295.4760	0.23141	0.0524
Expl	151.1437	1.34139	9.03276	304.2873	305.8137	0.36058	0.0003

Table 9: The measures $-\ell$, W^*, A^*, AIC, BIC, KS, and p-Value for fictitious water runoff data of the EPAIGW model with other CMls.

Model	$-\ell$	W^*	A^*	AIC	BIC	KS	p-Value
EPAIGW	365.5673	0.30763	2.49865	737.1346	744.7963	0.12329	0.1113
AIGW	367.6547	0.31114	2.64125	739.3093	744.4171	0.14065	0.0466
Weibull	366.8109	0.38106	2.57796	737.6218	742.7296	0.13690	0.0568
KmE	368.6557	0.48072	3.04412	743.3115	750.9731	0.15381	0.0223
EE	372.8252	0.78429	4.35233	749.6504	754.7581	0.19186	0.0018
APE	372.8963	1.13335	7.58829	749.7925	754.9003	0.21080	0.0004
Expl	391.9018	3.25262	20.7704	785.8036	788.3575	0.33620	9.419×10^{-10}

Table 10: Likelihood ratio test statistic for both cases of runoff data.

Model	Data	Hypotheses	LRT	p-Value
AIGW vs EPAIGW	water runoff	$H_0:\delta=1$ vs $H_1:H_0$ is false	4.66274	0.03082
AIGW vs EPAIGW	fictitious water runoff	$H_0:\delta=1$ vs $H_1:H_0$ is false	4.17470	0.04103

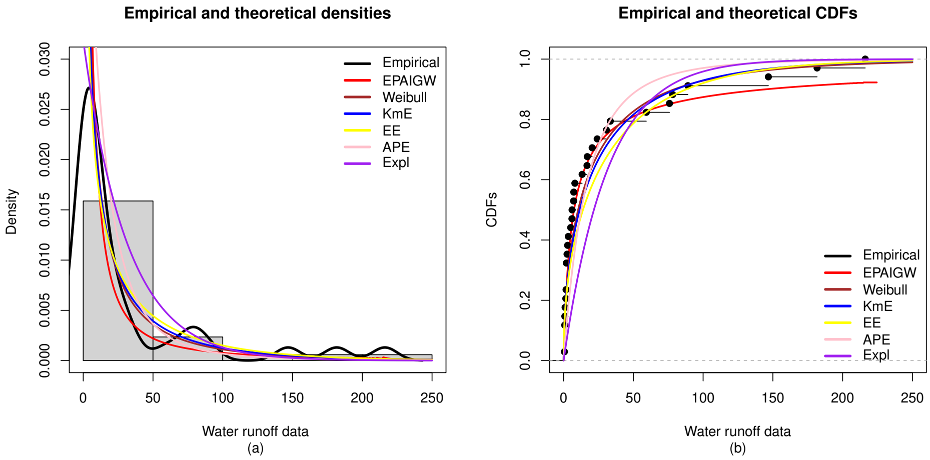

Fig. 15: Histogram, empirical and theoretical densities, and empirical and theoretical CDFs for all CMls for water runoff data.

10 Conclusions

In this manuscript, a new model "exponentiated power alpha index generalized Weibull (EPAIGW)" of three parameters is developed and studied. This model can also be employed to the distribution of extreme events. By fitting EPAIGW model to historical flood data, one can estimate return periods and analyze the potential strength of future floods. This information is useful for designing storm water management system. We derived several statistical properties, and MLMd is used for parameters estimates of EPAIGW model. Simulation study is also performed to assess the behavior of the estimates. GFt statistics have been compared with other CMls to check how well the developed EPAIGW model performed. Our developed model have superior numerical and graphical results. We hoped that our developed model attracts the researchers. Overall, the use of EPAIGW distribution converts uncertainty into measurable risk, empowering scientists, engineers and policy makers to make decisions that are supported by data, sound statistically, and sensitive to the unpredictability of nature.
The efficiency and applicability of the EPAIGW distribution can be improved through future research. First, to develop

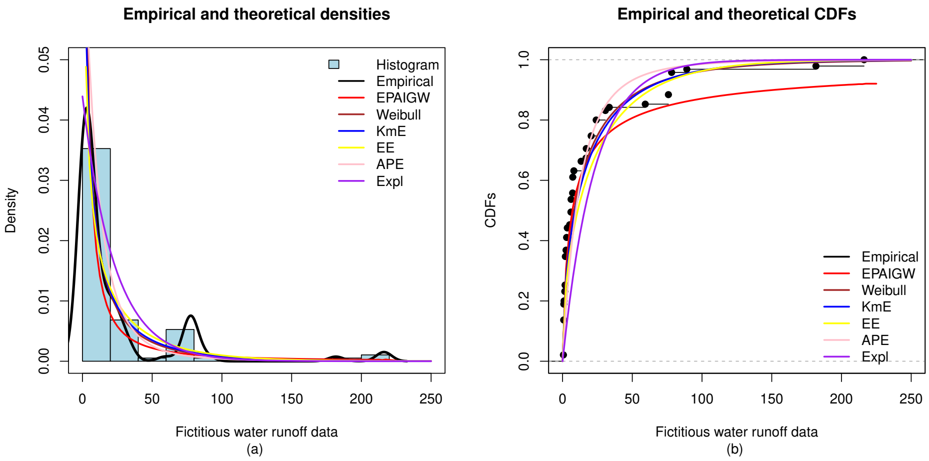

Fig. 16: Histogram, empirical and theoretical densities, and empirical and theoretical CDFs for all CMIs for fictitious water runoff data.

more techniques for parameter estimation. Second, the utility of the model in hydrological or other studies can be improved by it bivariate. Third, one can investigate Bayesian estimating techniques.

Declarations

Competing interests: The authors declare no competing interests.
Authors' contributions: **H. M. Kashif Rasheed**, Conceptualization, Data Curation, Visualization, Investigation, Formal Analysis; **Sajid Hussain**, Supervision, Methodology, Writing Original Draft, Software, Formal Analysis; **Shafqat Ali**, Project Administration, Writing Review & Editing; **Shazia Bibi Nadia**, Literature Review, Formal Analysis; **Muhammad Asghar**, Validation, Visualization; **Abid Khan**, Software, Writing-Review & Editing.
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