JJMS **19**, No. 1, 105-112 (2026)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/19.1.8

Estimations of Divergence Measures and Shannon Entropy Via Generalized Majorization Theorem Using Taylor's Formula and Green Functions

Awais Rasheed[1,*], *Khuram Ali Khan*[2], *Josip Pečarić*[3], *Đilda Pečarić*[4]

[1] Govt. Graduate College Sillanwali Sargodha, Pakistan. e-mail: awais10mathematicians@gmail.com
[2] Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan. e-mail: khuramsms@gmail.com
[3] Croatian Academy of Sciences and Arts, Zagreb, Croatia. e-mail:jopecaric@gmail.com
[4] Department of Communication, Media and Journalism University North, Trg Dr. Žarka Dolinara 1, Koprivnica, Croatia. e-mail: gildapeca@gmail.com

Received: Oct. 4, 2025 Accepted: Dec. 23, 2025

Abstract: In this paper, some results involving Csiszár f-divergence between two probability measures are presented with respect to generalized majorization theorem via Taylor's formula and Green functions. In seek of applications, the special cases of obtained results are deduced in terms of Shannon entropy, Kullback-Leibler divergence, Bhattacharyya coefficient, Jeffrey's distance and Triangular discrimination.

Keywords: Csiszár f-divergence; Shannon entropy; Kullback–Leibler distance; Taylor's formula.

2010 Mathematics Subject Classification. 26D15; 26D20; 94A17; 94A20.

1 Introduction and Preliminaries

Majorization is a reliable and practical mathematical technique that is frequently applied in many different domains. Those who are interested in majorization, should study Marshall *et al.*'s classic book [20], which examines both its theory and history in very extensive detail. It also contains additional explanation in it and a number of exploring ideas in many mathematical disciplines. A concise and comprehensive monograph on majorization is written by Alberti and Uhlmann [2] that would be intriguing to many readers. Moreover, a couple of excellent papers on majorization have been written by Ando [3,4], this offers a brief overview of the subject and discusses main developments that have occured since the publications of Marshall's book [20]. Latif *et al.* [16], generalized the majorization inequality associated with recently defined Green functions and Taylor's polynomial. Siddique *et al.* [29] used Lidstone interpolation and Green functions to obtain extended majorization results in the same year. Majorization theory is often used in a wide range of areas.

Information theory is an emerging branch of science that deals with the transferring, quantifying, and storing of data. It is not easy to quantify Information as it's an abstract object. In 1948, Claude Shannon [31] defined the presumption of information theory, in his pair of two essential results and developed the notion of information theory and suggested that entropy as the primary measure for information. The distance between two probability distributions is calculated using the divergence measure, which can help to alleviate a number of information theory issues. Information and divergence metrics are extremely beneficial and important in a number of fields, including Sensor networks [21], verifying the order in a Markov chain [13], finance [32], economics [34], and approximation of probability distributions [7].

Many researchers gave a lot of applications in information theory in terms Csiszár divergence, Shannon entropy and Kullback–Leibler divergence in recent years. Khan *et al.* [14] explained a lot of material on majorization as well as they gave applications in information theory. Sayyari [30] *et al.* derived the novel bounds for Shannon's entropy and

* Corresponding author e-mail: awais10mathematicians@gmail.com

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

calculated the lower and upper bounds of Jensen's inequality for uniform convex functions. In information theory, they also developed applications. In [1], Adeel *et al.* presented fruitful applications to information theory in terms of Csiszár divergence, Shannon entropy, Renyi entropy and the Zipf-Mandelbrot law. Rasheed [23] *et al.* established new 3-convex four-Green functions in 2023 and presented novel results for Csiszár divergence, Shannon entropy, Renyi entropy and the Zipf–Mandelbrot law. In [24,25], Rasheed *et al.* applied various interpolations to derive extended version of Levinson-type inequalities and also estimated various divergences and entropies results. In [6], Bilal *et al.* gave generalization of Shannon type inequalities via diamond integrals. In [5], Bilal *et al.* defined Csiszár's f-divergence for diamond integrals and also proved inequalities associated to Csiszár's f-divergence using diamond integrals.
An n-convex function [9, p. 15], is described in the following way:
n-convex function: A function $f:[z,u]\to\mathbb{R}$ is called n-convex $(0\leq m)$, for $u_0,\ldots,u_m\in[z,u]$, if and only if $[u_0,\ldots,u_m;f]\geq 0$ holds. Where

$$[u_\kappa;f]=f(u_\kappa),\quad \kappa=0,\ldots,m,$$

$$[u_0,\ldots,u_m;f]=\frac{[u_1,\ldots,u_m;f]-[u_0,\ldots,u_{m-1};f]}{u_m-u_0}.$$

Theorem 1.*(see [33])(Taylor's Formula) Let $n\in\mathbb{Z}^+$. Assume that $\chi^{(n-1)}$ is absolute continuous where $\chi:[\zeta_1,\zeta_2]\to\mathbb{R}$, then for each $v\in[\zeta_1,\zeta_2]$ formula of Taylor at the point $a\in[\zeta_1,\zeta_2]$ is*

$$\chi(v)=T_{n-1}(\chi;a,v)+R_{n-1}(\chi;a,v), \tag{1}$$

here $T_{n-1}(\chi;a,v)$ is $(n-1)$-th degree Taylor polynomial, which is

$$T_{n-1}(\chi;a,v)=\sum_{l=o}^{n-1}(v-a)^l\frac{\chi^{(l)}(a)}{l!}$$

and $R_{n-1}(\chi;a,v)$ is indicated by

$$R_{n-1}(\chi;a,v)=\frac{1}{(n-1)!}\int_a^v\chi^{(n)}(z)(v-z)^{n-1}dz.$$

At end points using formula of Taylor we get

$$\chi(v)=\sum_{l=o}^{n-1}\frac{\chi^{(l)}(\zeta_1)}{l!}(v-\zeta_1)^l+\frac{1}{(n-1)!}\int_{\zeta_1}^{\zeta_2}\chi^{(n)}(z)(v-z)_+^{n-1}dz, \tag{2}$$

$$\begin{aligned}\chi(v)=&\sum_{l=o}^{n-1}\frac{\chi^{(l)}(\zeta_2)}{l!}(\zeta_2-v)^l(-1)^l\\&-\frac{1}{(n-1)!}\int_{\zeta_1}^{\zeta_2}(-1)^{n-1}\chi^{(n)}(z)(z-v)_+^{n-1}dz,\end{aligned} \tag{3}$$

where $(v-z)_+$ is defined as

$$(v-z)_+=\begin{cases}v-z, & z\leq v,\\ 0 & z>v.\end{cases}$$

1.1 Majorization

In [22], majorization is defined as:
For fixed $m\geq 2$. Consider two non-increasing real-number sequences $\mathbf{x}=(\xi_1,\ldots,\xi_m)$, $\mathbf{y}=(\eta_1,\ldots,\eta_m)$. Then $\mathbf{x}$ majorizes $\mathbf{y}$ or $\mathbf{y}$ is majorized by $\mathbf{x}$, in symbol we have $\mathbf{x}\succ\mathbf{y}$, if we have

$$\sum_{l=1}^{j}y_l\leq\sum_{l=1}^{j}x_l, \tag{4}$$

for $j\in\{1,\ldots,m-1\}$ and

$$\sum_{l=1}^{m} x_l = \sum_{l=1}^{m} y_l. \tag{5}$$

The following theorem is given by Marshall-Olkin-Arnold [22, 17], which is also called Classical Majorization Theorem.

Theorem 2.*(Classical Majorization Theorem) Consider two non-increasing real m-tuples* $\mathbf{x} = (\xi_1, \ldots, \xi_m)$, $\mathbf{y} = (\eta_1, \ldots, \eta_m)$ *such that* $\xi_1, \eta_1 \in [\zeta_1, \zeta_2]$ *for* $l = 1, \ldots, m$*. Then* $\mathbf{y}$ *is majorized by* $\mathbf{x}$ *if and only if the following inequality holds for any convex, continuous function,* $\chi : [\zeta_1, \zeta_2] \to \mathbb{R}$

$$\sum_{l=1}^{m} \chi(x_l) \geq \sum_{l=1}^{m} \chi(y_l) \tag{6}$$

Fuchs [11, 22], proved following Weighted Majorization Theorem, that is generalization of Theorem 2.

Theorem 3.*(Weighted Majorization Theorem) Consider two non-increasing real m-tuples* $\mathbf{x} = (\xi_1, \ldots, \xi_m)$, $\mathbf{y} = (\eta_1, \ldots, \eta_m)$ *such that* $\xi_l, \eta_l \in [\zeta_1, \zeta_2]$ *for* $l = 1, \ldots, m$*. Let* $\mathbf{q} = (q_1, \ldots, q_m)$ *be real m-tuple such that*

$$\sum_{l=1}^{j} q_l y_l \leq \sum_{l=1}^{j} q_l x_l, \qquad for\, j \in \{1, \ldots, m-1\} \tag{7}$$

and

$$\sum_{l=1}^{m} q_l x_l = \sum_{l=1}^{m} q_l y_l. \tag{8}$$

Then the inequality

$$\sum_{l=1}^{m} q_l \chi(x_l) \geq \sum_{l=1}^{m} q_l \chi(y_l) \tag{9}$$

holds for every continuous convex function $\chi : [\zeta_1, \zeta_2] \to \mathbb{R}$.

*Remark.*If Theorem 3's assumptions are fulfilled, then

$$\mathfrak{M}(\mathbf{x}, \mathbf{y}, \mathbf{q}, \chi(.)) := \sum_{i=1}^{m} q_i \chi(x_i) - \sum_{i=1}^{m} q_i \chi(y_i) \geq 0, \tag{10}$$

for χ be convex and continuous function. Also $\mathfrak{M}(\mathbf{x}, \mathbf{y}, \mathbf{q}, \chi(.)) = 0$ when $\chi(x)$ is a linear function and constant.

Let $[\zeta_1, \zeta_2] \subset (-\infty, \infty)$ and $\hat{k} \in \{1, 2, 3, 4\}$. In [18], Pečarić *et al.* give new type of Green functions, $G_{\hat{k}} : [\zeta_1, \zeta_2]^2 \to \mathbb{R}$, which are given as:

$$G_1(w, v) = \begin{cases} \zeta_1 - v, & \zeta_1 \leq v \leq w, \\ \zeta_1 - w, & w \leq v \leq \zeta_2. \end{cases} \tag{11}$$

$$G_2(w, v) = \begin{cases} w - \zeta_2, & \zeta_1 \leq v \leq w, \\ v - \zeta_2, & w \leq v \leq \zeta_2. \end{cases} \tag{12}$$

$$G_2(w, v) = \begin{cases} w - \zeta_1, & \zeta_1 \leq v \leq w, \\ v - \zeta_1, & w \leq v \leq \zeta_2. \end{cases} \tag{13}$$

$$G_4(w, v) = \begin{cases} \zeta_2 - v, & \zeta_1 \leq v \leq w, \\ \zeta_2 - w, & w \leq v \leq \zeta_2. \end{cases} \tag{14}$$

In [16], Pečarić *et al.* gave the following generalized the majorization result in Taylor sense.

Theorem 4.*Assume that* $\chi : [\zeta_1, \zeta_2] \to \mathbb{R}$ *and* $(n-1)-th$ $(n \geq 3)$ *derivative of* χ *is absolutely continuous. Let* $\mathbf{q} = (q_1, \ldots, q_m)$ *be m-tuple and* $\mathbf{x} = (\xi_1 \ldots, \xi_m)$, $\mathbf{y} = (\eta_1 \ldots, \eta_m)$ *be non-increasing m-tuples such that* $\xi_i, \eta_i \in [\zeta_1, \zeta_2]$ *and* $q_i \in \mathbb{R}$ *for* $i = 1, \ldots, m$*, which satisfy conditions (7) and (8). Define* $G_{\hat{k}}(\hat{k} = 1, 2, 3, 4)$ *to be as in (11), (12), (13) and (14), respectively. If for even n,* χ *is n-convex, then*

$$\mathfrak{M}(\mathbf{x}, \mathbf{y}, \mathbf{q}, \chi(.)) \geq \sum_{k=0}^{n-3} \frac{(-1)^k \chi^{(k+2)}(\zeta_2)}{k!} \int_{\zeta_1}^{\zeta_2} \mathfrak{M}(\mathbf{x}, \mathbf{y}, \mathbf{q}, G_{\hat{k}}(., v))(\zeta_2 - v)^k dv. \tag{15}$$

2 Main results

In this section, inequality (15) is generalized in sense of information theory. First, by using Csiszár f-divergence more generalized results are achieved. In addition, novel results are presented in term of Shannon entropy and Kullback–Leibler (K-L) divergence. In the end, the results related to Bhattacharyya coefficient, Jeffrey's distance and Triangular discrimination are discussed.
The following presumptions are made throughout this section as:
$\mathbb{A}$: Assume that $f:[\zeta_1,\zeta_2]\to\mathbb{R}$ and $(n-1)$-th $(n\geq 3)$ derivative of f is absolutely continuous. Let $\mathbf{q}=(q_1,\ldots,q_m)$ be m-tuple and $\mathbf{x}=(\xi_1\ldots,\xi_m)$, $\mathbf{y}=(\eta_1\ldots,\eta_m)$ be non-increasing m-tuples such that $\xi_i,\eta_i\in[\zeta_1,\zeta_2]$ and $q_i\in\mathbb{R}$ for $i=1,\ldots,m$, which satisfy conditions (7) and (8).
$\mathbb{B}$: Assume that $\mathbf{p},\mathbf{r}\in\mathbb{R}^m_+$, $\mathbf{w}\in\mathbb{R}^m_+$ and

$$\sum_{i=1}^{\hat{k}} p_i \leq \sum_{i=1}^{\hat{k}} r_i, \tag{16}$$

for $\hat{k}\in\{1,2,...,m-1\}$ and

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} r_i, \tag{17}$$

with $\frac{pi}{w_i},\frac{r_i}{w_i}\in[\zeta_1,\zeta_2](i=1,...,m)$. Let $\frac{\mathbf{p}}{\mathbf{w}}$ and $\frac{\mathbf{r}}{\mathbf{w}}$ are decreasing and $G_{\hat{k}}(\hat{k}=1,2,3,4)$ be as defined in (11)-(14).

2.1 Csiszár divergence

In [8,10] Csiszár defined f-divergence functional is defined as follows:
Consider a convex function $f:(0,\infty)\to(0,\infty)$ and two positive probability distributions, $\mathbf{r},\mathbf{w}\in\mathbb{R}^m_+$ then f-divergence functional is defined by

$$I_f(\mathbf{r},\mathbf{w}) := \sum_{i=1}^{m} w_i f\left(\frac{r_i}{w_i}\right).$$

Horv́ath *et al.* [12] calculated the following functional, by using the f-divergence functional :
Let $J\subset\mathbb{R}$ and f be a real-valued convex function. Also, let $\mathbf{r}\in\mathbb{R}^m_+$ and $w\in\mathbb{R}^m_+$ be m-tuples so that $\frac{r_i}{w_i}\in J$, $i\in\{1,\ldots,m\}$. Then

$$\hat{I}_f(\mathbf{r},\mathbf{w}) := \sum_{i=1}^{m} w_i f\left(\frac{r_i}{w_i}\right), \tag{18}$$

where $\hat{I}_f(.)$ is a linear functional.
In the following theorem inequality given in Theorem 4 is generalized by Csiszár f-divergence.

Theorem 5.*Let $\mathbb{A}$ and $\mathbb{B}$ hold. If f is n-convex for even $n>3$, then we have the following inequality*

$$\hat{I}_f(\mathbf{p},\mathbf{w}) \geq \hat{I}_f(\mathbf{r},\mathbf{w}) + \sum_{k=0}^{n-3}\frac{(-1)^k f^{(k+2)}(\zeta_2)}{k!} \times \int_{\zeta_1}^{\zeta_2}\left(\hat{I}_{G_{\hat{k}}}(\mathbf{p},\mathbf{w},v)-\hat{I}_{G_{\hat{k}}}(\mathbf{r},\mathbf{w},v)\right)(\zeta_2-v)^k dv, \tag{19}$$

where

$$\hat{I}_{G_{\hat{k}}}(\mathbf{p},\mathbf{w},v) = \sum_{i=1}^{m} w_i G_{\hat{k}}\left(\frac{p_i}{w_i},v\right).$$

*Proof.*Taking $x_i=\frac{p_i}{w_i}$, $y_i=\frac{r_i}{w_i}$ and $q_i=w_i>0(i\in\{1,\ldots,m\})$ then the constraints (16) and (17) imply conditions (7) and (8), respectively. Thus using these substitutions in (15), we obtain (19).

*Remark.*If n is odd then inequality in (19) is reversed.

Theorem 6. *If $g : [\zeta_1, \zeta_2] \to \mathbb{R}$ and $f(x) = xg(x)$, $x \in [\zeta_1, \zeta_2]$ then inequality (19), for even $n > 3$ becomes:*

$$\check{I}_g(\mathbf{p}, \mathbf{w}) := \sum_{i=1}^{m} p_i g\left(\frac{p_i}{w_i}\right) \geq \check{I}_g(\mathbf{r}, \mathbf{w}) + \sum_{k=0}^{n-3} \frac{(-1)^k (xg)^{(k+2)}(\zeta_2)}{k!} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right) (\zeta_2 - v)^k dv. \quad (20)$$

Proof. Using $f(x) = xg(x)$ in Theorem 5, we get (20).

2.2 Shannon Entropy

The concepts of entropic disorder measurement and majorization are interlinked (see[27,28,26]). In this subsection, we prove the estimation of majorization inequality (15) via Shannon entropy.
In [19], for positively distributive probability $\mathbf{k} = (k_1, \ldots, k_n)$ the Shannon entropy is given by

$$H(\mathbf{k}) := -\sum_{\rho=1}^{n} k_\rho \log(k_\rho), \quad (21)$$

where $H(.)$ is a linear functional. b is the base of log and $\log x = \frac{\ln x}{\ln b}$ is used.

Corollary 1. *Consider $\mathbb{A}$, $\mathbb{B}$ holds and* log *has base greater than or equal to* 1. *If f is n-convex and $n > 3$ is even, then*

$$H(\mathbf{w}) \leq \sum_{i=1}^{m} w_i \log\left(\frac{r_i}{w_i}\right) - \sum_{k=0}^{n-3} \frac{(k+1)(\zeta_2)^{-(k+2)}}{\ln b} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right) (\zeta_2 - v)^k dv \quad (22)$$

holds.

Proof. For even $n > 3$, take n-convex function $f(x) = -\log x$ and $p_i = 1 (i \in \{1, \ldots, m\})$ in Theorem 5, then (19) becomes (22).

Remark. If the base $b \in (0, 1)$, or n is odd then inequality (22) is reversed.

Corollary 2. *Let A and B hold. Consider* log *has base b greater than 1 then for even $n > 3$, the following is the association between the Shannon entropy of* $\mathbf{p}$ *and* $\mathbf{r}$*:*

$$H(\mathbf{p}) \geq H(\mathbf{r}) - \sum_{k=0}^{n-3} \frac{(-1)^k (x \log x)^{(k+2)}(\zeta_2)}{k!} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right) (\zeta_2 - v)^k dv, \quad (23)$$

for $l = 1,\ 2,$

$$(x \log x)'(\zeta_l) = \frac{(1 + \ln(\zeta_l))}{\ln b}$$

and

$$(x \log x)^{(k+2)}(\zeta_1) = \frac{(-1)^{k+2} k!}{\zeta_1^{k+1} \ln b},\ k \geq 0.$$

Proof. For even $n > 3$, taking n-convex function $xg(x) := x \log x$ in (20), and $w_i = 1 (i \in \{1, \ldots, m\})$, then we obtain (23).

Remark. If $0 < b < 1$, or n is odd then inequality (23) is reversed.

2.3 Kullback–Leibler divergence

The generalized majorization-type inequality is described in terms of Kullback–Leibler (K-L) divergence in this subsection. For two positively distributive probabilities $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$, Kullback–Leibler [15], (K-L) divergence is defined by

$$L(\mathbf{r}, \mathbf{w}) := \sum_{i=1}^{m} r_i \log\left(\frac{r_i}{w_i}\right).$$

Corollary 3. *Under the assumptions of Corollary 1, the following estimates hold:*

$$\sum_{i=1}^{m} w_i \log\left(\frac{p_i}{w_i}\right) \leq \sum_{i=1}^{m} w_i \log\left(\frac{r_i}{w_i}\right) - \sum_{k=0}^{n-3} \frac{(\zeta_2)^{-(k+2)}(k+1)}{\ln b} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right)(\zeta_2 - v)^k dv. \tag{24}$$

If the base b of the log *is between 0 and 1, then inequality (24) is reversed.*

Proof. Putting n-convex function $f(x) := -\log x$ (even $n > 3$) in (19), then we get (24).

Corollary 4. *Consider all assumptions of Corollary 2 hold, then the K-L divergence of* $\mathbf{r}, \mathbf{w}$ *and* $\mathbf{p}, \mathbf{w}$ *is given as:*

$$L(\mathbf{p}, \mathbf{w}) \geq L(\mathbf{r}, \mathbf{w}) + \sum_{k=0}^{n-3} \frac{(-1)^k (x \log x)^{(k+2)}(\zeta_2)}{k!} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right)(\zeta_2 - v)^k dv, \tag{25}$$

for $l = 1,\ 2,$

$$(x \log x)'(\zeta_l) = \frac{(1 + \ln(\zeta_l))}{\ln b}$$

and

$$(x \log x)^{(k+2)}(\zeta_l) = \frac{(-1)^{k+2} k!}{\zeta_1^{k+1} \ln b},\ k \geq 0.$$

Proof. For even $n > 3$, take n-convex function $xg(x) := x \log x$ in (20), then inequality (20) becomes (25).

Remark. If $b \in (0, 1)$, or n is odd then inequality (25) holds in reverse direction.

2.4 Bhattacharyya coefficient

The Bhattacharyya coefficient is used to express generalised majorization-type inequality in this section. In [15], Bhattacharyya coefficient for two positive probability distributions $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ is given by

$$B(\mathbf{r}, \mathbf{w}) = \sum_{k=1}^{m} \sqrt{w_k r_k}. \tag{26}$$

Corollary 5. *Consider all assumptions of Corollary 1, is true then following inequality holds:*

$$\mathbb{B}(\mathbf{p}, \mathbf{w}) \leq \mathbb{B}(\mathbf{r}, \mathbf{w}) - \sum_{k=0}^{n-3} \frac{(\prod_{i=1}^{k}(2i-1))(\zeta_2)^{-(\frac{2k+1}{2})}(-1)^{2k+1}}{2^{k+1}} \times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v)\right)(\zeta_2 - v)^k dv. \tag{27}$$

Proof. Taking n-convex function $f(x) := -\sqrt{x}$ (even $n > 3$) in (19), then we get (27).

2.5 Jeffrey's distance

The generalised majorization-type inequality in terms of Jeffrey's distance is provided in this subsection. In [15], Jeffrey's distance for two positive probability distributions $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{w} = (w_1, \dots, w_m)$ is defined by

$$\mathbb{J}(\mathbf{r}, \mathbf{w}) = \sum_{k=1}^{m} \log(\frac{r_k}{w_k})(r_k - w_k). \tag{28}$$

Corollary 6. *If A and B hold and* log *has base b greater than 1 then for even* $n > 3$*, we get the following bound:*

$$\begin{aligned} \mathbb{J}(\mathbf{p}, \mathbf{w}) \geq \mathbb{J}(\mathbf{r}, \mathbf{w}) &+ \sum_{k=0}^{n-3} \frac{1+k+\zeta_2}{\ln b(\zeta_2)^{2+k}} \\ &\times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v) \right) (\zeta_2 - v)^k dv. \end{aligned} \tag{29}$$

If the base $b \in (0,1)$*, then the inequality (29) is reversed.*

Proof. Putting n-convex function $f(x) := (x-1)\log x$ (even $n > 3$) in (19), then we get the bound (29).

2.6 Triangular discrimination

In [15], Triangular discrimination for two positive probability distributions $\mathbf{w} = (w_1, \dots, w_m)$ and $\mathbf{r} = (r_1, \dots, r_m)$ is defined by

$$\mathbb{T}(\mathbf{r}, \mathbf{w}) = \sum_{k=1}^{m} \frac{(r_k - w_k)^2}{r_k + w_k}. \tag{30}$$

Corollary 7. *Under the assumptions of Corollary 1, the following inequality holds:*

$$\begin{aligned} \mathbb{T}(\mathbf{p}, \mathbf{w}) \geq \mathbb{T}(\mathbf{r}, \mathbf{w}) &+ \sum_{k=0}^{n-3} \frac{4(k+2)(k+1)}{(1+\zeta_2)^{k+3}} \\ &\times \int_{\zeta_1}^{\zeta_2} \left(\hat{I}_{G_{\hat{k}}}(\mathbf{p}, \mathbf{w}, v) - \hat{I}_{G_{\hat{k}}}(\mathbf{r}, \mathbf{w}, v) \right) (\zeta_2 - v)^k dv. \end{aligned} \tag{31}$$

Proof. Putting n-convex function $f(x) := \frac{(x-1)^2}{x+1}$ (even $n > 3$) in (19), then we obtain (31).

Conclusion The paper's main goal is to identify several applications of the majorization theorem in information theory, using various entropies and divergence measurements. This is accomplished by applying Shannon entropy and Csiszár divergence to majorization inequality in order to establish various novel conclusions. Additionally, several constraints are computed using other widely recognised information divergence measures, including: Kullback–Leibler divergence, Bhattacharyya coefficient, Jeffrey's distance and Triangular discrimination. The principal conclusions can be used in future research to analyse various divergences and distances, such as: the Zipf-Mandelbrot law, Rényi divergence and Rényi entropy etc.

Declarations

Conflicts of Interest:The authors declares that there is no conflict of interests regarding the publication of this paper.
Authors' contributions:AR initiated the work and made calculations. KAK supervised and validated the draft. JP dealt with the formal analysis and investigation. GP included the applications to information theory. All the authors read and approved the final manuscript.
Funding:There is no funding for this work.
Availability of data and materials:Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Acknowledgments:The authors wish to thank the editor for possible publication.

References

[1] Adeel, M., Khan, K. A., Pečarić, Đ., and Pečarić, J. (2019). Generalization of the Levinson inequality with applications to information theory. *J. Inequal. Appl.*, **2019**(1): 230.
[2] Alberti, P.M., Uhlmann, A., (1982). *Stochasticity and partial order: doubly stochastic maps and unitary mixing Dordrecht*, Boston.
[3] Ando, T., (1989). Majorization, doubly stochastic matrices and comparison of eigenvalues, *Linear Algebra Appl.*, **118**, 163–248.
[4] Ando, T., (1994). Majorizations and inequalities in matrix theory, *Linear Algebra Appl.*, **199**, 17–67.
[5] Bilal, M., Khan, K. A., Nosheen, A., and Pečarić, J. (2023). Some inequalities related to Csiszár divergence via diamond integral on time scales. *J. Inequal. Appl.,* **2023**(1), 55.
[6] Bilal, M., Khan, K. A., Nosheen, A., and Pečarić, J. (2023). Generalizations of Shannon type inequalities via diamond integrals on time scales. *J. Inequal. Appl.,* **2023**(1), 24.
[7] Chow, C. K. C. N. and Liu, C. (1968). Approximating discrete probability distributions with dependence trees. *IEEE transactions on Information Theory*, **14**(3), 462-467.
[8] Csiszár, I., (1978). *Information measures: a critical survey*. In: Tans. 7th Prague Conf. on Info. Th., Statist. Decis. Funct., Random Process and 8th European Meeting of Statist., **B**, Academia Prague, 73-86.
[9] Pečarić, J., Proschan, F. and Tong, Y. L. (1992). *Convex functions, Partial Orderings and Statistical Applications*. Academic Press, New York.
[10] Csiszár, I., (1967). Information-type measures of difference of probability distributions and indirect observations. Stud. Sci. Math. Hungar. **2**, 299-318.
[11] Fuchs, L., (2003). *A new proof of an inequality of Hardy-Littlewood-Polya*, Mat. Tidsskr, **B**.
[12] Horváth, L., Pečarić, Đ. and Pečarić, J. (2019). Estimations of f-and Rényi divergences by using a cyclic refinement of the Jensen's inequality, *Bull. Malays. Math. Sci. Soc.,* **42**, 933-946.
[13] Hibbard, L. S. (2004). Region segmentation using information divergence measures. *Med. Image Ana.*, **8**(3), 233-244.
[14] Khan, M.A., Bradanovic, S.I., Latif, N., Pečarić, . and Pečarić, J., (2019). Majorization Inequality and Information Theory, *Element, Zagreb*.
[15] Khan, M.A., Pečarić, ., Pečarić, J., (2021). A new refinement of the Jensen inequality with applications in information theory, *Bull. Malays. Math. Sci. Soc.*, **44**(1), 267-278.
[16] Latif, N., Siddique, N., and Pečarić, J., (2018). Generalization of majorization theorem-II, *J. Math. Inequal.*, **12**(3), 731–752.
[17] Liese, F., Vajda, I., (1987). Convex Statistical Distances, Teubner-Texte Zur Mathematik, Leipzig: Teubner, **95**.
[18] Mehmood, N., Agarwal, R. P., Butt, S. I. and Pečarić, J., (2017). New generalizations of Popoviciu type inequalities via new Green's functions and Montgomery identity, *J. Inequl. Appl.*, 1–21.
[19] Matic, M., Pearce, C. E. M. and Pečarić, J., (2000). *Shannon's and related inequalities in information theory, Survey on classical inequalities,* editor Themistocles M. Rassias, Kluwer Academic Publishers, 127–164.
[20] Marshall, A.W., Olkin, I., Arnold, B.C., (2011). *Inequalities: Theory of Majorization and Its Applications*, Springer Series in Statistics. Springer, New York, **3**.
[21] Pardo, L. (2018). *Statistical inference based on divergence measures.* Chapman and Hall/CRC.
[22] Pečarić, J.E, Proschan, F., Tong, Y.L. (1992). Convex Functions, Partial Orderings, and Statistical Applications. *Math. Sci. Eng.* **187**, Academic Press.
[23] Rasheed, A., Khan, K. A., Pečarić, J., and Pečarić, Đ. (2023). Generalization of the Levinson inequality via new Green functions with applications to information theory. *J. Inequal. Appl.*, **2023**(1): 124.
[24] Rasheed, A., Khan, K. A., Pečarić, J., and Pečarić, Đ. (2024). Generalizations of Levinson-type inequalities via new Green functions and Hermite interpolating polynomial. *J. Inequal. Appl.*, **2024**(1), 70.
[25] Rasheed, A., Khan, K. A., Pečarić, J., and Pečarić, Đ. Estimation of Shannon entropy and divergences by Levinson-type inequalities using new Green functions via Lidstone polynomial. *Hacet. J. Math. Stat.*, 1-23.
[26] Rasheed, A., Khan, K. A., Pečarić, J., and Pečarić, Đ. Estimations of divergence measures for majorization inequalities via Peano's representation of Hermite's polynomial. (2025). *Arab. J. Math. Sci.*, 1-16.
[27] Siddique, N., Imran, M., Khan, K, A., and Pečarić, J., (2020). Majorization inequalities via Green functions and Fink's identity with applications to Shannon entropy, *J. Inequal. Appl.*, 2020(**1**), 192.
[28] Siddique, N., Imran, M., Khan, K, A., and Pečarić, J., (2020). Difference equations related to majorization theorems via Montgomery identity and Green's functions with application to the Shannon entropy, *Adv. Differ. Equ.*, 2020(**1**), 430.
[29] Siddique, N., Latif, N., Pečarić, J., (2018). Generalized results of majorization inequality via Lidstone's polynomial and newly Green functions, *J. Nonlinear Sci. Appl.*, **11**(6), 812–831.
[30] Sayyari, Y. (2020). New entropy bounds via uniformly convex functions. *Chaos Solit. Fractals*, **141**, 110360.
[31] Shannon, C. E. (1948). A mathematical theory of communication. *Bell Syst. Tech. J.*, **27**(3), 379-423.
[32] Sen, A., Sen, M. A., Foster, J. E., Amartya, S., and Foster, J. E. (1997). *On economic inequality.* Oxford University Press.
[33] Spivak, M., (1994). *Calculus*, Houston, TX: Publish or Perish, **3**, p. 417, ISBN 0-914098-89-6.
[34] Theil, H. (1967). *Economics and Information Theory.* North-Holland, Amsterdam.