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A New Parametric Kernel Function Based On p-Generalized Sigmoid Function For The Primal-Dual Interior Point Method Of Linear Optimization

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Abstract: For the purpose of linear optimization, we propose a novel parametric kernel function situated within the context of the primal-dual interior point methodology. This function incorporates a logarithmic barrier term and a p-generalized sigmoid function. To evaluate the complexity associated with iterations of the algorithm, we consider various simple cases and mild conditions. The results show that the iteration bounds for the small- and large-update interior point methods constructed with these functions are, respectively, given by $O\left(\frac{\sqrt{n}}{p}\log\left(\frac{n}{\varepsilon}\right)\right)$ and $O\left(\frac{n}{p^3}\log\left(\frac{n}{\varepsilon}\right)\right)$. By selecting a specific parameter p, the primal-dual interior point methods based on this kernel function can achieve an optimal iteration bound of $O\left(\sqrt{n}\log(n)\log\left(\frac{n}{\varepsilon}\right)\right)$ for large update methods. This bound is consistent with the known complexity results for linear and semidefinite optimization problems obtained from self-regular kernel functions. In order to demonstrate the effectiveness of the new kernel function, we present numerical results from several test problems, which confirm that the optimal number of iterations has been achieved.

Keywords: kernel function, interior-point algorithms, linear optimization, primal-dual methods.

2010 Mathematics Subject Classification. 90C51; 90C05.

1 Introduction

In recent years, kernel functions have gained significant popularity. The utilization of kernel functions has proven to be crucial and highly advantageous in various domains of mathematical programming research. A kernel function derived from interior point methods (IPMs) is regarded as one of the most efficient methodologies for executing interior point analysis and addressing the linear optimization problem (LO).

In 1947, George Dantzig [17] developed the simplex algorithm, a method used to solve linear programming. However, this approach was not considered highly efficient from a theoretical perspective due to its exponential arithmetic complexity, characterized by a computational order of $O(2^n)$ operations. This approach has subsequently gained a reputation for numerous improvements. It was formulated by multiple researchers until the emergence of the ellipsoid method by Leonid Genrikhovitch Khachiyan in 1978 [40]. Khachiyan's work demonstrated that the method displays polynomial complexity.

In 1984, Narendra Karmarkar [25] made a significant discovery about interior point methods, which are capable of solving linear optimization problems in polynomial time. This discovery coincided with the development of the ellipsoid method, but Karmarkar's approach proved to be more efficient. The fundamental concept behind this method involves the use of convex barrier functions, as determined by the problem's definition. Differing from the simplex algorithm, this

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specific technique reaches the optimal solution via the examination of the interior area of the feasible solution set.

Following Karmarkar's announcement, there has been an increase in the number of research papers on interior point methods published by the scientific community. This has led to the emergence of various algorithms, such as the central trajectory method (CTM), which was introduced in the early 1990s. Interior point methods have recently been the subject of several monographs, including Cornelis Roos, Tamas Terlaky, and Jean-Pierre Vial [39]; Stephen J. Wright [42]; Yinyu Ye [46]; James Renegar [37]; and Yurii Nesterov and Arkadij Nemirovskij [30]. Recently, a significant amount of research has been conducted on interior point techniques based on the concept of kernel functions. These studies include the works of Mohamed ElGhami et al ([6], [5], [22], [19], [21]), Peyghami et al ([35], [36]), Yun-Qing Bai et al ([6], [4], [5]), Mousaab Bouafia [11], Xiaopeng Cai [13], Behrouz Kheirfam [26], Yurii Nesterov [30], and Jiming Peng et al [31]. In their research contribution, Jiming Peng et al. [31] formulated interior point strategies for the linear optimization problem (LO) contingent upon the implementation of barrier functions. Similarly, Yun-Qing Bai et al ([6] , [4], [5]) introduced a class of dual-primal type interior point techniques for (LO) problems that rely on various non-self-regular kernel functions. It should be noted that these techniques achieved the same favorable complexity as the ones presented by Jiming Peng et al. [31] for both large and small step algorithms. Furthermore, Xiuyu Wang et al. extended the results obtained for (LO) problems in their works (Xiuyu Wang, [42]; Xiuyu Wang and Yun-Qing Bai, [44]). Lastly, there are other interior point algorithms based on kernel functions that have been explored in the literature, including the works of Yun-Qing Bai [8], Michel Klein [27], Xiuyu Wang ([43], [42]), Yin Zhang and Xiuyu Wang [48], and Jun Zhao [49]. It should be noted that the most optimal iteration limit for algorithms that take large steps has been derived thus far. The order of this limit is $O\left(\sqrt{n}\log(n)\log\left(\frac{n}{\varepsilon}\right)\right)$.

The logistic function, first presented by Pierre François Verhulst in a series of publications from 1838 to 1847, represents the primary and most renowned sigmoid function. Initially devised by Pierre, the logistic function served as a model for population growth. Currently, a variety of disciplines utilize sigmoid functions, including epidemiology, agriculture, neural networks, physics and chemistry, demography, economics, statistics, signal processing, tumor growth models, and most recently, a new type of interpolation formula employing the sigmoid function has emerged. This refined interpolation technique has helped to create innovative integration formulas and methodologies to solve ordinary differential equations. (For more details, see [3, 1, 12, 15, 14, 16, 23, 24]).

Many kernel functions in the existing literature demonstrate similar complexities. Nevertheless, each kernel function presents unique advantages depending on various factors within the interior point method. One significant factor to consider is the step size α, which notably affects the reduction in iteration numbers and relies on the formulation of the kernel function. Sigmoid functions possess characteristics such as boundedness, monotonicity, and a first derivative that illustrates bell-shaped properties. Additionally, these functions are constrained by horizontal asymptotes as the input converges toward positive or negative infinity. Sigmoid functions, part of the trigonometric functions category, are found in multiple equations including logistic, hyperbolic tangent, and arctangent functions. When integrated with kernel functions that incorporate a logarithmic barrier, sigmoid functions maintain the eligibility of the kernel function. In our innovative kernel proposition, we introduce the novel p-generalized sigmoid function with the parameter p, merging it with the logarithmic barrier. Remarkably, the parameter p significantly contributes to reducing the iteration count in interior point methods, particularly when it corresponds to the value of the parameter step size α.

This work advances both the theoretical foundations and practical implementations of primal-dual interior-point methods through the introduction of a novel parametric kernel function derived from the p-generalized sigmoid function. The proposed approach achieves an improved iteration complexity of $\mathcal{O}(\sqrt{n}\log n\log(n/\varepsilon))$ for large-update schemes, contributing to the development of efficient numerical optimization algorithms. By bridging analytical innovation with computational performance, this research addresses fundamental challenges in **applied mathematics** and modern computational science.

The manuscript is structured in the following manner. In Section 2, we provide a review of the preliminaries. Section 3 is dedicated to the presentation of sigmoid functions, starting with the classical sigmoid function and its properties. We then proceed to generalize this function by introducing a parameter p. The properties of the generalized sigmoid function are also presented. Moving on, in Sections 4 and 5, we define a novel kernel function that consists of a logarithmic barrier function and a p-generalized sigmoid function. The properties of this function, which are crucial for the complexity analysis, are determined. Section 6 provides a detailed explanation of how to estimate the step size and how the new barrier function behaves when robustness is decreased. Section 7 presents the complexity outcomes for both the large-update and small-update methods. In Section 8, we present various numerical findings. We will then wrap up the paper with a concluding section.

Let's introduce some notations first. We denote by $\mathbb{R}^n$ the Euclidean space of dimension n, endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $||\cdot||$. Note that the ensemble of non-negative (positive) vectors of dimension n is represented by $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$). In the context of $x, s \in \mathbb{R}^n$, the notation $x_{\min}$ represents the smallest element of the vector x, while the componentwise multiplication of the vectors x and s is expressed as xs. $X = \text{diag}(x)$ is an $n \times n$ diagonal matrix, with diagonal elements that are the components of the vector $x \in \mathbb{R}^n$. Last but not least, the n-dimensional vector of ones is denoted by e. If f and g are two functions defined from $\mathbb{R}^n_{++}$ to $\mathbb{R}^n_{++}$, then we write $f(x) = O(g(x))$ if there exists a positive constant C_1 such that $f(x) \leq C_1 g(x)$. We write $f(x) = \Theta(g(x))$ if there exist two positive constants C_1 and C_2 such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$.

2 Preliminaries

Let (P) be the standard form of a linear optimization problem (LO).

$$\min \{ \langle c, x \rangle : Ax = b, x \geq 0 \}, \tag{P}$$

where $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m \leq n$, $b \in \mathbb{R}^m$, and $x, c \in \mathbb{R}^n$. Let (D) be its dual problem

$$\max \left\{ \langle b, y \rangle : A^T y + s = c, \, s \geq 0 \right\}. \tag{D}$$

Interior point methods (IPMs) have garnered considerable attention in research ever since the introduction of Karmarkar's innovative approach for solving LO [25], and they have been extensively explored in the academic literature. For further insights and a comprehensive understanding of the subject matter, consult the works of YQ. Bai et al. [7], J. Peng et al. [31], C. Roos et al. [39], and Y. Ye [47]. It is reasonable to conclude that both (P) and (D) meet the interior point condition (IPC), indicating the existence of (x^0, y^0, s^0) such that.

$$Ax^0 = b, \, x^0 > 0, \, A^T y^0 + s^0 = c, \, s^0 > 0. \tag{1}$$

The optimality condition necessitates solving the following nonlinear system.

$$Ax = b, \, x \geq 0, \, A^T y + s = c, \, s \geq 0, \, xs = 0. \tag{2}$$

The fundamental concept of primal-dual interior point methods is to substitute the third equation in (2) with the parametric equation $xs = \mu e$, where μ is a non-negative parameter. As a result, the system (2) is reformulated into a parametric system.

$$Ax = b, \, x \geq 0, \, A^T y + s = c, \, s \geq 0, \, xs = \mu e. \tag{3}$$

Should the IPC be verified, the parameterized system (3) yields a unique solution denoted as $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$. This particular solution is known as a μ-center for both (P) and (D). The assemblage of μ-centers is visualized through their central paths for both (P) and (D). The initial recognition of the central path's relevance to the LO problem was attributed to Nimrod Megiddo [28] and György Sonnevend [41]. As μ tends to zero, the central path converges, and its limit points satisfy the complementarity condition, providing optimal solutions for both the primal-dual problems (P) and (D). Employing Newton's method on the parameterized system (3) with a fixed $\mu > 0$ allows the determination of the search direction $(\Delta x, \Delta y, \Delta s)$.

$$A\Delta x = 0, \, A^T \Delta y + \Delta s = 0, \, s\Delta x + x\Delta s = \mu e - xs. \tag{4}$$

The expression (4) exclusively characterizes $(\Delta x, \Delta y, \Delta s)$ when both $x > 0$ and $s > 0$ are applicable, a consequence of the matrix A possessing a full row rank. Executing an iterative exploration in the direction of $(\Delta x, \Delta y, \Delta s)$ results in a fresh triplet denoted as (x_+, y_+, s_+).

$$x_+ = x + \alpha \Delta x, \, s_+ = s + \alpha \Delta s, \, y_+ = y + \alpha \Delta y, \tag{5}$$

The application of specific rules is employed to ascertain the step size $\alpha \in (0, 1]$ during the iterative process, ensuring that the newly obtained iteration adheres to the condition $(x_+, y_+, s_+) > 0$.

Currently, we will introduce the scaled vector v as well as the scaled search directions d_x and d_s.

$$v = \sqrt{\frac{xs}{\mu}}, d_x = \frac{v\Delta x}{x}, d_s = \frac{v\Delta s}{s}, \tag{6}$$

consequently, the Newton system (4) can be redefined as below

$$\overline{A}d_x = 0, \overline{A}^T\Delta y + d_s = 0, d_x + d_s = v^{-1} - v, \tag{7}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X$, $V = \text{diag}(v)$, $X = \text{diag}(x)$, $S = \text{diag}(s)$.

Due to the fact that d_s and d_x belong to the row space of the matrix $\overline{A}$ and the null space respectively, we conclude that

$$d_x = d_s = 0 \Leftrightarrow v - v^{-1} = 0 \Leftrightarrow x = x(\mu), s = s(\mu).$$

The system (7) can be changed as follows

$$\overline{A}d_x = 0, \overline{A}^T\Delta y + d_s = 0, d_x + d_s = -\nabla\Phi(v), \tag{8}$$

With the following definition of the barrier function $\Phi(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$

$$\Phi(v) = \Phi(x, s; \mu) = \sum_{i=1}^{n} \psi(v_i), \tag{9}$$

$$\psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{10}$$

We call $\psi(t)$ the kernel function of the logarithmic barrier function $\Phi(v)$ where $\psi(t) : \mathbb{R}^n_{++} \to \mathbb{R}_+$, and the following condition is satisfied by $\psi(t)$, which is twice differentiable.

$$\psi(1) = \psi'(1) = 0, \psi''(t) > 0, \lim_{t \Rightarrow 0^+} \psi(t) = \lim_{t \Rightarrow \infty} \psi(t) = \infty.$$

When threshold values $\mu > 0$ and τ are employed, the function $\Phi(v)$ serves as the proximity measure to assess the distance of the current iteration from the μ-center. Furthermore, the norm-based proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is determined as follows

$$\delta(v) = \frac{1}{2}||\nabla\Phi(v)|| = \frac{1}{2}||d_x + d_s||. \tag{11}$$

In this article, the function $\Phi(v)$ is replaced by a strictly convex function $\Phi_p(v)$, $v \in \mathbb{R}^n_{++}$. This will be described in section 4 in a manner such that $\Phi_p(v)$ is minimal at $v = e$ and $\Phi_p(e) = 0$. The generic primal-dual interior point executes in a similar manner to Algorithm 1, in which the parameters τ, θ and the step size α should be selected to ensure that the algorithm is optimized in the sense that the minimum number of iterations is required.

Both in IPM theory and practice, the barrier update parameter θ is a key consideration. Usually, the procedure is known as a large-update (or long-step) approach if θ is a constant that is independent of the dimension n of the problem, such as $\theta = \frac{1}{2}$. The technique is known as a small-update (or short-step) approach if θ depends on the problem's dimension, for example, $\theta = \frac{1}{\sqrt{n}}$. The selection of the step size α is another key factor in the algorithm's analysis. (D) and (P) must be chosen so that the iterates' proximity to the current μ-center improves sufficiently.

3 Sigmoid functions and their properties

3.1 Classical sigmoid function.

The sigmoid function, shown in Figure 1 and defined by the given formula, is commonly referred to as the classical sigmoid function due to its distinctive "S" shaped-curve.

$$S(t) = \frac{1}{1 + \exp(-t)} = \frac{\exp(t)}{1 + \exp(t)}, \quad t \in \mathbb{R} \tag{12}$$

Generic Primal-dual IPMs for LO

Input:
A proximity the function $\Phi_p(v)$,
a threshold parameter $\tau > 1$
an accuracy parameter $\varepsilon > 0$,
a fixed barrier update parameter $\theta, 0 < \theta < 1$,
begin
$x = e$, $s = e$, $\mu = 1$, $v = e$.
while $n\mu \geq \varepsilon$ do
begin (outer iteration)
$\mu = (1-\theta)\mu$,
while $\Phi(x,s;\mu) > \tau$ do
begin (inner iteration)
solve the system (8), $\Phi(v)$ replaced by $\Phi_{New}(v)$ to obtain $(\Delta x, \Delta y, \Delta s)$,
choose a suitable step size α,
$x = x + \alpha\Delta x, y = y + \alpha\Delta y, s = s + \alpha\Delta s$
$v = \sqrt{\frac{xs}{\mu}}$,
end (inner iteration)
end (outer iteration)
end.

Algorithm 1. Generic algorithm

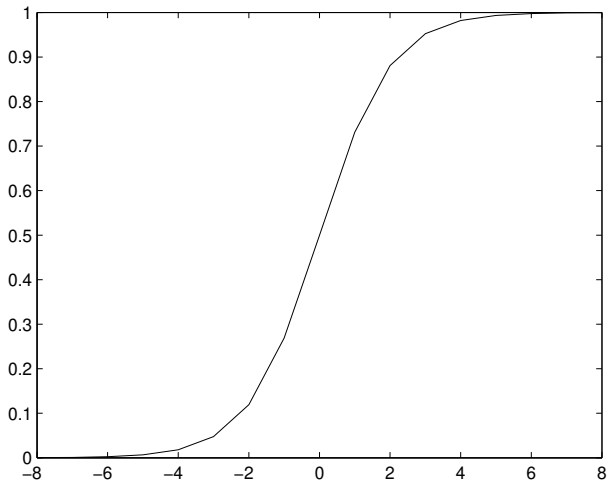

Fig. 1: The plot of logistic function $S(t)$.

3.1.1 Properties of classical sigmoid function

The properties of the classical sigmoid function are

1. The sigmoid function has a domain that covers the entire set of real numbers and produces output values within the interval $[0,1]$. It is characterized by its continuous and differentiable nature throughout its domain, with behavior that monotonically increases.
2. The sigmoid function has the following properties
   a) $\lim_{t\to-\infty} S(t) = 0$, $\lim_{t\to\infty} S(t) = 1$, $S(0) = \frac{1}{2}$, $S(-t) = 1 - S(t)$.

b) Its first and second derivatives are given as

$$S'(t) = \frac{\exp(t)}{(1+\exp(t))^2} = S(t)(1-S(t)), \tag{13}$$

$$S''(t) = S(t)(1-S(t))(1-2S(t)). \tag{14}$$

3. Additionally, the sigmoid function possesses the following properties:

$$S'(t) = S(t)S(-t), \qquad S'(t) = S'(-t), \tag{15}$$

$$\lim_{t\to-\infty} S'(t) = 0, \qquad \lim_{t\to\infty} S'(t) = 0, \tag{16}$$

$$\lim_{t\to 0} S'(t) = \frac{1}{4}, \qquad \int S(t)dt = \ln(1+\exp(t)) + C. \tag{17}$$

3.2 The p-generalized sigmoid function

It is clear from the definition of the classical sigmoid function that its range is the interval $]\frac{1}{2},1]$ for $t > 0$. To introduce more flexibility and control over the range of the function, a parameter p within the interval $[0,1]$ is introduced, leading to the p-generalized sigmoid function.

$$S_p(t) = \frac{p}{p+\exp(-pt)}, \qquad p > 0, \qquad t \in \mathbb{R} \tag{18}$$

Figure 2 illustrates the relationship between the range of $S_p(t)$ and the parameter p. Notably, the classical sigmoid function corresponds to a specific instance of the p-generalized sigmoid function, where $p = 1$.

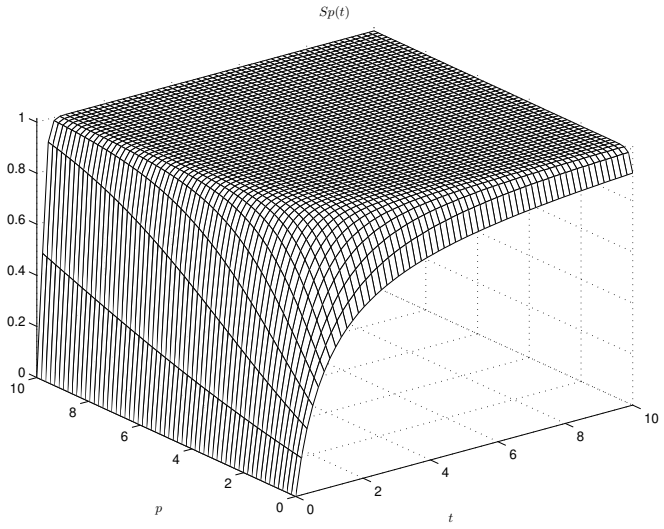

Fig. 2: The plot of the p-generalized Sigmoid function $S_p(t)$ in terms of p and t.

3.2.1 Properties of the p-generalized sigmoid function

1. The p-generalized sigmoid function retains the fundamental properties of the classical sigmoid function. It has a domain that spans all real numbers and a range confined to the interval $[0,1]$. Additionally, it is continuous and differentiable everywhere, exhibiting increasing behavior.
2. $\lim_{t\to-\infty} S_p(t) = 0$, $\lim_{t\to\infty} S_p(t) = 1$, $S_p(0) = \frac{p}{p+1}$.

3. The first and second derivatives of the p-generalized sigmoid function are given as

$$S'_p(t) = pS_p(t)(1 - S_p(t)), \tag{19}$$
$$S''_p(t) = pS_p(t)(1 - S_p(t))(1 - 2S_p(t)). \tag{20}$$

4. We also have, the following properties

$$\lim_{t\to-\infty} S'_p(t) = 0, \tag{21}$$
$$\lim_{t\to\infty} S'_p(t) = 0, \tag{22}$$
$$\lim_{t\to0} S'(t) = \frac{p^2}{(p+1)^2}, \tag{23}$$
$$\int S_p(t)dt = \frac{1}{p}\ln(1 + p\exp(pt)) + C. \tag{24}$$

4 The New Kernel Function and Its Properties

In this section, a novel kernel function is introduced, which is based on the p-generalized sigmoid function and utilized in the Generic Algorithm 1.

$$\psi_p(t) = pS_p(1)\frac{(t^2-1)}{2} - 2pS_p(1)\ln t + p\int_1^t S_p(x)\mathrm{d}x, \qquad t > 0, \qquad p > 1. \tag{25}$$

We note that the kernel function $\psi_p(t)$ is strictly convex for $t > 0$ and $p > 1$, since its second derivative $\psi''_p(t) = pS_p(1)(1 + 2/t^2) + pS'_p(t)$ is positive, as $S'_p(t) > 0$ and $S_p(1) = p/(p+1) > 0$.

To analyze the proposed kernel function $\psi_p(t)$, we first formalize two key properties essential for interior-point methods.

Definition 1(Strong Barrier Property). *A kernel function $\psi(t)$ has* strong barrier properties *if:*

$$\lim_{t\to0^+} \psi(t) = +\infty \quad \textit{and} \quad \lim_{t\to0^+} \frac{\psi(t)}{\tilde{\psi}(t)} = +\infty, \tag{26}$$

where $\tilde{\psi}(t)$ is a reference kernel. This ensures aggressive penalization near constraint boundaries, enforcing strict feasibility [5, 30].

Definition 2(Numerical Stability Near Constraints). *A kernel function $\psi(t)$ ensures* numerical stability near constraints *if*

$$|\psi'(t)| \to +\infty \quad \textit{and} \quad \psi''(t) \to +\infty \quad \textit{as} \quad t \to 0^+. \tag{27}$$

These conditions preserve the conditioning of Newton systems and mitigate round-off errors [38, 45].

*Remark.*A kernel function $\psi(t)$ has *strong barrier properties* if $\lim_{t\to0^+} \psi(t)/\tilde{\psi}(t) = \infty$ [33], and ensures *numerical stability* if $|\psi''(t)| \to \infty$ as $t \to 0^+$ [38]. Our $\psi_p(t)$ satisfies both; (i) $\lim_{t\to0^+} \psi_p(t)/\psi_{[30]}(t) = \infty$ implies stricter feasibility control, and (ii) rapid growth of $\psi''_p(t)$ improves conditioning near constraints. These properties theoretically predict; fewer iterations due to stronger centering (Table 1, Total It.) and better numerical stability in boundary regions (Table 2, Efficiency η).

Using the first and second derivatives of the p-generalized sigmoid function provided in the previous section, we can derive the three derivatives of $\psi_p(t)$ needed for further analysis.

$$\psi'_p(t) = pS_p(1)t - 2pS_p(1)t^{-1} + pS_p(t), \tag{28}$$
$$\psi''_p(t) = pS_p(1) + 2pS_p(1)t^{-2} + p^2S_p(t)(1 - S_p(t)), \tag{29}$$
$$\psi'''_p(t) = -4pS_p(1)t^{-3} + p^3S_p(t)(1 - S_p(t))(1 - 2S_p(t)). \tag{30}$$

We can easily verify that, as $t \to 0^+$ or $t \to +\infty$, then $\psi_p(t) \to +\infty$. We also have $\psi_p(1) = 0$ and $\psi'_p(1) = 0$, hence $\psi_p(t)$ is effectively a kernel function.

5 Eligibility of the New Kernel Function

In this section, we will present the eligibility conditions for the kernel function in the form of lemmas and propositions.

Lemma 1. *The new kernel function $\psi_p(t)$ defined by equation (25), satisfies the following properties for $t > 0$*

$$\psi_p'''(t) < 0, \tag{31}$$
$$\psi_p''(t) > 2, \tag{32}$$
$$t\psi_p''(t) + \psi_p'(t) > 0, \tag{33}$$
$$t\psi_p''(t) - \psi_p'(t) > 0. \tag{34}$$

Proof.

1. Based on the definition of the p-generalized sigmoid function, and for $t > 0$, it can be observed that

$$\frac{p}{p+1} < S_p(t) < 1, \tag{35}$$
$$0 < (1 - S_p(t)) < \frac{1}{p+1}, \tag{36}$$
$$-1 < (1 - 2S_p(t)) < \frac{-p}{p+1}, \tag{37}$$

   and from the formula of $\psi_p'''(t)$, given in (30), we conclude that (31) is satisfied.
2. As shown by the formula for $\psi_p''(t)$ in (29), it is clear that for $\psi_p''(t)$ to exceed 2, it is sufficient for the term $pS_p(1)$ to be greater than 2, given that the other terms are positive. From (35), we have $pS_p(1) > \frac{p^2}{p+1}$, so

$$\frac{p^2}{p+1} > 2 \Leftrightarrow p^2 - 2p - 2 > 0.$$

   Through an analysis of the roots of this polynomial in terms of p, while considering the positive nature of p, it is established that the polynomial is positive when $p > 1 + \sqrt{2}$.
3. From (28), (29), and the range of $S_p(t)$ for $t > 0$ we obtain

$$\begin{aligned} t\psi_p''(t) + \psi_p'(t) &= pS_p(1)t + 2pS_p(1)t^{-1} + p^2S_p(t)(1 - S_p(t))t \\ &\quad + pS_p(1)t - 2pS_p(1)t^{-1} + pS_p(t) \\ &= [2pS_p(1) + p^2S_p(t)(1 - S_p(t))]t + pS_p(t) \\ &> 0. \end{aligned}$$

4. From (28), (29), and the range of $S_p(t)$ for $t > 0$ we obtain

$$\begin{aligned} t\psi_p''(t) - \psi_p'(t) &= pS_p(1)t + 2pS_p(1)t^{-1} + p^2S_p(t)(1 - S_p(t))t \\ &\quad - pS_p(1)t + 2pS_p(1)t^{-1} - pS_p(t) \\ &= 4pS_p(1)t^{-1} + p^2S_p(t)(1 - S_p(t))t - pS_p(t) \\ &> 4pS_p(1)t^{-1} + p^2S_p(t)(1 - S_p(t))t - p, \qquad (S_p(t) < 1). \end{aligned}$$

   If we define $h(x) = x(1-x)$ and examine this function over the interval $[0, 1]$, we observe that it reaches its maximum value at $x = \frac{1}{2}$. Furthermore, since $S_p(t) > \frac{p}{p+1} > \frac{1}{2}$ for $p > 1$, we can deduce that:

$$4pS_p(1)t^{-1} + p^2S_p(t)(1 - S_p(t))t - p > 4pS_p(1)t^{-1} + \frac{p^2}{4}t - p. \tag{38}$$

   The function $g(t) = 4pS_p(1)t^{-1} + \frac{p^2}{4}t - p$ exhibits the following limits: $\lim_{t\to+\infty} g(t) = +\infty$ and $\lim_{t\to 0} g(t) = +\infty$. Thus, it can be concluded that g has a minimum and this minimum occurs when $g'(t_0) = 0$. By calculating the derivative

$g'(t)$, we find that $g'(t) = \frac{p^2}{4} - 4pS_p(1)t^{-2}$. The next step is to calculate t_0.

$$\begin{aligned} g'(t_0) = 0 \Leftrightarrow \frac{p^2}{4} - 4pS_p(1)t_0^{-2} &= 0 \\ \Leftrightarrow t_0^2 &= \frac{16pS_p(1)}{p^2} \\ \Leftrightarrow t_0 &= \frac{4\sqrt{pS_p(1)}}{p}. \end{aligned}$$

We now proceed to establish the positive nature of the minimum $g(t_0)$.

$$\begin{aligned} g(t_0) &= \frac{4p^2S_p(1)}{4\sqrt{pS_p(1)}} + \frac{4p^2\sqrt{pS_p(1)}}{4p} - p \\ &= p^2\sqrt{pS_p(1)} + p\sqrt{pS_p(1)} - p \\ &= p[p\sqrt{pS_p(1)} + \sqrt{pS_p(1)} - 1] \\ &> p[(p+1)\sqrt{\frac{p^2}{p+1}} - 1] \qquad \left(S_p(1) > \frac{p}{p+1}\right) \\ &= p[\frac{p(p+1)}{\sqrt{p+1}} - 1] \\ &> p[\frac{p(p+1)}{p+1} - 1] \qquad \left(\frac{1}{\sqrt{p+1}} > \frac{1}{p+1}\right) \\ &= p(p-1) > 0, \qquad (p > 1). \end{aligned}$$

Finally, we have proved that $g(t_0) > 0$, so (34) is satisfied.

Remark(Derivative Properties and Stability). The kernel $\psi_p(t)$ admits derivatives:

$$\psi_p'(t) = pS_p(1)\left(t - \frac{2}{t}\right) + pS_p(t), \tag{39}$$

$$\psi_p''(t) = pS_p(1)\left(1 + \frac{2}{t^2}\right) + pS_p'(t), \tag{40}$$

with critical stability properties

a)**Controlled growth**: $|\psi_p'(t)| \le p(1 + 2/t)$ (since $S_p(t) \in [0,1]$)
b)**Convexity**: $\psi_p''(t) > 2pS_p(1)/t^2 > 0$ (as $S_p'(t) > 0$)

*Proof.*For $t \to 0^+$, the dominant term $-2pS_p(1)/t$ in ψ_p' ensures barrier properties without explosion ($\sim \mathcal{O}(1/t)$). For ψ_p'', the term $2pS_p(1)/t^2$ guarantees convexity and prevents ill-conditioning. Exponential kernels [20] diverge as $e^{1/t}$, while trigonometric kernels [11] introduce oscillatory singularities.

Lemma 2. *For $\psi(t)$, we have*

$$(t-1)^2 \le \psi_p(t) \le \frac{1}{4}[\psi_p'(t)]^2, \qquad t > 0 \tag{41}$$

$$\psi_p(t) \le \frac{1}{2}[S_p(1)(3p + p^2(1 - S_p(1))](t-1)^2, \qquad t > 1 \tag{42}$$

Proof. For (41), using (32), we have

$$\psi_p(t) = \int_1^t \int_1^x \psi_p''(y)dydx \ge \int_1^t \int_1^x 2dydx = (t-1)^2.$$

$$\begin{aligned}\psi_p(t) &= \int_1^t \int_1^x \psi_p''(y)dydx \leq \int_1^t \int_1^x \frac{1}{2}\psi_p''(x)\psi_p''(y)dydx \\ &= \frac{1}{2}\int_1^t \psi_p''(x)\psi_p'(x)dx \\ &= \frac{1}{2}\int_1^t \psi_p'(x)d\psi_p'(x) = \frac{1}{4}\left[\psi_p'(t)\right]^2.\end{aligned}$$

By using Taylor's theorem and substituting $\psi_p(1) = \psi_p'(1) = 0$, $\psi_p'''(t) < 0$, we have

$$\begin{aligned}\psi_p(t) &= \psi_p(1) + \psi_p'(1)(t-1) + \frac{1}{2}\psi_p''(1)(t-1)^2 + \frac{1}{6}\psi_p'''(\xi)(t-1)^3 \\ &\leq \frac{1}{2}\psi_p''(1)(t-1)^2 \\ &= \frac{1}{2}\left[pS_p(1) + 2pS_p(1) + p^2S_p(1)(1-S_p(1))\right](t-1)^2, \\ &= \frac{1}{2}\left[S_p(1)(3p + p^2(1-S_p(1))\right](t-1)^2\end{aligned}$$

for some ξ, $1 \leq \xi \leq t$.

We now apply the steps explained in [4] to analyze the generic algorithm.

Step 1 Let $\sigma : [0,\infty[\to [1,+\infty[$ be the opposite function of $\psi_p(t)$ for $t \geq 1$ and $\rho : [0,\infty[\to]0,1]$ be the opposite function of $-\frac{1}{2}\psi_p'(t)$ for all $t \in]0,1]$. We will now introduce the subsequent proposition.

Lemma 3. *For $\psi_p(t)$, we have*

$$1 + \sqrt{\frac{2s}{S_p(1)(3p + p^2(1-S_p(1))}} \leq \sigma(s) \leq 1 + \sqrt{s}, \qquad s \geq 0 \tag{43}$$

$$\rho(z) \geq \frac{pS_p(1)}{z + pS_p(1)} > \frac{p^2}{(p+1)(p+z)}. \tag{44}$$

Proof. For (43), we put $s = \psi_p(t)$, $t \geq 1$, i.e., $\sigma(s) = t, t \geq 1$.
By (41) we have $s \geq (t-1)^2$, so $t = \sigma(s) \leq 1 + \sqrt{s}$
By (42), we have

$$\psi_p(t) \leq \frac{1}{2}[S_p(1)(3p + p^2(1-S_p(1))](t-1)^2, \qquad t \geq 1, \text{so}, t = \sigma(s) \geq 1 + \sqrt{\frac{2s}{S_p(1)(3p + p^2(1-S_p(1))}}$$

For (44), let $z = -\frac{1}{2}\psi_p'(t)$, $t \in]0,1]$. By the definition of $\psi_p'(t)$, we have:

$$\begin{aligned}2z &= -pS_p(1)t + 2pS_p(1)t^{-1} - pS_p(t) \\ (for\ t < 1 : \frac{p}{p+1} \leq S_p(t) \leq S_p(1) < 1) \quad &\geq -pS_p(1) + 2pS_p(1)t^{-1} - pS_p(1) \\ &= 2pS_p(1)\frac{1}{t} - 2S_p(1), \\ \text{so}, 2z + 2pS_p(1) &\geq 2pS_p(1)\frac{1}{t} \\ &\Rightarrow t = \rho(z) \geq \frac{pS_p(1)}{z + pS_p(1)} > \frac{p^2}{(p+1)(p+z)}.\end{aligned}$$

Step 2 Establish a lower bound for σ employing the p-generalized sigmoid function Φ_p.

Lemma 4. *Let* $\sigma : [0,\infty[\to [1,+\infty[$ *be the inverse function of* $\psi_p(t)$ *for* $t \geq 1$. *Then we have*

$$\Phi_p(\beta v) \leq n\psi_p\left(\beta\sigma\left(\frac{\Phi_p(v)}{n}\right)\right),\ v \in \mathbb{R}^*,\ \beta \geq 1.$$

Proof. The result can be derived using (31) and (34), along with Lemma 2.4 from [6].

Lemma 5. *For* $0 \leq \theta < 1$, *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$. *If* $\Phi_p(v) \leq \tau$, *then we have*

$$\Phi_p(v_+) \leq \frac{2\sqrt{\tau n} + \tau + \theta n}{(1-\theta)}.$$

Proof. Given that $\dfrac{1}{\sqrt{1-\theta}} \geq 1$ and $\sigma\left(\dfrac{\Phi_p(v)}{n}\right) \geq 1$, consequently $\dfrac{\sigma\left(\dfrac{\Phi_p(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. And for $t \geq 1$, we have $\psi_p(t) \leq t^2 - 1$. Using Lemma 4 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (43), and $\Phi_p(v) \leq \tau$, we have

$$\begin{aligned}
\Phi_p(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi_p(v)}{n}\right)\right)\\
&\leq n\left(\left[\frac{\sigma\left(\dfrac{\Phi_p(v)}{n}\right)}{\sqrt{1-\theta}}\right]^2 - 1\right)\\
&= \frac{n}{(1-\theta)}\left(\left[\sigma\left(\frac{\Phi_p(v)}{n}\right)\right]^2 - (1-\theta)\right)\\
&\leq \frac{n}{(1-\theta)}\left(\left[1+\sqrt{\frac{\Phi_p(v)}{n}}\right]^2 - (1-\theta)\right)\\
&\leq \frac{n}{(1-\theta)}\left(2\sqrt{\frac{\tau}{n}} + \frac{\tau}{n} + \theta\right)\\
&= \frac{2\sqrt{\tau n} + \tau + \theta n}{(1-\theta)}.
\end{aligned}$$

Now the proof is complete.

If we put

$$(\Phi_p)_0 = \frac{2\sqrt{\tau n} + \tau + \theta n}{(1-\theta)} = L(n,\theta,\tau), \tag{45}$$

consequently the upper bound for $\Phi_p(v_+)$ via the algorithm's execution is $(\Phi_p)_0$.

6 The Step Size Estimation

In this section, we examine the decrease in the barrier function following a damped step, while taking into consideration a default step size α. After a damped step, we have

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y,\ s_+ = s + \alpha\Delta s.$$

Using (6), we have

$$\begin{aligned}
x_+ &= x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x),\\
s_+ &= s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).
\end{aligned}$$

As a result, we obtain $v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v+\alpha d_x)(v+\alpha d_s)}$. Define for $\alpha > 0$, $f(\alpha) = \Phi_p(v_+) - \Phi_p(v)$.

For a given fixed value of μ, the function $f(\alpha)$ indicates the variation in closeness between a new iteration and its preceding counterpart. Utilizing equation (33), we obtain

$$\Phi_p(v_+) = \Phi_p\left(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\right) \leq \frac{1}{2}\left(\Phi_p(v+\alpha d_x) + \Phi_p(v+\alpha d_s)\right).$$

As a result, we obtain the relation $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}\left(\Phi_p(v+\alpha d_x) + \Phi_p(v+\alpha d_s)\right) - \Phi_p(v). \tag{46}$$

Evidently, $f(0) = f_1(0) = 0$. Using the first two derivatives of $f_1(\alpha)$ with respect to α, we obtain

$$f_1'(\alpha) = \sum_{i=1}^{n}\left(\psi_p'(v_i+\alpha d_{x_i})d_{x_i} + \psi_p'(v_i+\alpha d_{s_i})d_{s_i}\right),$$
$$f_1''(\alpha) = \sum_{i=1}^{n}\left(\psi_p''(v_i+\alpha d_{x_i})d_{x_i}^2 + \psi_p''(v_i+\alpha d_{s_i})d_{s_i}^2\right).$$

Using (6) and (11), we have

$$f_1'(0) = \frac{1}{2}\langle\nabla\Phi_p(v),(d_x+d_s)\rangle = -\frac{1}{2}\langle\nabla\Phi_p(v),\nabla\Phi_p(v)\rangle = -2\delta(v)^2.$$

For simplicity of application, we define $v_1 = \min(v)$, $\delta = \delta(v)$, $\Phi_p = \Phi_p(v)$.

Lemma 6. *Let (11) be the definition of $\delta(v)$. Next, we obtain*

$$\delta(v) \geq \sqrt{\Phi_p(v)}. \tag{47}$$

Proof. Using (41), we have

$$\Phi_p(v) = \sum_{i=1}^{n}\psi_p(v_i) \leq \sum_{i=1}^{n}\frac{1}{4}\left[\psi_p'(v_i)\right]^2 = \frac{1}{4}||\nabla\Phi_p(v)||^2 = \delta(v)^2,$$

so $\delta(v) \geq \sqrt{\Phi_p(v)}$.

Remark. In this study, we assume that $\tau \geq 1$. According to lemma 6 and the supposition that $\Phi_p(v) \geq \tau$, we have $\delta(v) \geq 1$.

Since $\psi_p(t)$ is a kernel function and $\psi_p''(t)$ is monotonically decreasing, we may derive the following Lemmas 4.3-4.67-10, from Lemmas 4.1-4.4 in [6] .

Lemma 7.*[Bai et al.[6]] Let $f_1(\alpha)$ be defined as in (46), and $\delta(v)$ be defined as in (11). We can observe that $f_1''(\alpha) \leq 2\delta^2\psi_p''(v_{min} - 2\alpha\delta)$. Since $f_1(\alpha)$ is convex, it implies that $f_1'(\alpha) \leq 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa.*

The three following lemmas are generated by the previous lemma:

Lemma 8.*[Bai et al.[6]] α must satisfy the inequality below for $f_1'(\alpha) \leq 0$ to hold.*

$$\psi_p'(v_{min}) - \psi_p'(v_{min} - 2\alpha\delta) \leq 2\delta \tag{48}$$

Lemma 9.*[Bai et al.[6]] Assuming that (48) is satisfied, the largest step size is expressed as follows*

$$\overline{\alpha} = \frac{\rho(\delta) - \rho(2\delta)}{2\delta}.$$

Lemma 10.*[Bai et al. [6]] Assume that $\overline{\alpha}$ is as described in lemma 9. Then*

$$\overline{\alpha} \geq \frac{1}{\psi_p''(\rho(2\delta))}.$$

Lemma 11. *By the definition of ρ and the inequality from Lemma 10, verified by $\overline{\alpha}$, when $\Phi_p(v) \geq \tau \geq 1$, we obtain*

$$\overline{\alpha} \geq \frac{pS_p(1)}{p^2S_p^2(1)+2(pS_p(1)+2\delta)^2+p^3S_p(1)(p+1)}$$

.

Proof. We have $\rho(2\delta) > \dfrac{pS_p(1)}{2\delta+pS_p(1)} > \dfrac{p^2}{(p+1)(p+2\delta)}$, so $\dfrac{1}{\rho(2\delta)^2} < \dfrac{(pS_p(1)+2\delta)^2}{p^2S_p^2(1)}$.
Using the previous lemma, replacing ψ_p'' and recalling that $0 < S_p(t) < 1$ we obtain

$$\begin{aligned}
\overline{\alpha} &\geq \frac{1}{\psi_p''(\rho(2\delta))} \\
&= \frac{1}{pS_p(1)+2pS_p(1)(\rho(2\delta))^{-2}+p^2S_p(\rho(2\delta))(1-S_p(\rho(2\delta))} \\
&\geq \frac{p^2S_p^2(1)}{p^3S_p^3(1)+2pS_p(1)(pS_p(1)+2\delta)^2+p^4S_p^2(1)(S_p(2\delta)(1-S_p(2\delta)))} \\
&\geq \frac{pS_p(1)}{p^2S_p^2(1)+2(pS_p(1)+2\delta)^2+p^3S_p(1)(p+1)}.
\end{aligned}$$

Denoting

$$\widetilde{\alpha} = \frac{pS_p(1)}{p^2S_p^2(1)+2(pS_p(1)+2\delta)^2+p^3S_p(1)(p+1)}. \tag{49}$$

We know that the default step size is $\widetilde{\alpha}$, and $\widetilde{\alpha} \leq \overline{\alpha}$ is the relation between these two values.

Lemma 12.*[Lemma 3.12 in [31]] If we consider a twice differentiable and convex function h that satisfies $h(0) = 0$ and $h'(0) < 0$, and attains its minimum at $t^* > 0$, where h'' is increasing for $t \in [0,t^*]$, then*

$$h(t) \leq \frac{th(0)}{2},\ 0 \leq t \leq t^*.$$

The next result is of interest in this context.

Lemma 13.*[Lemma 4.5 in [6]] When the step size α satisfies $\alpha \leq \overline{\alpha}$, we get*

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 14. *For $\Phi_p(v) \geq 1$ and let $\widetilde{\alpha}$ be the default step size as defined in (49), we obtain*

$$f(\widetilde{\alpha}) \leq -\frac{pS_p(1)}{\left(p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)\right)}. \tag{50}$$

Proof. Since $\Phi_p(v) \geq 1$, then from (47), we have

$$\delta \geq \sqrt{\Phi_p(v)} \geq 1.$$

Using Lemma 13 (Lemma 4.5 in [6]) with $\alpha = \widetilde{\alpha}$ and (49), we have

$$\begin{aligned}
\widetilde{\alpha}\delta^2 &= \frac{pS_p(1)\delta^2}{p^2S_p^2(1)+2(pS_p(1)+2\delta)^2+p^3S_p(1)(p+1)} \\
&\geq \frac{pS_p(1)\delta^2}{p^2S_p^2(1)\delta^2+2(pS_p(1)\delta^2+2\delta)^2+p^3S_p(1)(p+1)\delta^2} \\
&\geq \frac{pS_p(1)\delta^2}{\left(p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)\right)\delta^2} \\
&= \frac{pS_p(1)}{\left(p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)\right)}.
\end{aligned}$$

This completes the proof.

7 Iteration Complexity

7.1 Inner iteration bound

We now have,

$$\Phi_p(v_+) \leq (\Phi_p)_0 = \frac{2\sqrt{n\tau}+\tau+n\theta}{(1-\theta)} = L(n,\theta,\tau).$$

It is essential to monitor the count of inner iterations necessary to revert to a condition where $\Phi_p(v_+) \leq \tau$ after transitioning from μ to $(1-\theta)\mu$. After the adjustment of μ, the starting value of $\Phi_p(v)$ is represented as $(\Phi_p)_0$. The values that follow within the same outer iteration are documented as $(\Phi_p)_k$, where k ranges from 1 to K, with K representing the total number of inner iterations for that outer iteration. For every inner iteration, (50) shows the decrease in value. We can determine the appropriate values of κ and $\gamma \in]0,1]$ in [6], which are given, respectively, by

$$\kappa = \frac{\left(p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)\right)}{pS_p(1)}, \gamma = 1-0 = 1.$$

Lemma 15. *In the outer iteration, let K represent the total number of inner iterations. Next, we have*

$$K \leq \frac{pS_p(1)\,(\Phi_p)_0}{p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)}.$$

Proof. By Lemma 1.3.2 in [31], we have

$$K \leq \frac{\left[(\Phi_p)_0\right]^\gamma}{\kappa\gamma} = \frac{pS_p(1)\,(\Phi_p)_0}{p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)}.$$

This completes the proof.

7.2 Total iteration bound

The preceding formula $\frac{\log\left(\frac{n}{\varepsilon}\right)}{\theta}$ limits the number of outer iterations (see [39] Lemma II.17, page 116). We establish an upper bound for the total iterations by calculating the product of the outer iterations and the inner iterations. Thus,

$$\frac{pS_p(1)\,(\Phi_p)_0}{p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)}\frac{\log\left(\frac{n}{\varepsilon}\right)}{\theta}. \tag{51}$$

For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have

$$O\left(\frac{n}{p^3}\log\left(\frac{n}{\varepsilon}\right)\right) \text{ iterations complexity.}$$

It is important to emphasize that the parameter p satisfies $p > 1$, as it stems from the structure of the proposed generalized sigmoid-based kernel function. Therefore, the term $\frac{n}{p^3}$ is strictly less than n, implying that the derived iteration complexity

$$O\left(\frac{n}{p^3}\log\left(\frac{n}{\varepsilon}\right)\right)$$

is asymptotically better than the standard $O\left(n\log\left(\frac{n}{\varepsilon}\right)\right)$ bound obtained with classical kernel functions. This result highlights the theoretical improvement in efficiency provided by the proposed approach, especially when p remains independent of n and is appropriately chosen.

For a small-update method, we have $\tau = O(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. When we substitute these values into (51), the resulting bound is not optimal. An improved bound can be derived as follows.

By (42), (43) with $\psi_p(t) \leq \frac{1}{2}\left[S_p(1)\left(3p+p^2(1-S_p(1))\right)\right](t-1)^2$, $t>1$, we have

$$\begin{aligned}
\Phi_p(v_+) &\leq n\psi_p\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi_p(v)}{n}\right)\right)\\
&\leq \frac{n\left[S_p(1)\left(3p+p^2(1-S_p(1))\right)\right]}{2}\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi_p(v)}{n}\right)-1\right)^2\\
&= \frac{n\left[S_p(1)\left(3p+p^2(1-S_p(1))\right)\right]}{2(1-\theta)}\left(\sigma\left(\frac{\Phi_p(v)}{n}\right)-\sqrt{1-\theta}\right)^2\\
&\leq \frac{n\left[S_p(1)\left(3p+p^2(1-S_p(1))\right)\right]}{2(1-\theta)}\left(1+\sqrt{\frac{\Phi_p(v)}{n}}-\sqrt{1-\theta}\right)^2\\
&\leq \frac{S_p(1)\left(3p+p^2(1-S_p(1))\right)}{2(1-\theta)}\left(\theta\sqrt{n}+\sqrt{\tau}\right)^2,
\end{aligned}$$

where we also used that $1-\sqrt{1-\theta}=\frac{\theta}{1+\theta}\leq\theta$ and $\Phi_p(v)\leq\tau$. Using this upper bound for $(\Phi_p)_0$, we get the following iteration bound

$$\frac{pS_p(1)(\Phi_p)_0}{p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)}\frac{\log\left(\frac{n}{\varepsilon}\right)}{\theta}.$$

Note now that $(\Phi_p)_0 = O\left(S_p(1)\left(3p+p^2(1-S_p(1))\right)\right) = O\left(p^2\right)$, and the iteration bound becomes

$$O\left(\frac{pS_p^2(1)\left(3p+p^2(1-S_p(1))\right)}{p^2S_p^2(1)+2(pS_p(1)+2)^2+p^3S_p(1)(p+1)}\sqrt{n}\log\left(\frac{n}{\varepsilon}\right)\right) = O\left(\frac{\sqrt{n}}{p}\log\left(\frac{n}{\varepsilon}\right)\right)$$

iterations complexity.

It is worth noting that although $(\Phi_p)_0 = O(p^2)$, the denominator in the iteration bound expression grows as $O(p^3)$, resulting in the simplified total iteration complexity

$$O\left(\frac{\sqrt{n}}{p}\log\left(\frac{n}{\varepsilon}\right)\right),$$

which improves as p increases, consistent with the structure of the proposed kernel function.

8 Numerical Tests

We consider the problem given in the paper by Bouafia [11]

$$n=2m,\ A(i,j)=\begin{cases}0 \text{ if } i\neq j \text{ and } j\neq i+m\\ 1 \text{ if } i=j \text{ or } j=i+m\end{cases}$$

$c(i)=-1, c(i+m)=0, b(i)=2$, and the interior point condition (IPC), $x^0(i)=x^0(i+m)=1$, $y^0(i)=-2$, $s^0(i)=1$, $s^0(i+m)=2$ for $i=1,\ldots,m$.

The experiments were carried out on a standard personal computer featuring an Intel(R) Core(TM) i5-2520M Processor and 4GB of RAM. To showcase the efficacy of our novel kernel function and assess its influence on the algorithm's performance, we performed comparative numerical tests against alternative kernels utilizing Matlab R2013a. The parameters were configured with $\varepsilon=10^{-4}$ and $\mu^0=1.5$. A summary of the findings of this numerical analysis is presented in Table 1. Table 2 presents a summary of key performance improvements.

Inspired by established practices for algorithmic benchmarking in optimization [18,29], and building upon efficiency ratio concepts [2], we introduce two complementary indicators to evaluate performance relative to reference methods:

the efficiency ratio $\mathscr{E} = \dfrac{\text{Iter}_{\text{ref}}}{\text{Iter}_{\text{New}}} \times \dfrac{\text{Time}_{\text{ref}}}{\text{Time}_{\text{New}}}$.

These indicators jointly capture both computational efficiency and iteration reduction, providing an integrated measure to assess the practical advantage of the proposed $\psi_p(t)$ kernel relative to existing methods.

Kernel functions	θ	Inner It.	Outer It.	Total It.	Time (s)	η	$\mathscr{E}$	Reference
$\frac{t^2-1-\log(t)}{2} + \frac{t^{1-q}-1}{2(q-1)}$	0.25	9,412	50	470,600	2.3624	0.10	0.08	[10]
	0.5	16,182	21	339,822	3.8169	0.06	0.05	
	0.99	32,502	4	130,008	7.9097	0.03	0.02	
$\frac{1}{p+1}(t^{p+1}-1) + \frac{t^{1-q}-1}{q-1}$	0.25	1,441	50	72,050	0.3696	0.65	0.71	[5]
	0.5	1,758	21	36,918	0.4546	0.53	0.58	
	0.99	845	4	3,380	0.2767	1.11	0.85	
$\frac{1}{2}(t^2-1) + \frac{t^{1-q}-1}{q-1}$	0.25	1,638	50	81,900	0.2177	0.73	0.80	[34]
	0.5	1,948	21	40,908	0.2461	0.64	0.70	
	0.99	919	4	3,676	0.1213	1.36	1.02	
$(t^2-1) - \frac{t^{-2m+1}-1}{-2m+1} - \frac{t^{-m+1}-1}{-m+1}$	0.25	1,134	50	56,700	0.1801	1.05	1.15	[9]
	0.5	1,443	21	30,303	0.2215	0.86	0.94	
	0.99	830	4	3,320	0.1271	1.50	1.13	
$\frac{1}{p+1}(t^{p+1}-1) + \frac{\exp(-\sigma(t-1))-1}{\sigma}$	0.25	18,993	50	949,650	4.0729	0.06	0.05	[20]
	0.5	20,006	21	420,126	4.3084	0.05	0.04	
	0.99	7,066	4	28,264	1.5612	0.14	0.10	
$\frac{t^2-1}{2} - \frac{4}{p\pi}(\tan(\frac{\pi}{(2t+2)}))^p$	0.25	12,847	50	642,350	2.0395	0.09	0.08	[11]
	0.5	13,125	21	275,625	2.1449	0.09	0.07	
	0.99	11,596	4	46,384	1.6880	0.10	0.08	
$\psi_p(t)$	0.25	**892**	50	**44,600**	**0.2191**	**1.00**	**1.00**	New
	0.5	**926**	21	**19,446**	**0.2194**	**1.00**	**1.00**	
	0.99	**717**	4	**2,868**	**0.1854**	**1.52**	**1.15**	

Table 1: Performance comparison for $\varepsilon = 10^{-4}, n = 100$ with efficiency metrics η and $\mathscr{E}$.

Table 2: Performance comparison of the proposed method against state-of-the-art approaches.

Comparison Metric	**Value**	**Reference**	**Improvement**
Iteration Count ($\theta = 0.99$)	2,868	[34]	28% reduction
Efficiency Ratio η ($\theta = 0.99$)	1.52	[34]	52% improvement
Computation Time ($\theta = 0.25$)	0.1801 s	[9]	3% faster
Overall Effectiveness $\mathscr{E}$ ($\theta = 0.99$)	1.15	[9]	15% better

Note: All experiments were conducted with $\varepsilon = 10^{-4}$ on $n = 100$ dimensional problems. Baseline values for normalization: $\eta = 1.00$, $\mathscr{E} = 1.00$. Gray rows highlight the most significant improvements.

8.1 Discussion

Table 1 presents an enhanced comparative analysis of kernel functions applied to interior-point methods for linear programming, with $\varepsilon = 10^{-4}$ and $n = 100$. In addition to iteration counts and CPU time, two new indicators are included to better interpret performance: efficiency ratio η and effectiveness.

The proposed kernel function $\psi_p(t)$ consistently achieves the lowest total number of iterations and the best performance in terms of effectiveness across all tested values of θ. Notably, for $\theta = 0.99$, $\psi_p(t)$ requires only 2,868 iterations and 0.1854 seconds, outperforming all existing kernel functions in both metrics.

While it is true that methods from [9] and [34] exhibit slightly lower CPU times in isolated cases, these methods demand significantly more iterations to converge. The higher per-iteration simplicity of these kernels explains the reduced time, but their convergence behavior is less favorable.

The added η and effectiveness metrics demonstrate that $\psi_p(t)$ maintains a superior balance between computational efficiency and convergence speed. Particularly, for higher θ, the overall performance gains of $\psi_p(t)$ become more pronounced, confirming its robustness and practical advantage for large-scale optimization problems.

To better highlight specific advantages of the proposed $\psi_p(t)$ kernel, Table 2 summarizes key performance improvements based on the detailed numerical results.

The proposed method demonstrates significant reductions in total iterations, with 28% fewer iterations than the kernel from [9] for $\theta = 0.99$. Similarly, the η efficiency metric indicates a 52% improvement compared to the same reference, confirming that the lower CPU times observed for other kernels do not translate into better global efficiency.

While the kernel from [34] achieves slightly faster CPU times in isolated cases (notably for $\theta = 0.25$), $\psi_p(t)$ outperforms it in overall effectiveness by 15% for $\theta = 0.99$.

These findings confirm that $\psi_p(t)$ achieves a superior balance between computational cost per iteration and convergence behavior, providing tangible practical benefits for solving linear programming problems, particularly as θ increases.

This balanced performance stems from the carefully designed structure of the $\psi_p(t)$ kernel, which combines enhanced central path guidance with manageable per-iteration computational complexity. The result is an improved trade-off between convergence speed and computational efficiency, allowing the method to outperform conventional kernels in overall effectiveness, especially for larger values of θ.

9 Conclusion

In this paper, we analyzed both the large-update and small-update variants of the primal-dual interior-point algorithm, as described in Algorithm 1, incorporating a new parameterized kernel function defined in (25). We derived the theoretical iteration bounds, showing that for large-update methods, the complexity is $O\left(\frac{n}{p^3}\log\left(\frac{n}{\varepsilon}\right)\right)$, while for small-update methods, it is $O\left(\frac{\sqrt{n}}{p}\log\left(\frac{n}{\varepsilon}\right)\right)$.

Furthermore, by choosing the parameter p to depend on n, specifically $p = \left(\frac{\sqrt{n}}{\log(n)}\right)^{1/3}$, we recover the best-known iteration complexity for large-update interior-point methods, namely:

$$O\left(\sqrt{n}\log(n)\log\left(\frac{n}{\varepsilon}\right)\right).$$

The experimental results further demonstrate that the proposed $\psi_p(t)$ kernel significantly reduces the total number of iterations while maintaining competitive computational times, especially for larger values of θ. Although the per-iteration computational cost is marginally higher due to the kernel's structure, the substantial reduction in iterations results in superior overall efficiency.
These theoretical and numerical findings underline the potential of the proposed kernel for solving large-scale linear programming problems. Future work may explore its application to other classes of optimization problems, such as

quadratic programming, conic programming, and SDLCP problems, and investigate further improvements in complexity bounds.

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