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A Boolean Lattice of Isotone Derivations

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Abstract: We study the set of derivations that are isotone on a Boolean lattice. To that end, new properties of derivations on an arbitrary Boolean lattice are established. The complement of an isotone derivation is detected. In addition, a Boolean structure for the lattice of isotone derivations is introduced and characterized. As well as, a strong negation for this Boolean lattice is provided. Finally, we prove that their fixed sets form also a Boolean lattice.

Keywords: Boolean lattice, derivation, isotone derivation, fixed point, fixed set.

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1 Introduction

Derivations are recently investigated on ring structures (see e.g. [4] and [6,7]). In [15], derivations are defined on lattices L to be functions d satisfying:

1. $d(\alpha \wedge \beta) = (d(\alpha) \wedge \beta) \vee (\alpha \wedge d(\beta))$, and
2. $d(\alpha \vee \beta) = d(\alpha) \vee d(\beta)$, for every $\alpha, \beta \in L$.

Derivations have been further investigated in [9] for several classes of lattices. Then only the first condition (i) has been considered for the definition of a derivation since the second condition (ii) is always satisfied for isotone derivations on distributive lattices, see [18]. Furthermore, isotone derivations have been used to characterize distributive and modular lattices. Then the fixed set of a derivation on a lattice and its relation to ideals of a lattice has been studied in [17]. The concept of derivations is recently applied in several fields such as partially ordered sets in [1], [2], [3] and [23]; distributive lattices in [5,19,22]; bounded hyperlattices in [16]; residuated lattices in [11] and [21]; integration on lattices in [20]; the subgroup lattice of finite groups in [12]; and pseudo L-algebras in [10]. For general terminology about ordered sets and related concepts we refer to [8], [13] and [14].

Motivated by the above mentioned papers, we investigate isotone derivations in this work. After this introduction, we recall in Section 2 some basic concepts of lattices and their derivations. Then, in Section 3, we establish new properties of derivations on an arbitrary Boolean lattice. Section 4 is devoted to build a Boolean structure for the lattice of isotone derivations and to determine its strong negation. Finally in Section 5, we investigate the fixed sets of isotone derivations and show that they form a Boolean lattice.

2 Boolean lattices

Several concepts and properties of Boolean lattices that are going to be used in this paper, are revisit in this section. For more details, we refer to [8, 14].

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Definition 1(Lattice). *[8] An ordered set $(L,\leqslant)$ is a lattice if for each $\alpha,\beta \in L$, the infimum (greatest lower bound $\alpha\wedge\beta$) and the supremum (least upper bound $\alpha\vee\beta$) of α and β exist in L.*

Definition 2.*[14] A lattice $(L,\leqslant,\wedge,\vee)$ is distributive if it verifies one of the two equivalent conditions, for every $\alpha,\beta,\gamma\in L$:*
$(D1):\alpha\wedge(\beta\vee\gamma)=(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$.
$(D2):\alpha\vee(\beta\wedge\gamma)=(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$.

Definition 3.*[8] An lattice $(L,\leqslant,\wedge,\vee)$ is bounded if it possesses a smallest element usually denoted by 0, and a greatest element denoted by 1.*

Definition 4.*[14] A bounded lattice $(L,\leqslant,\wedge,\vee,0,1)$ is complemented if for each $\alpha\in L$, there is $\beta\in L$, where $\alpha\wedge\beta=0$ and $\alpha\vee\beta=1$. The element β is a complement of α in L.*

Definition 5.*[8] A bounded lattice $(B,\leqslant,\wedge,\vee,1,0)$ is a Boolean lattice if it is complemented and distributive.*

In a Boolean lattice every element α has a unique corresponding complement which is usually denoted by α'.

Proposition 1.*[14] Let $(B,\leqslant,\wedge,\vee,0,1)$ be a Boolean lattice. Then the following properties are satisfied.*

(i) $1^{'}=0$ *and* $0^{'}=1$;
(ii) $(\alpha^{'})^{'}=\alpha$, *for each* $\alpha\in B$;
(iii) $(\alpha\wedge\beta)^{'}=\alpha^{'}\vee\beta^{'}$ *and* $(\alpha\vee\beta)^{'}=\alpha^{'}\wedge\beta^{'}$, *for every* $\alpha,\beta\in B$ *(De Morgan's laws);*
(iv) *If* $\alpha\leqslant\beta$, *then* $\beta'\leqslant\alpha'$, *for each* $\alpha,\beta\in B$;
(v) $\alpha\leqslant\beta$, *if and only* $\alpha\wedge\beta^{'}=0$, *if and only if* $\alpha^{'}\vee\beta=1$, *for every* $\alpha,\beta\in B$.

Let $(B,\leqslant,\wedge,\vee,0,1)$ be a given bounded lattice and $\mathcal{N}:B\to B$ be a decreasing function. $\mathcal{N}$ is a negation if $\mathcal{N}(0)=1$ and $\mathcal{N}(1)=0$, and a strong negation if additionally $\mathcal{N}(\mathcal{N}(\alpha))=\alpha$, for each $\alpha\in B$.

Let $(L,\leqslant,\wedge,\vee)$ be an arbitrary lattice. We begin by reviewing the definition of a derivation on a lattice and the relevant properties required for this work. For further details, the reader may refer to [17,18].

Definition 6.*[18] A derivation on L is any function $d:L\longrightarrow L$ that satisfies the following condition:*

$$d(\alpha\wedge b)=(d(\alpha)\wedge\beta)\vee(\alpha\wedge d(\beta)),\ \textit{for every}\ \alpha,\beta\in L.$$

We often write $d\alpha$ instead of $d(\alpha)$.

Definition 7.*[18] A derivation d is isotone if*

$$d\alpha\leqslant d\beta\ \textit{whenever}\ \alpha\leqslant\beta\ \textit{and}\ \alpha,\beta\in L.$$

Proposition 2.*[18] The following statements hold true for every derivation d on L.*

(i) $d\alpha\leqslant\alpha$, *for every* $\alpha\in L$;
(ii) $d(d\alpha)=d\alpha$, *for every* $\alpha\in L$;
(iii) *if there is a least element* 0 *in L, then* $d0=0$;
(iv) *d becomes isotone if and only if* $d(\alpha\wedge\beta)=d\alpha\wedge d\beta$, *for every* $\alpha,\beta\in L$;
(v) *if d is isotone and L is distributive, then* $d(\alpha\vee\beta)=d\alpha\vee d\beta$, *for every* $\alpha,\beta\in L$.

Remark.[18] Let $Fix_d(L)=\{\alpha\in L\mid d\alpha=\alpha\}$ be the set of fixed points of a derivation d. Then $Fix_d(L)$ is a *down set*, i.e., if $\alpha\in Fix_d(L)$ and $\beta\leqslant\alpha$, then $\beta\in Fix_d(L)$. Further, if d is isotone, then $Fix_d(L)$ is an *ideal*.

3 Properties of Boolean derivations

In this section, we give new properties of derivations on an arbitrary Boolean lattice $(B,\leqslant,\wedge,\vee,0,1,')$.

Lemma 1. *If d is a derivation on B, then*

$$d(\alpha')\leqslant(d\alpha)',\ \textit{for every}\ \alpha\in B.$$

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Proof. Since d is a derivation on the Boolean lattice B, it follows from Proposition 2 (i) that $d(\alpha') \leqslant \alpha'$ and $d\alpha \leqslant \alpha$, for any $\alpha \in B$. Then $d(\alpha') \leqslant \alpha'$ and $\alpha' \leqslant (d\alpha)'$ by Proposition 1 (iv). Hence, $d(\alpha') \leqslant (d\alpha)'$.

Lemma 2. *Let d be an isotone derivation on B. Then d is the identity derivation of B if and only if there is an $\alpha \in B$ such that $d(\alpha') = (d\alpha)'$.*

Proof. We suppose that there is an $\alpha \in B$ such that $d(\alpha') = (d\alpha)'$. The fact that d is an isotone derivation and B is distributive implies that

$$\begin{aligned} d(1) &= d(\alpha \vee \alpha') \\ &= d(\alpha) \vee d(\alpha') \quad \text{(by Proposition 2 } (v)) \\ &= d(\alpha) \vee (d\alpha)' \\ &= 1 \end{aligned}$$

So $d(1) = 1$, then $1 \in Fix_d(B)$. Since $Fix_d(B)$ is a down set (see Remark 2) and $\beta \leqslant 1$ for every $\beta \in B$, it holds that $d(\beta) = \beta$. Consequently, d is the identity derivation of B.

From Lemmas 1 and 2, we establish the following theorem.

Theorem 1. *If d is an isotone derivation different from the identity derivation of B then*

$$d(\alpha') < (d\alpha)', \textit{ for each } \alpha \in L.$$

The following example illustrates the result of the above Theorem 1.

Example 1. Let the Boolean lattice $(D(30), |, gcd, lcm, 1, 30)$ and d be a principal derivation on $D(30)$ defined by $d\alpha = gcd(10, \alpha)$, for each $\alpha \in D(30)$. The complement operation $\alpha' = \dfrac{30}{\alpha}$, for every $\alpha \in D(30)$. As it is shown in the following table, we always have $d(\alpha') \leq (d\alpha)'$.

α	1	2	3	5	6	10	15	30
α'	30	15	10	6	5	3	2	1
$d\alpha$	1	2	1	5	2	10	5	10
$d(\alpha')$	10	5	10	2	5	1	2	1
$(d\alpha)'$	30	15	30	6	15	3	6	3

Theorem 2. *If d is an isotone derivation on B, then for any $\alpha, \beta \in B$ we have*

$$d(\alpha+\beta) \leqslant d\alpha + d\beta, \textit{ for each } \alpha, \beta \in B, \\ \textit{where } \alpha+\beta = (\alpha \vee \beta) \wedge (\alpha' \vee \beta').$$

Proof. Proposition 2 (iv) and Lemma 1 guarantee that

$$\begin{aligned} d(\alpha+\beta) &= d((\alpha \vee \beta) \wedge (\alpha' \vee \beta')) \\ &= d(\alpha \vee \beta) \wedge d(\alpha' \vee \beta') \\ &= (d\alpha \vee d\beta) \wedge (d(\alpha') \vee d(\beta')) \\ &\leqslant (d\alpha \vee d\beta) \wedge ((d\alpha)' \vee (dy)') \\ &= d\alpha + d\beta. \end{aligned}$$

4 Boolean lattice of isotone derivations

In this section, we build a Boolean structure to the lattice of isotone derivations. To that end, we need first this theorem.

Theorem 3.[17] *Let $(L, \leqslant, \wedge, \vee)$ be a given distributive lattice and d_1, d_2 be two isotone derivations on L. Define $(d_1 \sqcap d_2)(\alpha) = d_1\alpha \wedge d_2\alpha$ and $(d_1 \sqcup d_2)(\alpha) = d_1\alpha \vee d_2\alpha$, for each $\alpha \in L$. Then $(\mathfrak{D}(L), \preceq, \sqcap, \sqcup)$ is a distributive lattice also, where $\mathfrak{D}(L)$ denotes the set of all isotone derivations on L and $\preceq$ is the usual order of derivations.*

Inspired from that theorem, we can provide this corollary.

Corollary 1. *In a bounded lattice* $(L,\leqslant,\wedge,\vee,0,1)$*, the null derivation (denoted* $0_{\mathfrak{D}(L)}$*) and the identity derivation (denoted* $1_{\mathfrak{D}(L)}$*) are isotone. Moreover, they are the least and the greatest derivations of* $\mathfrak{D}(L)$ *respectively. Further, if* L *is distributive, then the structure* $(\mathfrak{D}(L),\preceq,\sqcap,\sqcup,0_{\mathfrak{D}(L)},1_{\mathfrak{D}(L)})$ *is a bounded distributive lattice.*

In the rest of the paper, we suppose that $(B,\leqslant,\wedge,\vee,',0,1)$ is a Boolean lattice and d is an arbitrary derivation on B.

Definition 8. *Let* B *be a Boolean lattice and* d *a derivation on* B*. We define an associated function* $d^*:B\to B$ *to* d *as follows*

$$d^*(\alpha)=\alpha\wedge(d\alpha)',\ \textit{for every}\ \alpha\in B.$$

Theorem 4. *Let* B *be a Boolean lattice,* d *a derivation on* B *and* d^* *the associate function. Then the following hold true:*

1. d^* *is isotone, and*
2. *if* d *is isotone, then* d^* *is also an isotone derivation on* B*.*

Proof. Let $\alpha,\beta\in B$.

1. Using the definition of d^* we obtain:

$$\begin{aligned}
d^*(\alpha\wedge\beta)&=(\alpha\wedge\beta)\wedge(d(\alpha\wedge\beta))'\\
&=(\alpha\wedge\beta)\wedge((d\alpha\wedge\beta)\vee(\alpha\wedge d\beta))'\\
&=(\alpha\wedge\beta)\wedge((d\alpha)'\vee\beta')\wedge(\alpha'\vee(d\beta)')\\
&=\beta\wedge((d\alpha)'\vee\beta')\wedge\alpha\wedge(\alpha'\vee(d\beta)')\\
&=((\beta\wedge(d\alpha)')\vee(\beta\wedge\beta'))\wedge((\alpha\wedge\alpha')\vee(\alpha\wedge(d\beta)'))\\
&=(\beta\wedge(d\alpha)')\wedge(\alpha\wedge(d\beta)')\\
&=(\alpha\wedge(d\alpha)')\wedge(\beta\wedge(d\beta)')\\
&=d^*(\alpha)\wedge d^*(\beta).
\end{aligned}$$

   It follows that d^* is a $\wedge$-morphism. Hence, d^* is increasing, i.e., isotone.

2. Assume that d is isotone. Then Proposition 2 (iv) guarantees that:

$$\begin{aligned}
d^*(\alpha\wedge\beta)&=(\alpha\wedge\beta)\wedge(d(\alpha\wedge\beta))'\\
&=(\alpha\wedge\beta)\wedge(d\alpha\wedge d\beta)'\\
&=(\alpha\wedge\beta)\wedge((d\alpha)'\vee(d\beta)')\\
&=(\alpha\wedge\beta\wedge(d\alpha)')\vee(\alpha\wedge\beta\wedge(d\beta)')\\
&=(\alpha\wedge(d\alpha)'\wedge\beta)\vee(\alpha\wedge\beta\wedge(d\beta)')\\
&=(d^*(\alpha)\wedge\beta)\vee(\alpha\wedge d^*(\beta)).
\end{aligned}$$

   Thus, d^* is a derivation.

The next lemma is a key result to build a Boolean structure for $\mathfrak{D}(B)$.

Lemma 3. *If* d *is isotone, then* d^* *becomes a complement of* d *in* $\mathfrak{D}(B)$*, i.e.,*

$$d\sqcap d^*=0_{\mathfrak{D}(B)}\ \textit{and}\ d\sqcup d^*=1_{\mathfrak{D}(B)}.$$

Proof. Let $\alpha\in B$, we have

$$\begin{aligned}
(d\sqcap d^*)(\alpha)&=d\alpha\wedge d^*(\alpha)\\
&=d\alpha\wedge\alpha\wedge(d\alpha)'\\
&=0,
\end{aligned}$$

then $(d\sqcap d^*)(\alpha)=0_{\mathfrak{D}(B)}(\alpha)$. Thus, $d\sqcap d^*=0_{\mathfrak{D}(B)}$.

We also have

$$\begin{aligned}(d \sqcup d^*)(\alpha) &= d\alpha \vee d^*(\alpha)\\ &= d\alpha \vee (\alpha \wedge (d\alpha)')\\ &= (d\alpha \vee \alpha) \wedge (d\alpha \vee (d\alpha)')\\ &= \alpha \wedge 1\\ &= \alpha\\ &= 1_{\mathfrak{D}(B)}(\alpha).\end{aligned}$$

Therefore, $d \sqcup d^* = 1_{\mathfrak{D}(B)}$.

Based on Corollary 1 and Lemma 3, we are able to present a Boolean lattice for the isotone derivations.

Theorem 5. *Let B be a Boolean lattice and $\mathfrak{D}(B)$ the set of isotone derivations on B. Then the structure $(\mathfrak{D}(B), \preceq, \sqcap, \sqcup, 0_{\mathfrak{D}(B)}, 1_{\mathfrak{D}(B)})$ is a Boolean lattice.*

We recall that a function f on a Boolean lattice B is called strong negation if it is decreasing and $f(f(d)) = d$ for every derivation d on B. Here, we define a strong negation for the Boolean lattice of isotone derivations.

Definition 9. *We define a function $\eta : \mathfrak{D}(B) \to \mathfrak{D}(B)$ for each $d \in \mathfrak{D}(B)$ as follows:*

$$\eta(d) = d^*.$$

Theorem 6. *The function η is a strong negation of $\mathfrak{D}(B)$.*

Proof. Let $\alpha \in B$ and $d_1, d_2 \in \mathfrak{D}(B)$ such that $d_1 \preceq d_2$. Then $(d_2(\alpha))' \leqslant (d_1(\alpha))'$, so $\alpha \wedge (d_2(\alpha))' \leqslant \alpha \wedge (d_1(\alpha))'$. Thus, $d_2^*(\alpha) \leqslant d_1^*(\alpha)$. Hence, $d_2^* \preceq d_1^*$. Therefore, $\eta(d_2) \preceq \eta(d_1)$. Consequently, η is decreasing.

Now, we show that $\eta(\eta(d)) = d$, for each $d \in \mathfrak{D}(B)$. Let $d \in \mathfrak{D}(B)$ and $\alpha \in B$, then

$$\begin{aligned}d^{**}(\alpha) &= \alpha \wedge (d^*(\alpha))'\\ &= \alpha \wedge (\alpha \wedge (d\alpha)')'\\ &= \alpha \wedge (\alpha' \vee d\alpha)\\ &= (\alpha \wedge \alpha') \vee (\alpha \wedge d\alpha)\\ &= (\alpha \wedge d\alpha)\\ &= d\alpha.\end{aligned}$$

Thus $d^{**} = d$. We have $\eta(\eta(d)) = d^{**}$, so $\eta(\eta(d)) = d$. Therefore, η is a negation on $\mathfrak{D}(B)$. Consequently, η is a strong negation of $\mathfrak{D}(B)$.

Using Theorems 4 and 6, we obtain the following result.

Corollary 2. *$d \in \mathfrak{D}(B)$ is equivalent to $d^* \in \mathfrak{D}(B)$.*

We now recall the definition of a principal derivation.

Definition 10.[17] *Let $(L, \wedge, \vee)$ be a lattice and a be an element of L. The function d_a defined on L as*

$$d_a(x) = a \wedge x, \text{ for any } x \in L,$$

is an isotone derivation on L called a principal derivation.

Proposition 3. *Let d_α be a principal derivation on B. Then the complement of d_α is $\eta(d_\alpha) = d_{\alpha'}$.*

Proof. Let $\beta \in B$, so $\eta(d_\alpha) = d_\alpha^*$. Then

$$\begin{aligned}d_\alpha^*(\beta) &= \beta \wedge (d_\alpha(\beta))'\\ &= \beta \wedge (\alpha \wedge \beta)'\\ &= \beta \wedge (\alpha' \vee \beta')\\ &= (\beta \wedge \alpha') \vee (\beta \wedge \beta')\\ &= \beta \wedge \alpha'\\ &= d_{\alpha'}(\beta).\end{aligned}$$

Thus, $d_\alpha^* = d_{\alpha'}$. Therefore, $\eta(d_\alpha) = d_{\alpha'}$.

Corollary 3. *The complement of a principal derivation d_α on B is the principal derivation $d_{\alpha'}$.*

5 A Boolean lattice of fixed sets

Finally, we show that the fixed sets of isotone derivations form a Boolean lattice. Before that, we need to prove the following property.

Proposition 4. *An element* $\alpha \in B$ *is a fixed point of* d^* *if and only if* $d\alpha = 0$.

Proof. Let α an element of B. We suppose that α is a fixed point of d^*. Then $d^*(\alpha) = \alpha \wedge (d\alpha)' = \alpha$, so $\alpha \leqslant (d\alpha)'$. We know from Proposition 2 (1) that $\alpha' \leqslant (d\alpha)'$. Thus, $\alpha \vee \alpha' \leqslant (d\alpha)'$. Hence, $(d\alpha)' = 1$. Therefore, $d\alpha = 0$.

Conversely, we assume that $d\alpha = 0$. Then $(d\alpha)' = 1$, so $\alpha \leqslant (d\alpha)'$. Thus $d^*(\alpha) = \alpha \wedge (d\alpha)' = \alpha$. Therefore, α is a fixed point of d^*.

Corollary 4. $Fix_{d^*}(B) = Ker(d)$ *and* $Fix_d(B) = Ker(d^*)$.

Theorem 7.[17] *Let* L *be a distributive lattice and* $\mathscr{F} =: \{Fix_d(L) \mid d \in \mathfrak{D}(L)\}$ *be the set of fixed sets of isotone derivations on* L. *Define for every* $Fix_{d_1}(L)$, $Fix_{d_2}(L) \in \mathscr{F}$:

$$Fix_{d_1}(L) \wedge Fix_{d_2}(L) = Fix_{d_1 \wedge d_2}(L) \text{ and } Fix_{d_1}(L) \vee Fix_{d_2}(L) = Fix_{d_1 \vee d_2}(L).$$

Then $(\mathscr{F}, \wedge, \vee)$ *is a distributive lattice isomorphic to* $(\mathfrak{D}(L), \preceq, \sqcap, \sqcup)$.

By the combinition of Theorems 5, 7, and Corollary 4, we obtain the following results.

Theorem 8. *If* L *is a Boolean lattice, then the lattice* $(\mathscr{F}, \wedge, \vee)$ *is also a Boolean lattice isomorphic to the Boolean lattice* $(\mathfrak{D}(B), \preceq, \sqcap, \sqcup, 0_{\mathfrak{D}(B)}, 1_{\mathfrak{D}(B)})$, *where*

$$0_{\mathscr{F}} = Fix_{(0_{\mathfrak{D}(B)})}(B), \text{ and } 1_{\mathscr{F}} = Fix_{(1_{\mathfrak{D}(B)})}(B).$$

Furthermore, the complement of $Fix_d(B)$ *is*

$$(Fix_d(B))' = Fix_{d^*}(B) = Ker(d), \text{ for every } Fix_d(B) \in \mathscr{F}.$$

Finally, based on Corollary 4, we conclude a characterization of the Boolean lattice $\mathscr{F}$.

Theorem 9. *Let* L *be a distributive lattice,* $\mathfrak{D}(L)$ *the set of all derivations on* L *and* $\mathscr{F}$ *the set of fixed sets of isotone derivations on* L. *Then*

$$\mathscr{F} =: \{Ker(d) \mid d \in \mathfrak{D}(L)\}.$$

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