



# Numerical Solutions Of Linear And Nonlinear Fredholm Integro-Differential Equations By Using The Weighted Integral Mean Value Methods

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**Abstract:** In this work, we present a numerical algorithm for approaching the solutions of a specific form of linear and nonlinear Fredholm integro-differential equations. This algorithm is constructed by applying the weighted integral mean value methods, which allow us to convert the proposed equations into a system of algebraic equations, then by solving this system we obtain the approximate solutions. Moreover, we provide some illustrative examples to show the effectiveness and simplicity of the algorithm.

**Keywords:** Fredholm equations; Integro-differential equations; Degenerate kernels; Weighted integral mean value theorem.

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## 1 Introduction

Integral equations have become as an important branch in mathematics, because of their useful applications in many diverse areas such as: fluid dynamics, biological models, electrical engineering, solid state physics, control technology, chemistry etc, for interested readers see [9, 16, 13, 1, 15, 4]. In most cases, it is very difficult to solve these integral equations analytically. Due to this, many numerical techniques have been developed [5, 22, 7, 8, 10, 18, 12] for finding the approximate solutions of them.

In the past several years, a new useful technique, called integral mean value method (**IMVM**) or generalized integral mean value method (**GIMVM**) have been proposed for obtaining the approximate solutions for a different classes of Fredholm integral equations, for example: Linear and nonlinear equations [6], High dimensional equations [14], Toeplitz plus Hankel equations [19, 3], Integro-differential equations [21, 2]. Furthermore, a new technique which called multistage integral mean value has been developed in this field by [17, 11, 20], where the authors proved that this new method gives a good results.

The aim of this paper is to apply both the **GIMVM** and **IMVM** to solve two specific classes of Fredholm integro-differential equations, which are described as follows:

**Nonlinear Fredholm integro-differential equations:**

$$u(t) = \int_a^b k(s,t)\varphi(u(s) + u'(s))ds + f(t), \quad t \in [a, b].$$

**Linear Fredholm integro-differential equations:**

$$u(t) = \int_a^b k(s,t)(u(s) + u'(s))ds + f(t), \quad t \in [a, b].$$

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Where,  $k$ ,  $f$  and  $\varphi$  are given nonlinear functions,  $u$  is the unknown solution which to be determined in the space  $C^1([a, b])$ . We point out that these classes of equations have an interesting form, because the unknown solution ( $u$ ) and its derivative ( $u'$ ) are found inside the integral operator.

The paper is structured as follows: In section 2, we recall the principles of the **GIMVM** and **IMVM** and provide some necessary hypotheses concerning our equations. In section 3, we apply the both methods **GIMVM** and **IMVM** to solve the nonlinear Fredholm integro-differential equations. In section 4, we apply the same mentioned methods to solve the linear Fredholm integro-differential equations. In section 5, we provide some examples to illustrate the algorithm of solving these classes of equations.

## 2 Preliminaries

In this section, we introduce some basic definitions, properties and assumptions which we will utilize throughout the work.

**Definition 1.**[17] Let  $g(s)$  be a continuous function on the closed interval  $[a, b]$ , then there is a number  $c \in [a, b]$  such that

$$\int_a^b g(s)ds = (b-a)g(c).$$

**Definition 2.**[2] Let  $g(s), h(s)$  be continuous functions on the closed interval  $[a, b]$ . If  $h(s)$  never changes sign in  $[a, b]$ , then there exists a number  $c \in [a, b]$  such that

$$\int_a^b g(s)h(s)ds = g(c) \int_a^b h(s)ds.$$

Before starting the study of the proposed nonlinear and linear equations, we need to combine them with another auxiliary equations. For this reason, by differentiating both sides of the previous equations and applying Leibniz's integral rule [5], we obtain the following systems:

**System of nonlinear Fredholm integro-differential equations:**

$$u(t) = \int_a^b k(s, t)\varphi(u(s) + u'(s))ds + f(t), \quad (1)$$

$$u'(t) = \int_a^b \partial_t k(s, t)\varphi(u(s) + u'(s))ds + f'(t). \quad (2)$$

**System of linear Fredholm integro-differential equations:**

$$u(t) = \int_a^b k(s, t)(u(s) + u'(s))ds + f(t), \quad (3)$$

$$u'(t) = \int_a^b \partial_t k(s, t)(u(s) + u'(s))ds + f'(t). \quad (4)$$

Moreover, we need to make the following assumptions on the kernel functions as follows:

- $k(s, t)$ ,  $\partial_t k(s, t)$ ,  $\varphi(s)$ ,  $f(t)$  and  $f'(t)$  are continuous functions for all  $t, s \in [a, b]$ .
- $k(s, t) \times \partial_t k(s, t) \geq 0$  (or,  $k(s, t) \times \partial_t k(s, t) \leq 0$ ) for all  $t, s \in [a, b]$ .
- $k(s, t)$  and  $\partial_t k(s, t)$  are degenerate functions (or can be degenerated), which means that  $k(s, t) = x(s)y(t)$  and  $\partial_t k(s, t) = \tilde{x}(s)\tilde{y}(t)$ .

We want to point out that we have added these assumptions in order to permit us to apply **GIMVM** and **IMVM**. On the other hand, we have chosen the kernels  $k(s, t)$  and  $\partial_t k(s, t)$  as a degenerate functions in order to simplify the application of the integral mean value methods and to allow them to produce a number  $c$  rather than a function  $c(t)$  in Definitions 1 and 2. Up to now, there are no studies in which the function  $c(t)$  has been used to approach the integral equations.

## 3 Solving nonlinear Fredholm integro-differential equations

In Section 3, we apply both the **GIMVM** and **IMVM** to solve nonlinear Fredholm integro-differential equations.

### 3.1 Solving via GIMVM

Consider the system of equations (1)-(2), by applying **GIMVM** on this system, we get:

$$\begin{aligned} u(t) &= \varphi(u(c_1) + u'(c_1)) \int_a^b k(s,t) ds + f(t), \\ &= \varphi(\alpha_1 + \beta_1) \gamma(t) + f(t), \end{aligned} \quad (5)$$

$$\begin{aligned} u'(t) &= \varphi(u(c_2) + u'(c_2)) \int_a^b \partial_t k(s,t) ds + f'(t), \\ &= \varphi(\alpha_2 + \beta_2) \delta(t) + f'(t), \end{aligned} \quad (6)$$

where,  $\gamma(t) = \int_a^b k(s,t) ds$ ,  $\delta(t) = \int_a^b \partial_t k(s,t) ds$ ,  $\alpha_1 = u(c_1)$ ,  $\alpha_2 = u(c_2)$ ,  $\beta_1 = u'(c_1)$  and  $\beta_2 = u'(c_2)$ . Then, by substituting  $c_1$  and  $c_2$  for  $t$  in (5) and (6), respectively, we get:

$$\alpha_1 = \varphi(\alpha_1 + \beta_1) \gamma(c_1) + f(c_1), \quad (7)$$

$$\beta_2 = \varphi(\alpha_2 + \beta_2) \delta(c_2) + f'(c_2). \quad (8)$$

Notice that equations (7) and (8) have 6 unknowns:  $c_1$ ,  $c_2$ ,  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_2$ , so, 4 more equations are needed. First, we just substitute  $c_2$  and  $c_1$  for  $t$  in (5) and (6), respectively, to obtain 2 other equations as follows:

$$\alpha_2 = \varphi(\alpha_1 + \beta_1) \gamma(c_2) + f(c_2), \quad (9)$$

$$\beta_1 = \varphi(\alpha_2 + \beta_2) \delta(c_1) + f'(c_1). \quad (10)$$

Now, to find the last two equations, we replace the following functions:

$$u(s) = \varphi(\alpha_1 + \beta_1) \gamma(s) + f(s), \quad u'(s) = \varphi(\alpha_2 + \beta_2) \delta(s) + f'(s),$$

in (1) and (2), respectively, which gives us:

$$u(t) = \int_a^b k(s,t) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f(t), \quad (11)$$

$$u'(t) = \int_a^b \partial_t k(s,t) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f'(t). \quad (12)$$

So, by replacing  $t$  with  $c_1$  in (11) and  $c_2$  in (12) we find:

$$\alpha_1 = \int_a^b k(s, c_1) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f(c_1), \quad (13)$$

$$\beta_2 = \int_a^b \partial_t k(s, c_2) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f'(c_2). \quad (14)$$

Consequently, by combining these equations (7), (8), (9), (10), (13) and (14) together, a nonlinear algebraic system of 6 equations with 6 unknowns is obtained as follows:

$$\alpha_1 = \varphi(\alpha_1 + \beta_1) \gamma(c_1) + f(c_1),$$

$$\beta_2 = \varphi(\alpha_2 + \beta_2) \delta(c_2) + f'(c_2),$$

$$\alpha_2 = \varphi(\alpha_1 + \beta_1) \gamma(c_2) + f(c_2),$$

$$\beta_1 = \varphi(\alpha_2 + \beta_2) \delta(c_1) + f'(c_1),$$

$$\alpha_1 = \int_a^b k(s, c_1) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f(c_1),$$

$$\beta_2 = \int_a^b \partial_t k(s, c_2) \varphi[\varphi(\alpha_1 + \beta_1) \gamma(s) + \varphi(\alpha_2 + \beta_2) \delta(s) + f(s) + f'(s)] ds + f'(c_2).$$

Finally, we need to solve this system numerically, then we switch the values of  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_2$  in formula (5) and (6) to get the approximate solution with its derivative of our nonlinear Fredholm equation.

### 3.2 Solving via IMVM

In this partition, we apply **IMVM** on the nonlinear system of equations (1)-(2) to get:

$$u(t) = \varphi(\alpha_1 + \beta_1)k(c_1, t)(b-a) + f(t), \quad (15)$$

$$u'(t) = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, t)(b-a) + f'(t). \quad (16)$$

By substituting  $c_1$  and  $c_2$  for  $t$  in (15) and (16), respectively, we obtain:

$$\alpha_1 = \varphi(\alpha_1 + \beta_1)k(c_1, c_1)(b-a) + f(c_1), \quad (17)$$

$$\beta_2 = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, c_2)(b-a) + f'(c_2). \quad (18)$$

Similarly as before, the equations (17) and (18) have 6 unknowns. So, we must combine them with another 4 equations. First, if we substitute  $c_2$  and  $c_1$  for  $t$  in (15) and (16), respectively, we obtain the following two equations:

$$\alpha_2 = \varphi(\alpha_1 + \beta_1)k(c_1, c_2)(b-a) + f(c_2), \quad (19)$$

$$\beta_1 = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, c_1)(b-a) + f'(c_1). \quad (20)$$

Now, for getting the last two equations, we consider the functions:

$$u(s) = \varphi(\alpha_1 + \beta_1)k(c_1, s)(b-a) + f(s), \quad u'(s) = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)(b-a) + f'(s),$$

then we replace them in (1) to find:

$$u(t) = \int_a^b k(s, t)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f(t), \quad (21)$$

and in (2) to find:

$$u'(t) = \int_a^b \partial_t k(s, t)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f'(t). \quad (22)$$

So, by taking  $c_1$  and  $c_2$  for  $t$  in (21) and (22) we obtain:

$$\alpha_1 = \int_a^b k(s, c_1)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f(c_1), \quad (23)$$

$$\beta_2 = \int_a^b \partial_t k(s, c_2)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f'(c_2). \quad (24)$$

Consequently, the equations (17), (18), (19), (20), (23) and (24) gives us the following nonlinear algebraic system:

$$\alpha_1 = \varphi(\alpha_1 + \beta_1)k(c_1, c_1)(b-a) + f(c_1),$$

$$\beta_2 = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, c_2)(b-a) + f'(c_2),$$

$$\alpha_2 = \varphi(\alpha_1 + \beta_1)k(c_1, c_2)(b-a) + f(c_2),$$

$$\beta_1 = \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, c_1)(b-a) + f'(c_1),$$

$$\alpha_1 = \int_a^b k(s, c_1)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f(c_1),$$

$$\beta_2 = \int_a^b \partial_t k(s, c_2)\varphi[(b-a)[\varphi(\alpha_1 + \beta_1)k(c_1, s) + \varphi(\alpha_2 + \beta_2)\partial_t k(c_2, s)] + f(s) + f'(s)ds + f'(c_2).$$

Finally, when we solve this system numerically, then by switching the values of parameters  $c_1$ ,  $c_2$ ,  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_2$  in (15) and (16) we find an approximate solution with its derivative of our nonlinear Fredholm equation.

## 4 Solving linear Fredholm integro-differential equations

In the same way that we have followed in the previous section, we will solve the system of linear equations (3)-(4) using **GIMVM** then **IMVM**, respectively. Moreover, as the linear system is a special case of the nonlinear system, we just replace the function  $\varphi(u(s) + u'(s)) = I(u(s) + u'(s)) = u(s) + u'(s)$  in what we have done before in order to avoid the redundant calculations.

#### 4.1 Solving via GIMVM

Consider the linear system (3)-(4). It is easy to show that the application of **GIMVM** allows us to create the following nonlinear algebraic system:

$$\alpha_1 = (\alpha_1 + \beta_1)\gamma(c_1) + f(c_1),$$

$$\beta_2 = (\alpha_2 + \beta_2)\delta(c_2) + f'(c_2),$$

$$\alpha_2 = (\alpha_1 + \beta_1)\gamma(c_2) + f(c_2),$$

$$\beta_1 = (\alpha_2 + \beta_2)\delta(c_1) + f'(c_1),$$

$$\alpha_1 = (\alpha_1 + \beta_1) \int_a^b k(s, c_1)\gamma(s)ds + (\alpha_2 + \beta_2) \int_a^b k(s, c_1)\delta(s)ds + \int_a^b k(s, c_1)(f(s) + f'(s))ds + f(c_1),$$

$$\beta_2 = (\alpha_1 + \beta_1) \int_a^b \partial_t k(s, c_2)\gamma(s)ds + (\alpha_2 + \beta_2) \int_a^b \partial_t k(s, c_2)\delta(s)ds + \int_a^b \partial_t k(s, c_2)(f(s) + f'(s))ds + f'(c_2).$$

So, after getting the values of  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_2$  we can find the approximate solutions of our linear Fredholm equation.

#### 4.2 Solving via IMVM

In this case, by applying **IMVM** on the system of equations (3)-(4), we can build the following nonlinear algebraic system:

$$\alpha_1 = (\alpha_1 + \beta_1)k(c_1, c_1)(b - a) + f(c_1),$$

$$\beta_2 = (\alpha_2 + \beta_2)\partial_t k(c_2, c_2)(b - a) + f'(c_2),$$

$$\alpha_2 = (\alpha_1 + \beta_1)k(c_1, c_2)(b - a) + f(c_2),$$

$$\beta_1 = (\alpha_2 + \beta_2)\partial_t k(c_2, c_1)(b - a) + f'(c_1),$$

$$\alpha_1 = (\alpha_1 + \beta_1)(b - a) \int_a^b k(s, c_1)k(c_1, s)ds + (\alpha_2 + \beta_2)(b - a) \int_a^b k(s, c_1)\partial_t k(c_1, s)ds$$

$$+ \int_a^b k(s, c_1)(f(s) + f'(s))ds + f(c_1),$$

$$\beta_2 = (\alpha_1 + \beta_1)(b - a) \int_a^b \partial_t k(s, c_2)k(c_1, s)ds + (\alpha_2 + \beta_2)(b - a) \int_a^b \partial_t k(s, c_2)\partial_t k(c_2, s)ds$$

$$+ \int_a^b \partial_t k(s, c_2)(f(s) + f'(s))ds + f'(c_2).$$

So, after solving this system, we use the values of  $c_1$ ,  $c_2$ ,  $\alpha_1$ ,  $\beta_1$ ,  $\alpha_2$  and  $\beta_2$  to get the approximate solutions of our linear Fredholm equation.

*Remark.* We point out that both the **GIMVM** and **IMVM** are based on finding the ideal value of the constant  $c$  shown in Definitions 1 and 2. Therefore, by applying these methods, we have transformed the proposed integro-differential equations into an algebraic systems just of 6 equations, and by solving them we obtain the approximate solutions with their derivatives of the proposed equations. With regard to the other well-known numerical schemes (e.g., collocation, Galerkin methods, or product integration methods [5]), we can see that all these methods are depended primarily on dividing the interval  $[a, b]$  using the step  $h$ , then by applying these numerical schemes we obtain a big algebraic systems of  $n$  equations which give us the approximate solutions of the proposed equations. In addition, the convergence analysis of these approximate methods is fully related by vanishing the step  $h$  towards zero. For this reason, we conclude that both the **GIMVM** and **IMVM** are the easiest, simplest and most effective when compared to the other numerical methods.

### 5 Illustrative examples

In this section, we provide 4 examples: two linear equations and two nonlinear equations, then we solve them using **GIMVM** and **IMVM**. As well as, we introduce the absolute error function:

$$E(t) = |u_{ex}(t) - u_{ap}(t)| + |u'_{ex}(t) - u'_{ap}(t)|,$$

in order to compare between the exact solution  $u_{ex}(t)$  and the approximate solution  $u_{ap}(t)$ . Moreover, to highlight the performance of the presented methods **GIMVM** and **IMVM**, we compare them with the Nyström-Collocation method [5]. Where, we approach the solutions of the same examples at the same node points by using the Nyström-Collocation method and include the results in the comparison tables.

*Example 1.* Consider the first linear Fredholm integro-differential equation:

$$u(t) = \int_0^1 e^{s+t}(u(s) + u'(s))ds + t^2 - e^{t+1},$$

where the exact solution for this example and its derivative are given by:

$$u_{ex}(t) = t^2, \quad u'_{ex}(t) = 2t, \quad \forall t \in [0, 1].$$

An application of the **GIMVM** produces us:

$$\begin{aligned} \alpha_1 &= e^{c_1}(\alpha_1 + \beta_1)(e - 1) + c_1^2 - e^{c_1+1}, \\ \beta_2 &= e^{c_2}(\alpha_2 + \beta_2)(e - 1) + 2c_2 - e^{c_2+1}, \\ \alpha_2 &= e^{c_2}(\alpha_1 + \beta_1)(e - 1) + c_2^2 - e^{c_2+1}, \\ \beta_1 &= e^{c_1}(\alpha_2 + \beta_2)(e - 1) + 2c_1 - e^{c_1+1}, \\ \alpha_1 &= \frac{1}{2}(\alpha_1 + \beta_1 + \alpha_2 + \beta_2)(e - 1)(e^2 - 1)e^{c_1} + c_1^2 + e^{c_1+1} - e^{c_1+3}, \\ \beta_2 &= \frac{1}{2}(\alpha_1 + \beta_1 + \alpha_2 + \beta_2)(e - 1)(e^2 - 1)e^{c_2} + 2c_2 + e^{c_2+1} - e^{c_2+3}. \end{aligned}$$

We solve the arising system by using the MATLAB R2014a (version 8.3) software, then we get the approximate values:

$$\begin{aligned} c_1 &= 0.606852904202905, & \alpha_1 &= 0.368270715443063, & \beta_1 &= 1.213706076533204, \\ c_2 &= 0.606852859557245, & \alpha_2 &= 0.368270661254951, & \beta_2 &= 1.213705987240341. \end{aligned}$$

Which give us the approximate solution  $u_{ap}(t)$  and its derivative  $u'_{ap}(t)$  by:

$$\begin{aligned} u_{ap}(t) &= e^t(1.581976791976267)(e - 1) + t^2 - e^{t+1}, \\ u'_{ap}(t) &= e^t(1.581976648495292)(e - 1) + 2t - e^{t+1}. \end{aligned}$$

Now, **IMVM** leads us to the following algebraic system:

$$\begin{aligned} \alpha_1 &= e^{2c_1}(\alpha_1 + \beta_1) + c_1^2 - e^{c_1+1}, \\ \beta_2 &= e^{2c_2}(\alpha_2 + \beta_2) + 2c_2 - e^{c_2+1}, \\ \alpha_2 &= e^{c_1+c_2}(\alpha_1 + \beta_1) + c_2^2 - e^{c_2+1}, \\ \beta_1 &= e^{c_2+c_1}(\alpha_2 + \beta_2) + 2c_1 - e^{c_1+1}, \\ \alpha_1 &= \frac{1}{2}[(\alpha_1 + \beta_1)e_1^c + (\alpha_2 + \beta_2)e_2^c](e^2 - 1)e^{c_1} + c_1^2 + e^{c_1+1} - e^{c_1+3}, \\ \beta_2 &= \frac{1}{2}[(\alpha_1 + \beta_1)e_1^c + (\alpha_2 + \beta_2)e_2^c](e^2 - 1)e^{c_2} + 2c_2 + e^{c_2+1} - e^{c_2+3}. \end{aligned}$$

Solving this system, we get:

$$\begin{aligned} c_1 &= 0.585458052993885, & \alpha_1 &= 0.342761130812170, & \beta_1 &= 1.170916105719592, \\ c_2 &= 0.585458053109000, & \alpha_2 &= 0.342761131068858, & \beta_2 &= 1.170916106109908. \end{aligned}$$

Then, we obtain:

$$\begin{aligned} u_{ap}(t) &= e^{0.585458052993885t}(1.513677236531762) + t^2 - e^{t+1}, \\ u'_{ap}(t) &= e^{0.585458053109000t}(1.513677237178766) + 2t - e^{t+1}. \end{aligned}$$

*Example 2.* Consider the second linear Fredholm integro-differential equation:

$$u(t) = \int_0^{\frac{\pi}{2}} st(u(s) + u'(s))ds + \sin(t) - \frac{\pi}{2}t,$$

where the exact solution for this example and its derivative are given by:

$$u_{ex}(t) = \sin(t), \quad u'_{ex}(t) = \cos(t), \quad \forall t \in \left[0, \frac{\pi}{2}\right].$$

Using **GIMVM** we get:

$$\alpha_1 = (\alpha_1 + \beta_1) \frac{\pi^2}{8} c_1 + \sin(c_1) - \frac{\pi}{2} c_1,$$

$$\beta_2 = (\alpha_2 + \beta_2) \frac{\pi^2}{8} + \cos(c_2) - \frac{\pi}{2},$$

$$\alpha_2 = (\alpha_1 + \beta_1) \frac{\pi^2}{8} c_2 + \sin(c_2) - \frac{\pi}{2} c_2,$$

$$\beta_1 = (\alpha_2 + \beta_2) \frac{\pi^2}{8} + \cos(c_1) - \frac{\pi}{2},$$

$$\alpha_1 = c_1(\alpha_1 + \beta_1) \frac{\pi^5}{192} + c_1(\alpha_2 + \beta_2) \frac{\pi^4}{64} - \frac{\pi^4}{48} c_1 - \frac{\pi^3}{16} c_1 + \sin(c_1),$$

$$\beta_2 = (\alpha_1 + \beta_1) \frac{\pi^5}{192} + (\alpha_2 + \beta_2) \frac{\pi^4}{64} - \frac{\pi^4}{48} - \frac{\pi^3}{16} + \cos(c_2).$$

Solving this system, we get:

$$c_1 = 1.235698751630932, \quad \alpha_1 = 0.944378225434312, \quad \beta_1 = 0.328861319300813, \\ c_2 = 1.235698751623657, \quad \alpha_2 = 0.944378225428763, \quad \beta_2 = 0.328861319306613.$$

Then,

$$u_{ap}(t) = (1.273239544735125) \frac{\pi^2}{8} t + \sin(t) - \frac{\pi}{2} t,$$

$$u'_{ap}(t) = (1.273239544735376) \frac{\pi^2}{8} + \cos(t) - \frac{\pi}{2}.$$

Now, by using **IMVM** we find:

$$\alpha_1 = c_1^2(\alpha_1 + \beta_1) \frac{\pi}{2} + \sin(c_1) - \frac{\pi}{2} c_1,$$

$$\beta_2 = c_2(\alpha_2 + \beta_2) \frac{\pi}{2} + \cos(c_2) - \frac{\pi}{2},$$

$$\alpha_2 = c_1 c_2(\alpha_1 + \beta_1) \frac{\pi}{2} + \sin(c_2) - \frac{\pi}{2} c_2,$$

$$\beta_1 = c_2(\alpha_2 + \beta_2) \frac{\pi}{2} + \cos(c_1) - \frac{\pi}{2},$$

$$\alpha_1 = c_1^2(\alpha_1 + \beta_1) \frac{\pi^4}{48} + c_1 c_2(\alpha_2 + \beta_2) \frac{\pi^3}{16} - \frac{\pi^4}{48} c_1 - \frac{\pi^3}{16} c_1 + \sin(c_1),$$

$$\beta_2 = c_1(\alpha_1 + \beta_1) \frac{\pi^4}{48} + c_2(\alpha_2 + \beta_2) \frac{\pi^3}{16} - \frac{\pi^4}{48} c_1 - \frac{\pi^3}{16} c_1 + \cos(c_2).$$

Solving this system, we get:

$$c_1 = 0.709166368006409, \quad \alpha_1 = 0.651201349876294, \quad \beta_1 = 0.758905001909804, \\ c_2 = 0.709166368006813, \quad \alpha_2 = 0.651201349879968, \quad \beta_2 = 0.758905001907650.$$

Which give us:

$$u_{ap}(t) = (0.999999999998915) \frac{\pi}{2} t + \sin(t) - \frac{\pi}{2} t,$$

$$u'_{ap}(t) = (1.000000000000562) \frac{\pi}{2} + \cos(t) - \frac{\pi}{2}.$$

*Example 3.* Consider the first nonlinear Fredholm integro-differential equation:

$$u(t) = \int_0^2 s e^t (u(s) + u'(s))^2 ds + t - 1 - 4e^t.$$

The exact solution of this equation and its derivative are:

$$u_{ex}(t) = t - 1, \quad u'_{ex}(t) = 1, \quad \forall t \in [0, 2].$$

By using **GIMVM** we get to the following algebraic system:

$$\begin{aligned}\alpha_1 &= 2e^{c_1}(\alpha_1 + \beta_1)^2 + c_1 - 1 - 4e^{c_1}, \\ \beta_2 &= 2e^{c_2}(\alpha_2 + \beta_2)^2 + 1 - 4e^{c_2}, \\ \alpha_2 &= 2e^{c_2}(\alpha_1 + \beta_1)^2 + c_2 - 1 - 4e^{c_2}, \\ \beta_1 &= 2e^{c_1}(\alpha_2 + \beta_2)^2 + 1 - 4e^{c_1}, \\ \alpha_1 &= \left[ (\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2 - 4 \right]^2 (3e^4 + 1)e^{c_1} + 8 \left[ (\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2 - 4 \right] (e^2 - 1)e^{c_1} + c_1 - 1, \\ \beta_2 &= \left[ (\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2 - 4 \right]^2 (3e^4 + 1)e^{c_2} + 8 \left[ (\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2 - 4 \right] (e^2 - 1)e^{c_2} + 1.\end{aligned}$$

Solving this system, we get the approximate values:

$$\begin{aligned}c_1 &= 1.414213562373014, & \alpha_1 &= 0.414213562373117, & \beta_1 &= 0.9999999999999990, \\ c_2 &= 1.414213562372904, & \alpha_2 &= 0.414213562373006, & \beta_2 &= 1.0000000000000098.\end{aligned}$$

Then we obtain the approximate solutions:

$$\begin{aligned}u_{ap}(t) &= 2e^t(1.414213562373107)^2 + t - 1 - 4e^t, \\ u'_{ap}(t) &= 2e^t(1.414213562373104)^2 + 1 - 4e^t.\end{aligned}$$

Now, **IMVM** gives us:

$$\begin{aligned}\alpha_1 &= 2c_1e^{c_1}(\alpha_1 + \beta_1)^2 + c_1 - 1 + 4e^{c_1}, \\ \beta_2 &= 2c_2e^{c_2}(\alpha_2 + \beta_2)^2 + 1 + 4e^{c_2}, \\ \alpha_2 &= 2c_1e^{c_2}(\alpha_1 + \beta_1)^2 + c_2 - 1 + 4e^{c_2}, \\ \beta_1 &= 2c_2e^{c_1}(\alpha_2 + \beta_2)^2 + 1 + 4e^{c_1}, \\ \alpha_1 &= \left[ c_1(\alpha_1 + \beta_1)^2 + c_2(\alpha_2 + \beta_2)^2 - 4 \right]^2 (3e^4 + 1)e^{c_1} + 8 \left[ c_1(\alpha_1 + \beta_1)^2 + c_2(\alpha_2 + \beta_2)^2 - 4 \right] (e^2 - 1)e^{c_1} + c_1 - 1, \\ \beta_2 &= \left[ c_1(\alpha_1 + \beta_1)^2 + c_2(\alpha_2 + \beta_2)^2 - 4 \right]^2 (3e^4 + 1)e^{c_2} + 8 \left[ c_1(\alpha_1 + \beta_1)^2 + c_2(\alpha_2 + \beta_2)^2 - 4 \right] (e^2 - 1)e^{c_2} + 1.\end{aligned}$$

Solving this system, we get:

$$\begin{aligned}c_1 &= 1.259921049894873, & \alpha_1 &= 0.259921049894872, & \beta_1 &= 1.0000000000000001, \\ c_2 &= 1.259921049894873, & \alpha_2 &= 0.259921049894873, & \beta_2 &= 1.0000000000000000.\end{aligned}$$

Thus,

$$\begin{aligned}u_{ap}(t) &= 2(1.259921049894873)e^t(1.259921049894873)^2 + t - 1 + 4e^t, \\ u'_{ap}(t) &= 2(1.259921049894873)e^t(1.259921049894873)^2 + 1 + 4e^t.\end{aligned}$$

*Example 4.* Consider the second nonlinear Fredholm integro-differential equation:

$$u(t) = \int_1^2 s\sqrt{t}u(s)u'(s)ds + \frac{\sqrt{t}}{4}.$$

The exact solution of this equation and its derivative are:

$$u_{ex}(t) = \sqrt{t}, \quad u'_{ex}(t) = \frac{1}{2\sqrt{t}}, \quad \forall t \in [1, 2].$$

Using **GIMVM** we get:

$$\alpha_1 = \frac{3}{2}\sqrt{c_1}\alpha_1\beta_1 + \frac{\sqrt{c_1}}{4},$$

$$\beta_2 = \frac{3}{4\sqrt{c_2}}\alpha_2\beta_2 + \frac{1}{8\sqrt{c_2}},$$

$$\alpha_2 = \frac{3}{2}\sqrt{c_2}\alpha_1\beta_1 + \frac{\sqrt{c_2}}{4},$$

$$\beta_1 = \frac{3}{4\sqrt{c_1}}\alpha_2\beta_2 + \frac{1}{8\sqrt{c_1}},$$

$$\alpha_1 = \frac{27}{16}\sqrt{c_1}\alpha_1\beta_1\alpha_2\beta_2 + \frac{9}{32}\sqrt{c_1}(\alpha_1\beta_1 + \alpha_2\beta_2) + \frac{3}{64}\sqrt{c_1} + \frac{\sqrt{c_1}}{4},$$

$$\beta_2 = \frac{1}{2\sqrt{c_2}} \left[ \frac{27}{16}\alpha_1\beta_1\alpha_2\beta_2 + \frac{9}{32}(\alpha_1\beta_1 + \alpha_2\beta_2) + \frac{3}{64} \right] + \frac{1}{8\sqrt{c_2}}.$$

Solving this system, we get the approximate values:

$$c_1 = 0.999999999986196, \quad \alpha_1 = 1.00000000002869, \quad \beta_1 = 0.50000000001443,$$

$$c_2 = 0.999999999986235, \quad \alpha_2 = 1.00000000002889, \quad \beta_2 = 0.50000000001444.$$

Then we obtain the approximate solutions:

$$u_{ap}(t) = (0.50000000002877) \frac{3}{2}\sqrt{t} + \frac{\sqrt{t}}{4},$$

$$u'_{ap}(t) = (0.50000000002888) \frac{3}{4\sqrt{t}} + \frac{1}{8\sqrt{t}}.$$

Now, **IMVM** leads to the following algebraic system:

$$\alpha_1 = c_1\alpha_1\beta_1\sqrt{c_1} + \frac{\sqrt{c_1}}{4},$$

$$\beta_2 = c_2\frac{1}{2\sqrt{c_2}}\alpha_2\beta_2 + \frac{1}{8\sqrt{c_2}},$$

$$\alpha_2 = c_1\alpha_1\beta_1\sqrt{c_2} + \frac{\sqrt{c_2}}{4},$$

$$\beta_1 = c_2\frac{1}{2\sqrt{c_1}}\alpha_2\beta_2 + \frac{1}{8\sqrt{c_1}},$$

$$\alpha_1 = \sqrt{c_1} \left[ \frac{3}{4}c_1c_2\alpha_1\beta_1\alpha_2\beta_2 + \frac{3}{16}(c_1\alpha_1\beta_1 + c_2\alpha_2\beta_2) + \frac{3}{64} \right] + \frac{\sqrt{c_1}}{4},$$

$$\beta_2 = \frac{1}{2\sqrt{c_2}} \left[ \frac{3}{4}c_1c_2\alpha_1\beta_1\alpha_2\beta_2 + \frac{3}{16}(c_1\alpha_1\beta_1 + c_2\alpha_2\beta_2) + \frac{3}{64} \right] + \frac{1}{8\sqrt{c_2}}.$$

Solving this system, we get:

$$c_1 = 1.224744866295188, \quad \alpha_1 = 1.106681932507557, \quad \beta_1 = 0.553340966252873,$$

$$c_2 = 1.224744866294298, \quad \alpha_2 = 1.106681932506823, \quad \beta_2 = 0.553340966253380.$$

Thus,

$$u_{ap}(t) = (0.750000014236843)\sqrt{t} + \frac{\sqrt{t}}{4},$$

$$u'_{ap}(t) = (0.750000014236487) \frac{1}{2\sqrt{t}} + \frac{1}{8\sqrt{t}}.$$

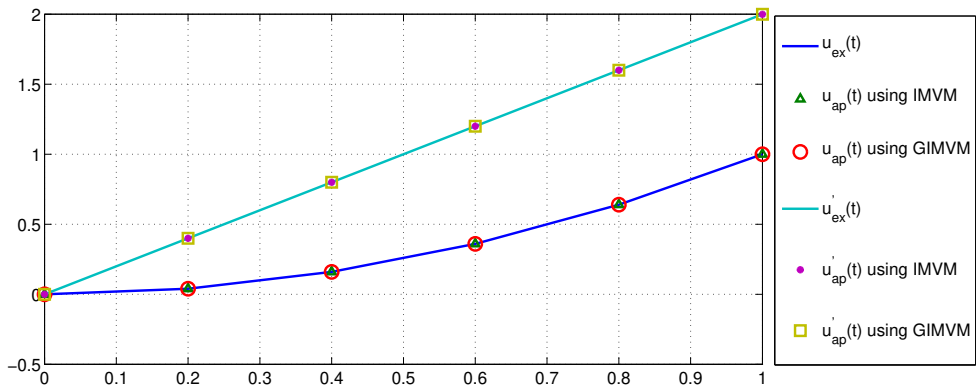
**Discussion:** In Figures 1, 2, 3 and 4, the graphical representations of the exact and numerical solutions of Example 1, 2, 3 and 4, respectively, appear that are almost identical. Furthermore, Tables 1 and 2, show us that the absolute error function  $E(t)$  approaches towards zero very quickly when using the methods **GIMVM** and **IMVM**, conversely, it approaches towards zero slowly when using the Nyström-Collocation method. Consequently, the comparison results confirm the efficiency and accuracy of the presented methods **GIMVM** and **IMVM** for solving the proposed Fredholm integro-differential equations.

$t$	$E(t)$ using (a)	$E(t)$ using (b)	$E(t)$ using (c)	$t$	$E(t)$ using (a)	$E(t)$ using (b)	$E(t)$ using (c)
0.2	3.01E-7	2.37E-9	1.26E-2	$\pi/10$	2.78E-13	1.41E-12	2.62E-3
0.4	3.67E-7	2.89E-9	1.54E-2	$\pi/5$	2.92E-13	1.95E-12	5.24E-3
0.6	4.49E-7	3.53E-9	1.88E-2	$3\pi/10$	3.07E-13	2.49E-12	7.86E-3
0.8	5.48E-7	4.32E-9	2.30E-2	$2\pi/5$	3.21E-13	3.02E-12	8.34E-3
Absolute errors for Example 1				Absolute errors for Example 2			

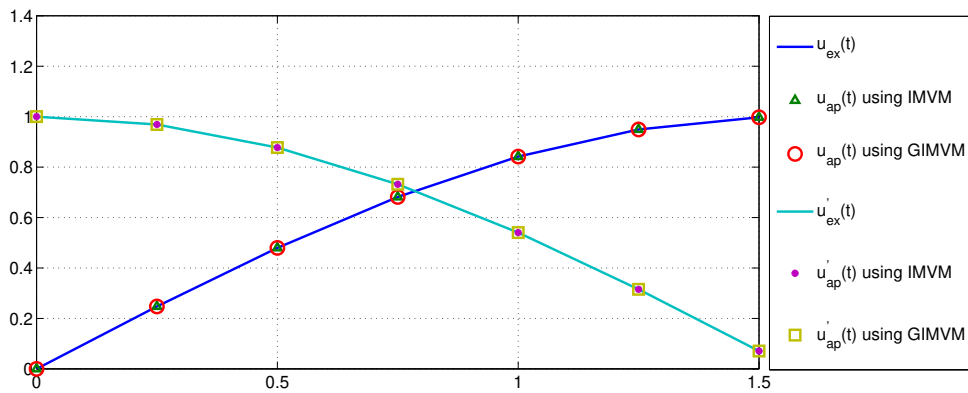
**Table 1:** Errors analysis using GIMVM (a), IMVM (b) and Nyström-Collocation method[5] (c).

$t$	$E(t)$ using (a)	$E(t)$ using (b)	$E(t)$ using (c)	$t$	$E(t)$ using (a)	$E(t)$ using (b)	$E(t)$ using (c)
0.4	1.75E-13	2.99E-14	4.38E-1	1.2	7.10E-12	2.33E-8	3.04E-1
0.8	2.61E-13	4.21E-14	6.54E-1	1.4	7.67E-12	2.52E-8	2.81E-1
1.2	3.88E-13	5.99E-14	9.76E-1	1.6	8.19E-12	2.70E-8	2.63E-1
1.6	5.80E-13	1.28E-13	1.45E-0	1.8	8.69E-12	2.86E-8	2.48E-1
Absolute errors for Example 3				Absolute errors for Example 4			

**Table 2:** Errors analysis using GIMVM (a), IMVM (b) and Nyström-Collocation method[5] (c).



**Fig. 1:** Plot of exact solutions versus numerical solutions of Example 1.



**Fig. 2:** Plot of exact solutions versus numerical solutions of Example 2.

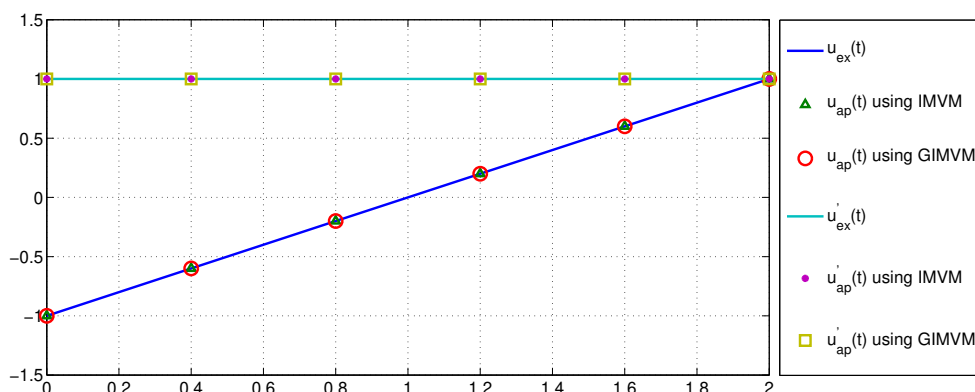


Fig. 3: Plot of exact solutions versus numerical solutions of Example 3.

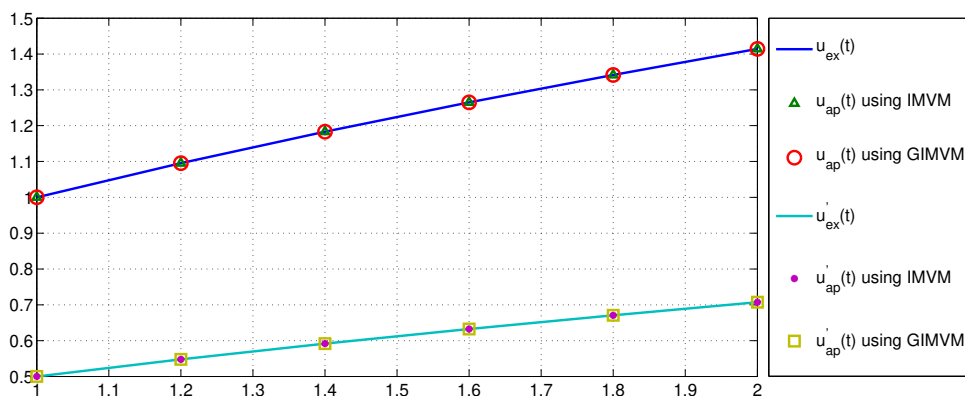


Fig. 4: Plot of exact solutions versus numerical solutions of Example 4.

## 6 Conclusion

In this work, based on the technique of the integral mean value theorem, we have proposed a numerical algorithm to solve a special classes of linear and nonlinear Fredholm integro-differential equations. Moreover, the illustrative examples confirm us the validity, simplicity and accuracy of this numerical algorithm.

## Declarations

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## References

- [1] M.A. Abdou, On a symptotic methods for Fredholm-Volterra integral equation of the second kind in contact problems, *J. Comput. Appl. Math.*, **154**(2003), 431–446.
- [2] A. Altürk, Numerical solution of linear and nonlinear Fredholm integral equations by using weighted mean-value theorem, *Springer Plus.*, **5**(2016), 1–15.
- [3] A. Altürk, S. Sahin, An application of the weighted mean value method to Fredholm integral equations with Toeplitz plus Hankel kernels, *J. Int. Approx. Sci. Comput.*, **2**(2017), 1–9.
- [4] I.K. Argyros, On a class of nonlinear integral equations arising in neutron transport, *Aequ. Math.*, **36**(1988), 99–111.
- [5] K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge university press, (1997).
- [6] Z. Avazzadeh, M. Heydari, G.B. Loghmani, Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem, *Appl. Math. Model.*, **35**(5)(2011), 2374–2383.
- [7] I. Aziz, S.U. Islam, New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, *J. Comput. Appl. Math.*, **239**(2013), 333–345.
- [8] E. Babolian, A. Shamsavaran, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, *J. Comput. Appl. Math.*, **225**(2009), 89–95.
- [9] K. Chanda, P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, (1989).
- [10] P. De Angelis, R. De Marchis, A.L. Martire, A new numerical method for a class of Volterra and Fredholm integral equations, *J. Comp. Appl. Math.*, **379**(2020), 112944.
- [11] P. De Angelis, R. De Marchis, A.L. Martire, S. Patri, Non-standard Volterra integral equations: a mean-value theorem numerical approach, *Appl. Math. Sci.*, **14**(2020), 423–432.
- [12] E. Haq, A.S. Khan, M. Ali, An efcient approach for solving the linear and nonlinear integro-diferential equation, *CRPASE.*, **06**(2020), 104–107.
- [13] J.H. He, Some applications of non linear fractional differential equations and their approximations, *Bull. Sci. Technol.*, **15**(1999), 86–90.
- [14] M. Heydari, Z. Avazzadeh, H. Navabpour, G.B. Loghmani, Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem II. High dimensional problems, *Appl. Math. Model.*, **37**(2013), 432–442.
- [15] S. Hu, M. Khavanin, W.A.N. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.*, **34**(1989), 261–266.
- [16] V. Lakshmikantham, M. Rao, *Theory of Integro-Differential Equations*, Gordon & Breach, London, (1995).
- [17] H. Li, J. Huang, A novel approach to solve nonlinear Fredholm integral equations of the second kind, *Springer Plus.*, **5**(2016), 1–9.
- [18] Z. Mahmoodi, J. Rashidinia, E. Babolian, B-spline collocation method for linear and nonlinear Fredholm and Voltterraintegro-diferential equations, *Appl. Anal.*, **92**(2012), 1787–1802.
- [19] N. Mikaeilvand, S. Noeiaghdam, Mean value theorem for integrals and its application on numerically solving of Fredholm integral equation of second kind with Toeplitz plus Hankel Kernel, *Int. J. Ind. Math.*, **6**(2014), 351–360.
- [20] A. Molabahrami, S. Niasadegh, Multistage Integral Mean Value Method for the Fredholm Integral Equations of the Second Kind, *Thai J. Math.*, **18**(2020), 751–763.
- [21] S. Noeiaghdam, Numerical solution of  $n$ -th order Fredholm integro-differential equations by integral mean value theorem method, *Int. J. Pure Appl. Math.*, **99**(2015), 277–287.
- [22] A.M. Wazwaz, *Linear and nonlinear integral equations*, Berlin, Springer, (2011).