Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/19.1.2

On Proximity and Fixed Point Theorem for semi contractions of E-Metric space

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Received: Jan. 1, 2025 Accepted : Jan. 5, 2026

Abstract: In this paper, we investigate some theorems on the existence and convergence of best proximity point for a semi-cyclic contraction pair (S,T) in the setting of non solid cone E-metric space with empty interior containing semi-interior points which poses larger class of metric spaces. further we apply our finding results in the E-cone metric space to prove the existence of the solution of certain integral equation.

Keywords: Positive non solid cone; semi-interior point; fixed point; best proximity point.

2010 Mathematics Subject Classification. 47H10; 46B40; 54H25.

1 Introduction

The importance of the fixed point theorem and the Banach contraction principle due to the fact that it is an interesting tool to prove the existence and uniqueness of the solution of differential equations, integro-differential equations, functional equations which represent a mathematical modeling of real life problems.

The theory of fixed point goes back to the early 1880s, where a French mathematician, Poincare **[19]**, studied nonlinear equations. Banach **[7]** proved that on a complete metric space, every contraction has a unique fixed point which is now known as Banach contraction mapping principle or Banach fixed point theorem. In 1969, Kannan **[15]** defined a new type of contraction mapping and proved a fixed point result for this type.

Kirk and coauthors proposed cyclic contraction in metric space in 2003, and examined the existence of a proximity point and a fixed point for cyclic contraction mapping **[16]**. Since then, plenty of new results have emerged in this subject. Sanker Raj and Veeramani demonstrated the existence of an optimal proximity point for non-expansive maps in 2006 **[22]**, and bring up a problem about the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space. Al-Thagafi and Shahzad proposed a positive answer to this problem.**[1]**. Amount of work have been carried to improve both underlying space and contractive conditions used by Banach 1922 under the effect the metric structure or the contractive conditions considering the cyclic contractive mappings in addition to using the other definition of distances that have discovered a lot of fascinating results in fixed point theory, see [1]-[6], [11]-[15] and references therein.

Huang et al **[13]** established the principle of cone metric space, and they studied the convergent in cone metric space. More over Cauchy sequences in cone metric space with interior points in connection with the principal of cone partial ordering is mentioned in **[20]** . Later, fixed point problems in such spaces have attracted the attention of many mathematicians. They also redefined the principle of K-metric spaces which are initially defined by Kurepa **[17]** and modified the definition of convergence in Banach space E with solid cone S. Razapour and Hamlbarani **[21]** proved the same concepts and results while exclusive of the normality assumption on the cone S.

The concept of the semi-interior point is presented by Basile et al **[8]** in 2017, and they proved that any semi-interior point of positive cone is an interior point in the sense of another norm. New results of fixed point in cone metric spaces with

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semi-interior points were obtained. Recently, in [10, 14, 18] the authors derived several fixed point theorems in E-metric spaces with such cones.

In this paper, we prove some theorems on the convergence and existence of best proximity point for a semi-cyclic contraction pair (S,T) in E-metric space with empty interior that containing semi-interior points. We begin this paper by some preliminaries.

2 Preliminaries

Definition 1.[13]. *Let S be a subset of a real Banach space E, if S satisfies the following conditions, then it is called a positive cone*

(1) *S is closed, nonempty and $S \neq \{0_E\}$.*
(2) *For any c and d in $\mathbb{R}$ with $c,d \geq 0$, and any z,w in S, then $cz+dw \in S$.*
(3) *If $w \in S$ and $-w \in S$, then $w = 0_E$.*

A partial ordering on E is define with respect to a cone S as $z \preceq w$ iff $w-z \in S$, and $z \prec w$ means that $z \preceq w$ but $z \neq w$, while $z \ll w$ will stand for $w - z \in int(S)$. Clearly, $z \in S$ if and only if $0_E \preceq z$.

Definition 2. [13]. *Let E be a Banach space and S be a cone in $\{E, \|.\|\}$. Then S is called*

(*i*) *Normal with normal constant N if a number $N > 0$ exists, and for all $z,w \in S$, with*

$$0 \preceq z \preceq w \quad implies \quad \|z\| \leq N\|w\|.$$

(*ii*) *Solid if $int(S) \neq \phi$*
(*iii*) *Generating if $E = S - S$.*

Example 1.[9]. Let D be the compact interval $[0,1]$ and $C(D)$ be the set of continuous functions on D, and the supremum norm is defined as $\|z\|_\infty = \sup_{t\in D}|z(t)|$. If

$$S = \{z \in C(D) \mid z(t) \geq 0, \forall t \in D\},$$

then S is a solid, normal, and generating cone.

Definition 3. [13]. *Let E be a Banach space and S be a cone in $\{E, \|.\|\}$.if every increasing bounded sequence is convergent, then the cone S is regular in $\{E, \|.\|\}$, that is, suppose that the sequence (z_n) is ordered as*

$$z_1 \preceq z_2 \preceq \ldots \preceq z_n \preceq \ldots \preceq w$$

for fixed $w \in E$, then a point $z_0 \in E$ exists with $lim_{n\to\infty}||z_n - z_0|| = 0$.

Assume that $(\aleph, d)$ is a metric space and G, F are subsets in $\aleph$, then define

$$d(G,\ F) = \inf\{d(g,\ v):\ g \in G,\ v \in F\}.$$

Definition 4. [16]. *Let G and F be nonempty subsets of a metric space $(\aleph, d)$. Then $T : G\cup F \to G\cup F$ is called cyclic mapping if $T(G) \subseteq F$ and $T(F) \subseteq G$.*

Definition 5. [1]. *Let G and F be nonempty subsets of a metric space $(\aleph, d)$, and $T : G\cup F \to G\cup F$ is cyclic mapping. Then*

(*i*) *T is called cyclic contraction if*

$$d(Tg,\ Tv) \leq \delta\ d(g,v) + (1-\delta)d(G,\ F),$$

where δ is a real number in the interval $(0,1)$ and for all $g \in G$, and $v \in F$.
(*ii*) *A point s in $G\cup F$ is best proximity point for T if $d(s, Ts) = d(G, F)$.*

Definition 6. [12]. *Let G and F be nonempty closed subsets of a complete metric space $(\aleph, d)$, and let S,T be two self mappings on $G\cup F$. Then (S,T) is called a semi-cyclic contraction pair if the following are satisfied*

(*i*) *$S(G) \subseteq F$, $T(F) \subseteq G$;*
(*ii*) *A constant δ exists in the interval $(0,1)$ with*

$$d(Tg,\ Tv) \leq \delta\ d(g,v) + (1-\delta)d(G,\ F),$$

for any $g \in G$, and $v \in F$.

Proposition 1. *[11]. Suppose that G and F are nonempty subsets of a metric space $\aleph$, and $T : G \cup F \to G \cup F$ is a cyclic contraction map. Then for any u_0 in $G \cup F$, we have $d(u_n, Tu_n) \to d(G,F)$ where $u_{n+1} = Tu_n$, for $n = 0,1,2,\ldots$.*

Proposition 2.*[12]. Suppose that (S,T) is a semi-cyclic contraction pair. Let $g_0 \in G$ and define*

$$\begin{cases} g_{n+1} = Tv_n \\ v_n = Sg_n \end{cases} n = 0,1,2,\ldots. \tag{1}$$

Then $\{g_n\}$ is a sequences in G, and $\{v_n\}$ is a sequences in F, and

$$d(g_n, Sg_n) \to d(G,F), \text{ and } d(v_n, Tv_n) \to d(G,F).$$

Proposition 3. *[11]. Let G and F be closed nonempty subsets of $\aleph$, and $(\aleph, d)$ is a complete metric space. Suppose that $T : G \cup F \to G \cup F$ is a cyclic contraction map, and $g_0 \in G$. Define $g_{n+1} = Tg_n$. If the sequence $\{g_{2n}\}$ has a convergent subsequence in G, then a point c exists in G with $d(c,Tc) = d(G,F)$.*

Proposition 4. *[12]. Assume that (S,T) is semi-cyclic contraction pair, $\{g_n\}$ and $\{v_n\}$ are the sequences defined by (1). If $\{g_n\}$ and $\{v_n\}$ have convergent subsequences in G and F, respectively, then a point c in G and a point v in F exists with*

$$d(c,Sc) = d(G,F) = d(v,Tv).$$

Proposition 5.*[12]. If (S,T) is semi-cyclic contraction pair, then the sequences $\{g_n\}$ and $\{v_n\}$ defined in (1) are bounded.*

Theorem 1. *[12]. Suppose that (S,T) is a semi-cyclic contraction pair on $G \cup F$ in a complete metric space $(\aleph, d)$. If at least one of the sets G and F is boundedly compact, then $w \in G \cup F$ exists with either $d(w,Sw) = d(G,F)$ or $d(w,Tw) = d(G,F)$.*

3 Non-solid Cone With Semi-Interior Points

Basile and coworkers [8] established the concept of semi-interior point. They showed that the class of cones with semi-interior point and empty interior is varied than the class with nonempty interior by providing some examples, and proved several fixed-point theorems in non-solid cones with semi-interior points in E-metric spaces. At the same time, fixed-point theorems for ordered normed spaces are valid for non-soild cones with semi-interior points.
Consider E an ordered normed space ordered by the positive cone E^+, *and* let the zero of E denoted by 0_E. Furthermore, let

$$U = \{z \in E : \|z\| \leq 1\}$$

denote the closed unit ball of E, and the positive portion of the unit ball (U_+) is defined as

$$U_+ = U \cap E^+.$$

Definition 7. ***[8].*** *Let z_0 be a point in E, z_0 is said to be a semi-interior point of E^+ if there exists $\delta \gneqq 0$ with $z_0 - \delta U_+ \subseteq E^+$. And $(E^+)^\theta$ denote the set of all semi-interior points of E^+. Moreover for any $z,w \in E^+$,we will say $z \lll w$ if and only if $z - w \in (E^+)^\theta$.*

Recall the definition of normed ordered space $(E, \preceq)$, and recall an E-cone metric space.

Definition 8. ***[18].*** *Let E be an ordered normed space with $(E^+)^\theta$ is non-empty set, and assume that $\aleph$ is non-empty. An E-cone metric on $\aleph$ is a map $d^E : \aleph \times \aleph \to \mathbb{R}^+$ that meet following conditions*;

1. *For all ν, τ, $d^E(\nu,\tau) \succeq 0_E$, and $d^E(\nu,\tau) = 0_E$ iff $\nu = \tau$.Γ*
2. *For all ν, τ, $d^E(\nu,\tau) = d^E(\tau,\nu)$.*
3. *For all ν, τ, ρ, $d^E(\nu,\tau) \preceq d^E(\nu,\rho) + d^E(\rho,\tau)$.*

An E-cone metric space is a pair $(\aleph, d^E)$ with non-empty set $\aleph$ and d^E is an E-cone metric on the set $\aleph$.

Example 2. **[13].** Let $S = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha,\beta \succcurlyeq 0_E\} \subset \mathbb{R}^2 = E$, and let $\aleph = \mathbb{R}$. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined as

$$d(\alpha,\beta) = (|\alpha - \beta|, \mu\, |\alpha - \beta|),$$

where μ is a nonnegative number. Therefore $(\aleph, d)$ is an ordered cone metric space.

The following provides the definition of e-convergence in ordered normed space E , and the definition of e-Cauchy convergence.

Definition 9. *[18]. Let $(E^+)^\theta \neq \phi$, $(\aleph, d^E)$ is an E-metric space. Suppose that $\{z_n\}$ is a sequence in $\aleph$ and $z \in \aleph$. Then:*
(i) $\{z_n\}$ is e-converges to z if for any $e \ggg 0_E$, a number N_0 exists with $d^E(z_n, z) \lll e$ for $n > N_0$.
(ii) $\{z_n\}$ is e-Cauchy converges to z if for any $e \ggg 0_E$, a number N_0 exists with $d^E(z_n, z_m) \lll e$ for $n, m > N_0$.
(iii) The E-metric space is named e-complete if any e-Cauchy sequence in $\aleph$ is an e-convergent sequence.
(iv) The sequence $\{z_n\}$ is said to be bounded if there exists $z_0 \in \aleph$ and $\sigma \in E$ with $d^E(z_n, z_0) \lll \sigma$ for all n.

Fact: An interior point of E^+ is a semi-interior point of E^+, while not all semi-interior points are interior points.

Proposition 6. ***[8].*** *Let E be a complete ordered normed space with generating closed cone E^+. If z_0 is a semi-interior point of E^+ then it is an interior point of E^+.*

Definition 10.*An ordered vector space Y is called a vector lattice if $y_1 \vee y_2 = \sup\{y_1, y_2\}$ and $y_1 \wedge y_2 = \inf\{y_1, y_2\}$ are both exist for any $y_1, y_2 \in Y$.*

Vector lattices are called linear lattices or Riesz spaces, [23]. An example of a vector lattice is $\mathbb{R}$, with usual operations.

Assume that $(\aleph, d^E)$ is an $E-$cone metric space with closed cone E^+ also assume $(E^+)^\theta \neq \emptyset$. If E is a vector lattice and G, F are two sets in $\aleph$, we define

$$d(G,F) = \inf\left\{d^E(g,\ f) :\ g \in G,\ f \in F\right\}.$$

4 Main Results

We will prove several results in this section about best proximity points of cyclic maps in a non solid E-metric space having semi-interior points. Now we begin by the following theorem

Theorem 2.*Consider* E *to be an ordered normed space, $(E^+)^\theta$is nonempty set and $(\aleph, d^E)$ is E-complete metric space. Let Φ and Ω be closed non-empty subsets of $(\aleph, d^E)$. If $T : \Phi \cup \Omega \to \Phi \cup \Omega$ is a cyclic mapping such that*

$$d^E(T\phi, Tz) \preceq \delta d^E(\phi, z)$$

for all $\phi \in \Phi$, $z \in \Omega$ and $0 < \delta < 1$, then T has a unique fixed point in $\Phi \cap \Omega$.

*Proof.*Consider $T : \Phi \cup \Omega \to \Phi \cup \Omega$ be a cyclic map, then

$$\begin{aligned} d^E(T^2(\phi), T(\phi)) &= d^E(T(T(\phi)), T(\phi)) \\ &\preceq \delta d^E(T(\phi), \phi). \end{aligned} \tag{2}$$

So,

$$\begin{aligned} d^E(T^3(\phi), T^2(\phi)) &= d^E(T(T^2(\phi)), T(T(\phi)) \\ &\preceq \delta\left[d^E(T^2(\phi), T(\phi))\right] \\ &\preceq \delta\left[\delta d^E(T(\phi), \phi)\right] \\ &\preceq \delta^2 d^E(T(\phi), \phi). \end{aligned} \tag{3}$$

Now if and $\lambda = d^E(T\phi, \phi)$, then for all $n \in \mathbb{N}$, we get $d^E(T^{n+1}(\phi), T^n(\phi)) \preceq \delta^n \lambda$. Let $n, m \in \mathbb{N}$ with $m > n$. Consider

$$\begin{aligned} d^E(T^m\phi, T^n\phi) &\preceq \left[d^E(T^m\phi, T^{m-1}\phi) + d^E(T^{m-1}\phi, T^n\phi)\right] \\ &\preceq \left[d^E(T^m\phi, T^{m-1}\phi) + d^E(T^{m-1}\phi, T^{m-2}\phi) + d^E(T^{m-2}\phi, T^n\phi)\right] \\ &\preceq d^E(T^m\phi, T^{m-1}\phi) + d^E(T^{m-1}\phi, T^{m-2}\phi) + d^E(T^{m-2}\phi, T^{m-3}\phi) \\ &\qquad + d^E(T^{m-3}\phi, T^n\phi) + \ldots + d^E(T^{m-n}\phi, T^n\phi) \\ &\preceq \delta^{m-1}\lambda + \delta^{m-2}\lambda + \delta^{m-3}\lambda + \ldots + \delta^{m-1}\lambda + \delta^{n+1}\lambda + \delta^n\lambda \\ &= \delta^n\lambda\left[1 + \delta + \delta^2 + .. + \delta^{m-n-1}\right] \\ &= \delta^n\lambda\left[\frac{1-\delta^{m-n}}{1-\delta}\right] \end{aligned}$$

Taking the limits as $m \to \infty$, we get $d^E(T^m\phi, T^n\phi) \to 0$ as $\delta < 1$. Let $e \ggg 0_E$ be given, and $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and choose a number h_1 with $\delta^n \lambda \frac{1-\delta^{m-n}}{1-\delta} \in \frac{\rho}{2} U_+$ for all $m, n \geq h_1$. Therefore $e - \delta^n \delta \frac{1}{1-\delta} - \frac{\rho}{2} U_+ \subseteq e - \rho U_+ \subseteq E^+$

$$d^E(T^m\phi, T^n\phi) \preceq \delta^n \lambda \frac{1}{1-\delta} \lll e \text{ for all } n, m \geq h_1,$$

therefore $\phi_n = T^n(\phi)$ is a Cauchy sequence because $\aleph$ is an $e-$complete. Which implies that a point $\phi \in \Phi \cup \Omega$ exists with $\phi_n \xrightarrow{e} \phi$. For fixed $e \ggg 0_E$, take $h_2 \in \mathbb{N}$, with $d^E(\phi, \phi_n) \lll \frac{e}{2}$ for $n \geq h_2$, then

$$\begin{aligned} d^E(\phi, T\phi) &\preceq d^E(\phi, \phi_n) + d^E(\phi_n, T\phi) \\ &\preceq d^E(\phi, \phi_n) + \delta d^E(\phi_n, \phi_{n-1}) \\ &\lll e. \end{aligned}$$

Now if $n \to \infty$, then $d^E(\phi, T\phi) = 0$. Which implies $\phi = T\phi$.

Note that the sequence $\{T^{2n}\phi\}$ in Φ and $\{T^{2n-1}\phi\}$ in Ω and both sequences approaches to the limit ϕ. Since both Φ and Ω are closed, then $\phi \in \Phi \cap \Omega$.

To show that ϕ is a unique fixed point of T. Assume that w is additional fixed point of T, then $Tw = w$. The contractive condition implies

$$d^E(\phi, w) = d^E(T\phi, Tw) \preceq \delta d^E(\phi, w),$$

therefore $(1-\delta) d^E(\phi, w) \preceq 0$. Which implies that $d^E(\phi, w) = 0$, therefore ϕ is a unique fixed point of T.

Now we generalize Propositions 3.1 and 3.3 of [11].

Theorem 3.*Assume that E is a vector lattice and $(\aleph, d^E)$ is an $E-$complete cone metric space with closed positive cone E^+ with semi-interior points. If Φ and Ω are two closed non-empty subsets of $(\aleph, d^E)$ and $T : \Phi \cup \Omega \to \Phi \cup \Omega$ is a cyclic contraction map. If $u_{n+1} = Tu_n$, $n = 0, 1, 2,..$ starting with any u_0 in $\Phi \cup \Omega$ we will have $\lim_{n \to \infty} d^E(u_n, Tu_n) = d(\Phi, \Omega)$.*

*Proof.*Let u_0 be any point in $\Phi \cup \Omega$, then for k in $(0, 1)$

$$\begin{aligned} d^E(u_n, u_{n+1}) &= d^E(Tu_{n-1}, Tu_n) \preceq k\, d^E(u_{n-1}, u_n) + (1-k) d(\Phi, \Omega) \\ &= k\, d^E(Tu_{n-2}, Tu_{n-1}) + (1-k) d(\Phi, \Omega) \\ &\preceq k\, \left[k\, d^E(u_{n-2}, u_{n-1}) + (1-k) d(\Phi, \Omega)\right] + (1-k) d(\Phi, \Omega) \\ &= k^2\, d^E(u_{n-2}, u_{n-1}) + (k - k^2) d(\Phi, \Omega) + (1-k) d(\Phi, \Omega) \\ &= k^2\, d^E(u_{n-2}, u_{n-1}) + (1-k^2) d(\Phi, \Omega) \\ &\preceq \ldots \preceq k^n\, d^E(u_0, u_1) + (1-k^n) d(\Phi, \Omega). \end{aligned} \tag{3.1.1}$$

Now assume that $e \ggg 0_E$ be given, $\rho > 0$ and $e - \rho U_+ \subseteq E^+$, choose a natural number h_1 with $k^n\, d^E(u_0, u_1) \in \frac{\rho}{2} U_+$ for any $n \geq h_1$. Which implies

$$e - k^n d^E(u_0, u_1) - \frac{\rho}{2} U_+ \subseteq e - \rho U_+ \subseteq E^+.$$

Therefore, $d^E(u_n, Tu_n) \to d(\Phi, \Omega)$ as $n \to \infty$.

Proposition 7.*Suppose that the sequence $\{u_{2n}\}$ has a convergent subsequence in Φ. Then a point u in Φ exists with $d^E(u, Tu) = d(\Phi, \Omega)$.*

*Proof.*Assume that $\{u_{2n_k}\}$ is a subsequence of $\{u_{2n}\}$ that converges to $u \in \Phi$. Now

$$d(\Phi, \Omega) \preceq d^E(u, u_{2n_{k-1}}) \preceq d^E(u, u_{2n_k}) + d^E(u_{2n_k}, u_{2n_{k-1}}).$$

Therefore $d^E(u, u_{2n_{k-1}})$ converges to $d(\Phi, \Omega)$, since

$$d(\Phi, \Omega) \preceq d^E(u_{2n_k}, Tu) = d^E(Tu_{2n_{k-1}}, Tu) \preceq k\, d^E(u_{2n_{k-1}}, u) + (1-k) d(\Phi, \Omega),$$

and as $n_k \to \infty$

$$d^E(u, Tu) = d(\Phi, \Omega).$$

Finally we give a generalization of Propositions 3.1 and 3.2 of [12] and Theorem 3.5 of [12] by assuming the space is E-metric space with empty interior.

Definition 11. *Assume that $(\aleph, d)$ is a metric space. A subset G is called boundedly compact if each bounded sequence in G has a convergent subsequence.*

Theorem 4. *Assume that E is a vector lattice and $\left(\aleph, d^E\right)$ is an $E-$complete cone metric space with nonempty closed positive cone E^+ with semi-interior points. G and F are closed non-empty subsets of $\aleph$, and $T,S : G\cup F \to G\cup F$ are semi-cyclic contraction pair of mappings in $\aleph$. If one of the subsets G and F is boundedly compact, then there exists w in $G\cup F$ with either $d^E(w,Sw) = d(G,F)$ or $d^E(w,Tw) = d(G,F)$.*

Proof. Assume S and T are semi-cyclic contraction mappings, choose $v_0 \in G$, and define

$$\begin{cases} u_{n+1} = Tv_n \\ v_n = Su_n \end{cases} n = 0,1,2,...$$

Then $\{u_n\}$ and $\{v_n\}$ are sequences in G and F, respectively. Moreover

$$d^E(u_n,Su_n) \to d(G,F),\ d^E(v_n,Tv_n) \to d(G,F).$$

We first show that $d^E(v_n,Tv_n)$ converges to $d(G,F)$;

$$\begin{aligned} d^E(v_n,Tv_n) &= d^E(Su_n,Tv_n) \preceq k\, d(u_n,v_n) + (1-k)d(G,F) \\ &= k\, d^E(Tv_{n-1},Su_n) + (1-k)d(G,F) \\ &\preceq k\left[k\, d^E(v_{n-1},u_n) + (1-k)d(G,F)\right] + (1-k)d(G,F) \\ &= k^2\, d^E(u_{n-1},v_{n-1}) + (k-k^2)d(G,F) + (1-k)d(G,F) \\ &= k^2\, d^E(Tv_{n-2},v_{n-1}) + (1-k^2)d(G,F) \\ &\preceq ... \preceq k^{2n}\, d^E(v_0,v_1) + (1-k^{2n})d(G,F). \end{aligned} \tag{4}$$

As $n\to\infty$, then $d^E(v_n,Tv_n) \to d(G,F)$ for $0 < k < 1$. Let $e \ggg 0_E$ be given, $\rho > 0$ with $e-\rho U_+ \subseteq E^+$. Let $h_1 \in \mathbb{N}$ such that $k^{2n}d^E(v_0,v_1) \in \frac{\rho}{2}U_+$ for any $n \geq h_1$. Therefore

$$e - k^{2n}d^E(v_0,v_1) - \frac{\rho}{2}U_+ \subseteq e-\rho U_+ \subseteq E^+.$$

Therefore, $d^E(v_n,Tv_n) \to d(G,F)$ for $n\to\infty$. Similarly $d^E(u_n,Su_n) \to d(G,F)$.

Then we show that $\{v_n = Su_n\}$ is bounded in F. Suppose that $\{Su_n\}$ is not bounded in F, then there exists $N_0 \in \mathbb{N}$ and $M \ggg 0_E$ such that

$$d^E(u_1,Su_{N_0}) \ggg M \text{ and } d^E(u_1,Su_{N_0-1}) \lll M,$$

where

$$M \ggg \max(\frac{2d^E(u_0,u_1)}{1/k^2-1} + \frac{1}{1-k^2}d(G,F), d^E(u_1,Sv_0)).$$

As S and T are semi-cyclic contraction mappings, we get

$$\begin{aligned} M &\lll d^E(u_1,Su_{N_0}) = d^E(Tv_0,Su_{N_0}) \preceq kd^E(v_0,u_{N_0}) + (1-k)d(G,F) \\ &= kd^E(Su_0,Tv_{N_0-1}) + (1-k)d(G,F) \\ &\preceq k\left[k\, d^E(u_0,v_{N_0-1}) + (1-k)d(G,F)\right] + (1-k)d(G,F) \\ &= k^2d^E(u_0,v_{N_0-1}) + (1-k^2)d(G,F). \end{aligned}$$

Therefore,

$$\begin{aligned} (1/k^2)(M-(1-k^2)d(G,F) &\prec d^E(u_0,Su_{N_0-1}) \\ &\preceq d^E(u_0,Tv_0) + d^E(Tv_0,Su_{N_0-1}) \preceq d^E(u_0,Tv_0) + M \\ &\preceq d^E(u_0,Su_0) + d^E(Su_0,Tv_0) + M \\ &\preceq d^E(u_0,Su_0) + kd^E(u_0,v_0) + (1-k)d(G,F) + M \\ &\preceq 2d^E(u_0,v_0) + d(G,F) + M. \end{aligned}$$

Which implies that
$$M(1/k^2-1) \lll 2d^E(u_0,v_0)+1/k^2 d(G,F),$$
or equivalently
$$M \lll \frac{2d^E(u_0,v_0)}{1/k^2-1}+\frac{1}{1-k^2}d(G,F),$$
which is a contradiction.

Now we will assume that F is boundedly compact, then $\{v_n\}$ has a convergent subsequence say $v_{n_m} \to v$.
$$d(G,F) \preceq d^E(Tv_{n_m},v) \preceq d^E(v,v_{n_m})+d^E(v_{n_m},Tv_{n_m})$$
Thus $d^E(v,v_{n_m})$ converges to $d(G,F)$. Since
$$\begin{aligned} d(G,F) &\preceq d^E(v_{n_m},Tv)=d^E(Su_{n_m},Tv) \preceq kd^E(v,u_{n_m})+(1-k)d(G,F) \\ &= kd^E(v,Tv_{n_{m-1}})+(1-k)d(G,F), \end{aligned}$$
as $m \to \infty$ we have $d^E(Tv,v)=d(G,F)$.

Similarly if G is boundedly compact, then u in G exists with $d^E(u,Su)=d(G,F)$.

5 Application

Let $X=L^1[0,1]$, the space of all real valued Lebesque integrable functions on $[0,1]$ endowed with the metric, $d(u,v)=\int_0^1|u(t)-v(t)|\,dt$. Consider the integral equation
$$v(s)=\int_0^1 K(s,t)h(t,v(t))\,dt,$$
for all $s\in[0,1]$, where $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $K:[0,1]\times[0,1]\to[0,\infty)$ are continuous functions with $\sup_{s,t\in[0,1]} K(s,t)\le 1$.

Now, for $f,g\in X$, let $a,b\in\mathbb{R}$ be such that
$$a\le f(s)\le g(s)\le b,$$
for all $s\in[0,1]$. Assume also that for all $s\in[0,1]$, we have
$$f(s)\le\int_0^1 K(s,t)h(t,g(t))\;dt \quad\text{and}\quad g(s)\ge\int_0^1 K(s,t)h(t,f(t))\;dt.$$
Further, assume that for all $t\in[0,1]$, $h(t,.)$ be a non increasing function on $\mathbb{R}$, that is,
$$for\, x,y\in\mathbb{R},\; x\ge y\Rightarrow h(t,x)\le h(t,y),$$
and for all $s\in[0,1]$, for all $x,y\in\mathbb{R}$ with ($x\le b$ and $y\ge a$) or ($x\ge a$ and $y\le b$), we have
$$|h(s,x)-h(s,y)|\le\rho\left(|x-y|\right),$$
where ρ is a real valued continuous function satisfying $\rho(t)\le\delta, 0<\delta\le 1$.

Now consider the map $T:L^1[0,1]\to L^1[0,1]$ with
$$T(u(s)=\int_0^1 K(s,t)h(t,u(t))\,dt.$$
Further choose $d^E:L^1[0,1]\times L^1[0,1]\to P$, the cone E-metric, $d^E(u,v)=z\int_0^1|u(s)-v(s)|\,ds$ as in Example 2.9 in [10].

Let A_1,A_2 be two closed subsets of $L^1[0,1]$ as
$$A_1=\{u\in L^1[0,1]:u\le g\}\text{ and }A_2=\{u\in L^1[0,1]:u\ge f\}.$$
First, we will show that $T:A_1\cup A_2\to A_1\cup A_2$ is a cyclic map.

Let $u \in A_1$. Then for all $t \in [0,1]$, we have $u(t) \leq g(t)$. Now, since $h(s,.)$ is a non increasing function on $\mathbb{R}$ and $K(s,t) \geq 0$ for all $t,s \in [0,1]$, we get

$$K(s,t)h(t,u(t)) \geq K(s,t)h(t,g(t)),$$

for all $t,s \in [0,1]$. Consequently, we have

$$\int_0^1 K(s,t)h(t,u(t))\ dt \geq \int_0^1 K(s,t)h(t,g(t))\ dt \geq f(s),$$

for all $s \in [0,1]$. Hence, $Tu \in A_2$. Similarly, if $u \in A_2$,, then

$$\int_0^1 K(s,t)h(t,u(t))\,dt \leq \int_0^1 K(s,t)h(t,f(t))\,dt \leq g(s)$$

for all $s \in [0,1]$ and hence, $Tu \in A_1$. Thus, T is a cyclic map from $A_1 \cup A_2$ into $A_1 \cup A_2$.

Now for $u,v \in [0,1]$, we have

$$\begin{aligned}
d^E(Tu,Tv) &= z\int_0^1 |Tu(s)-Tv(s)|\ dt \\
&= z\int_0^1 \left|\int_0^1 K(s,r)h(r,u(r))\ dr - \int_0^1 K(s,r)h(r,v(r))\,dr\right| ds \\
&= z\int_0^1 \left|\int_0^1 (K(s,r)h(r,u(r)) - K(s,r)h(r,v(r)))\,dr\right| ds \\
&\leq z\int_0^1 \int_0^1 |K(s,r)(h(r,u(r)) - h(r,v(r)))|\,drds \\
&\leq z\int_0^1 \rho(|u-v|)\,dr \\
&= \rho d^E(u,v).
\end{aligned}$$

Applying theorem we conclude that the integral equation

$$v(s) = \int_0^1 K(s,t)h(t,v(t))\,dt,$$

with the above conditions on K and h has a unique solution $w \in \left\{v \in L^1[0,1] : f(t) \leq v(t) \leq g(t), t \in [0,1]\right\}$.

Declarations

Conflicts of Interest:The author declares no conflict of interest.
Funding: Not applicable.
Availability of data and materials: Not Applicable.
Ethical approval Statement Not applicable.

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