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The Level Cardinality Of Fuzzy Module Under $\mathbb{Z}$-Module Homomorphism On $\mathbb{Z}_m$ Into $\mathbb{Z}_n$ Where $gcd(m,n)$ Is Product Of Powers Of Primes

Manju Varghese[1], *Shery Fernandez*[2,*]

[1] Mar Thoma College for Women, Perumbavoor,
Cochin University of Science and Technology
[2] Cochin University of Science and Technology

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Abstract: The homomorphic image of a fuzzy module over an R-module is itself a fuzzy module, yet explicit bounds on level cardinalities under specific structural constraints remain largely unexplored. In prior work, such bounds were established for $\mathbb{Z}$-module homomorphisms $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ when $gcd(m,n)$ is either a prime or the product of two distinct primes, getting maxima of 3 and 4 respectively. In this paper, we extend these results to the cases $gcd(m,n) = p^s$, where p is a prime and $s \in \mathbb{Z}^+$ and further to the case $gcd(m,n) = p_1^{s_1}.p_2^{s_2}\dots p_k^{s_t}$, with distinct primes p_i and $s_i \in \mathbb{Z}^+$, $i = 1,2,\dots t$. Our results contribute to a deeper understanding of the behavior of fuzzy modules under structural mappings between cyclic modules.

Keywords: Fuzzy module homomorphism; level cardinality.

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1 Introduction

Classical set theory, with binary membership values 0 and 1, is inadequate for modeling the uncertainty present in many real-world problems. The introduction of fuzzy sets addressed this limitation, leading to the development of fuzzy groups by Rosenfeld in 1971 [5] and fuzzy modules by Negoita and Ralescu in 1975 [1]. A module homomorphism is a structure-preserving map between modules, and it is known that the homomorphic image of a fuzzy module is itself a fuzzy module.

Previous work [4] determined bounds on the level cardinality of the image of a fuzzy module under a $\mathbb{Z}$-module homomorphism $\mathbb{Z}_m \to \mathbb{Z}_n$ when $gcd(m,n)$ is a prime or the product of two distinct primes. In this paper, we generalise these results to the case where $gcd(m,n)$ is a power of a prime and, more generally, to a product of prime (distinct) powers. We also try to get a deeper understanding of the behavior of fuzzy modules under structural mappings between cyclic modules.

The methodology of this work is, when we consider the general case $gcd(m,n) = p_1^{s_1}p_2^{s_2}\dots p_t^{s_t} = \beta$, where $s_i's$ are non negative integers and not all $s_i's$ are simultaneosly zeros, then there are β homomorphisms, $\Gamma_k : \mathbb{Z}_m \to \mathbb{Z}_n$ where the values of k ranges from 0 to $\beta - 1$. Next we consider the total number of $\prod_{i-1}^{n}(s_i+1)$ cases of $gcd(\beta,k)$ separately with the values 0, $p_1^{r_1}p_2^{r_2}\dots p_t^{r_t}$ with $0 \le r_i \le s_i$, $i = 1,2,\dots,t$ and $r = r_1 + r_2 + \dots + r_t \le s_1 + s_2 + \dots s_t - 1 = s - 1$ and find out the level cardinality of the image of the fuzzy modules on $\mathbb{Z}_n$ under the fuzzy module homomorphism as '**s+2**'.

* Corresponding author e-mail: sheryfernandezaok@gmail.com

2 Preliminaries

Theorem 1.[4] *Let Γ be a $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_m$ into $\mathbb{Z}_n$ where $m,n \in \mathbb{Z}^+$, then the module homomorphism is determined by the value of $\Gamma(1) \in \mathbb{Z}_n$.*

Theorem 2.[4] *The level cardinality of any fuzzy module of a $\mathbb{Z}$-module $\mathbb{Z}_m$ is atmost $r+1$ where r is the sum of the integral powers of prime factorisation of m.*

Theorem 3.[4] *Let Γ be a $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_m$ into $\mathbb{Z}_n$ and μ be a fuzzy module on $\mathbb{Z}_m$ with level cardinality r then level cardinality of the fuzzy module $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost to $r+1$.*

Remark. The Euler totient function $\phi(n)$ is the number of positive integers not exceeding n that are coprime to n.

3 Fuzzy module homomorphism

Theorem 4.[4] *Let $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n) = p$, a prime and let μ be a fuzzy module on $\mathbb{Z}_m$ then level cardinality of $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost 3.*

Theorem 5.[4] *Let $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n) = pq$, where p and q are distinct primes and let μ be a fuzzy module on $\mathbb{Z}_m$ then level cardinality of $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost 4.*

Theorem 6. *Let $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n) = p^2$, p a prime and let μ be a fuzzy module on $\mathbb{Z}_m$ then level cardinality of fuzzy module $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost 4.*

Proof. Here as $gcd(m,n) = p^2$, the possible $\mathbb{Z}$-module homomorphisms are $\Gamma_k(x) = \frac{n}{p^2}kx \pmod{n}$ where $k = 0,1,2,\dots,p^2-1$. Let μ be a fuzzy modules on $\mathbb{Z}_m$, then $\Gamma_k(\mu)$ is a fuzzy module on $\mathbb{Z}_n$. According to the values of $gcd(p^2,k)$ the $\mathbb{Z}$-module homomorphisms can be divided into 3 cases. Let r be the greatest power of p in the prime factorisation of m.

Case 1 : When $k = 0$ (i.e $gcd(p^2,k) = 0$).
$\Gamma_0(x) = 0$ for all $x \in \mathbb{Z}_m$. Let μ be an arbitrary fuzzy module on $\mathbb{Z}_m$ then its homomorphic image $\Gamma_0(\mu)$ is defined by,

$$\Gamma_0(\mu)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 0 & \text{if } y \in <1> \setminus <0> \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_0(\mu)$ on $\mathbb{Z}_n$ is 2.

Case 2 : When $gcd(p^2,k) = p$.
The $\mathbb{Z}$-module homomorphisms $\Gamma_k(x) = \frac{n}{p^2}kx \pmod{n}$ can be written as $\Gamma_k(x) = \frac{n}{p}k'x \pmod{n}$ where $k = k'p$ and $gcd(k',p) = 1$. As k' can have $\phi(p) = p-1$ values, we have $p-1$ such $\mathbb{Z}$-module homomorphisms. Under these $\mathbb{Z}$-module homomorphisms, $\Gamma_k(x) = 0$ if and only if x is a multiple of p or 0. Hence the submodule $<p> = \{0, p, 2p, \dots, (\frac{m}{p}-1)p\}$ of $\mathbb{Z}_m$ of order $\frac{m}{p}$ is mapped to 0 in $\mathbb{Z}_n$ under Γ_ks and the elements in $\mathbb{Z}_m$ with orders, that are not divisors of $\frac{m}{p}$ has non zero images under these Γ_ks. Under these Γ_ks, lp^r where $l \mid (\frac{m}{p^r})$, $l \in \mathbb{Z}^+$, are the orders of elements in $\mathbb{Z}_m$ having non zero images. Now by Theorem 1 order of $\Gamma_k(h)$ divides both n and order of h for all $h \in \mathbb{Z}_m$, so $|f(h)|$ divides $gcd(m,n) = p^2$. Also, in the $\mathbb{Z}$-module homomorphisms $\Gamma_k(x) = \frac{n}{p}k'x \pmod{n}$ where $k = k'p$ and $gcd(k',p) = 1$, the order of $\frac{n}{p}k'x \pmod{n}$ is either 1 or p for all $x \in \mathbb{Z}_m$. So the elements of order 1 and p in $\mathbb{Z}_n$ only have preimages under these Γ_ks and $\Gamma_k(\mu)(y) = 0$ for all $y \in <1> \setminus <\frac{n}{p}> \subset \mathbb{Z}_n$ for every fuzzy module μ on $\mathbb{Z}_m$. So if $x \in <p>$ *i.e* $|x| = \frac{m}{p}$ or its divisor then $\frac{n}{p}k'x \pmod{n} = 0 \in \mathbb{Z}_n$ and if $x \in <1> \setminus <p> \subset \mathbb{Z}_m$ or $|x| = lp^r$ such that $l \mid (\frac{m}{p^r})$ then $\frac{n}{p}k'x \pmod{n} = a$ where a is any element in $<\frac{n}{p}> \setminus <0>$ in $\mathbb{Z}_n$ and the submodule $<\frac{n}{p}> = \{0, \frac{n}{p}, 2\frac{n}{p}, \dots, (p-1)\frac{n}{p}\}$ of $\mathbb{Z}_n$ of order p only have preimage under these Γ_k's.

$$\Gamma_k(x) = \begin{cases} 0 & \text{if } |x| = \frac{m}{p} \text{ or its divisors} \\ a & \text{if } |x| = lp^r \text{ with } l \mid (\frac{m}{p^r}) \end{cases}$$

Now let μ be an arbitrary fuzzy module on $\mathbb{Z}_m$ then the level cardinalities of the homomorphic image $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ are as follows
Let $t = \vee\{\mu(x) \,/\, x \in \mathbb{Z}_m,\, |x| = lp^r,\, l \mid (\frac{m}{p^r})\}$

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

1.If $\mu(0) = t \neq 0$ then $\Gamma_k(\mu)$ is a fuzzy module with the orders of level submodules p and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < \frac{n}{p} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p} > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 2.

2.If $\mu(0) \neq t = 0$ then $\Gamma_k(\mu)$ is a fuzzy module with the orders of level submodules 1 and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < 0 > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 2.

3.If $\mu(0) \neq t \neq 0$ where $t = \vee\{\mu(x) \,/\, x \in \mathbb{Z}_m,\ |x| = lp^r,\ l \mid (\frac{m}{p^r})\}$ then $\Gamma_k(\mu)$ is a fuzzy module with the orders of level submodules 1, p and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t & \text{if } y \in < \frac{n}{p} > \setminus < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p} > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 3.

Hence the homomorphic image $\Gamma_k(\mu)$ of an arbitrary fuzzy module μ has maximum level cardinality 3 when $gcd(p^2,k) = p$.

Case 3 : When $gcd(p^2,k) = 1$.

Consider the $\mathbb{Z}$-module homomorphisms are $\Gamma_k(x) = \frac{n}{p^2}kx \pmod{n}$ with $gcd(k,p^2) = 1$. As $\phi(p^2) = p(p-1)$ values for k, there exists $p(p-1)$ such $\mathbb{Z}$-module homomorphisms also. $\Gamma_k(x) = 0$ if and only if x is a multiple of p^2 or 0 and under these Γ_k, the submodule of $\mathbb{Z}_m$, $< p^2 >= \{0, p^2, 2p^2, \ldots, (\frac{m}{p^2}-1)p^2\}$ of order $\frac{m}{p^2}$ is mapped to 0 in $\mathbb{Z}_n$. The elements of $\mathbb{Z}_m$ under Γ_k with non zero images have orders lp^{r-1} or lp^r with $l \mid (\frac{m}{p^r})$, $l \in \mathbb{Z}^+$. Also by Theorem 1 order of $\Gamma_k(h)$ divides both n and order of h for all $h \in \mathbb{Z}_m$, hence $|f(h)| \mid gcd(m,n) = p^2$. So $1, p, p^2$ are the possible orders of $f(h)$. Hence under Γ_k, the elements in $\mathbb{Z}_n$ of orders $1, p, p^2$ can only have preimages *i.e* the elements in the submodule $< \frac{n}{p^2} >= \{0, \frac{n}{p^2}, 2\frac{n}{p^2}, \ldots, (p^2-1)\frac{n}{p^2}\}$ of $\mathbb{Z}_n$ of order p^2 only have preimages in $\mathbb{Z}_m$ and $\Gamma_k(\mu)(y) = 0$ for all $y \in < 1 > \setminus < \frac{n}{p^2} >$ for every fuzzy module μ on $\mathbb{Z}_m$, if $n \neq p^2$. The homomorphism is onto only if $n = p^2$. If $x \in < p^2 >$ *i.e* $|x| = \frac{m}{p^2}$ or its divisors then $\frac{n}{p^2}kx = 0 \in \mathbb{Z}_n$, if $x \in < p > \setminus < p^2 >$ or $|x| = lp^{r-1}$ with $l \mid (\frac{m}{p^r})$ then $\frac{n}{p^2}kx \pmod{n} = a_1$ where a_1 is any element in $< \frac{n}{p} > \setminus < 0 >$ and if $x \in < 1 > \setminus < p >$ or $|x| = lp^r$ with $l \mid (\frac{m}{p^r})$ then $\frac{n}{p^2}kx \pmod{n} = a_2$ where a_2 is any element in $< \frac{n}{p^2} > \setminus < \frac{n}{p} >$. Then

$$\Gamma_k(x) = \begin{cases} 0 & \text{if } |x| = \frac{m}{p^2} \text{ or its divisor} \\ a_1 & \text{if } |x| = lp^{r-1} \text{ with } l \mid (\frac{m}{p^r}) \\ a_2 & \text{if } |x| = lp^r \text{ with } l \mid (\frac{m}{p^r}) \end{cases}$$

Now let μ be an arbitrary fuzzy module on $\mathbb{Z}_m$, then the level cardinalities of the homomorphic image $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ are defined as follows

Let $t_1 = \vee\{\mu(x_1) \,/\, x_1 \in \mathbb{Z}_m,\ |x_1| = lp^{r-1},\ l \mid (\frac{m}{p^r})\}$ and $t_2 = \vee\{\mu(x_2) \,/\, x_2 \in \mathbb{Z}_m,\ |x_2| = lp^r,\ l \mid (\frac{m}{p^r})\}$

1.If $\mu(0) = t_1 = t_2 \neq 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules p^2 and n (if $p^2 \neq n$), $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < \frac{n}{p^2} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^2} > \end{cases}$$

So level cardinality of $\Gamma_k(\mu)$ is 2 if $n \neq p^2$ and has level cardinality 1 if $n = p^2$.

2.If $\mu(0) = t_1 \neq t_2 = 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules p and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < \frac{n}{p} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p} > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 2.

3.If $\mu(0) \neq t_1 = t_2 = 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules 1 and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < 0 > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 2.

4.If $\mu(0) \neq t_1 = t_2 \neq 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules 1, p^2 and n (if $n \neq p^2$), $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{n}{p^2} > \setminus < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^2} > \end{cases}$$

So level cardinality of $\Gamma_k(\mu)$ is 3 if $n \neq p^2$ and has level cardinality 2 if $n = p^2$.

5.If $\mu(0) = t_1 \neq t_2 \neq 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules p, p^2 and n (if $n \neq p^2$), $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < \frac{n}{p} > \\ t_2 & \text{if } y \in < \frac{n}{p^2} > \setminus < \frac{n}{p} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^2} > \end{cases}$$

Hence the level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 3 if $n \neq p^2$ and has level cardinality 2 if $n = p^2$.

6.If $\mu(0) \neq t_1 \neq t_2 = 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules 1, p and n, $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{n}{p} > \setminus < 0 > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p} > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 3.

7.If $\mu(0) \neq t_1 \neq t_2 \neq 0$ then $\Gamma_k(\mu)$ is a fuzzy module with orders of level submodules 1, p, p^2 and n (if $n \neq p^2$), $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{n}{p} > \setminus < 0 > \\ t_2 & \text{if } y \in < \frac{n}{p^2} > \setminus < \frac{n}{p} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^2} > \end{cases}$$

So level cardinality of the homomorphic image $\Gamma_k(\mu)$ is 4 if $n \neq p^2$ and has level cardinality 3 if $n = p^2$.

Hence, if $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ is a $\mathbb{Z}$-module homomorphism and μ an arbitrary fuzzy module on $\mathbb{Z}_m$, the maxima for the level cardinality of its homomorphic image $\Gamma(\mu)$ on $\mathbb{Z}_n$ is 4.

*Example 1.*In this example, we are considering the $\mathbb{Z}$-module homomorphisms between $\mathbb{Z}_{72}$ and $\mathbb{Z}_{20}$. Since $gcd(72,20) = 4 = 2^2$ the $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_{72}$ into $\mathbb{Z}_{20}$ are $\Gamma_k(x) = 5kx \pmod{20}\ \forall x \in \mathbb{Z}_{72},\ \ k = 0,1,2,3$. By Theorem 2, any fuzzy module on $\mathbb{Z}_{72}$ have level cardinality atmost 6. Below we are considering a fuzzy module μ_6 on $\mathbb{Z}_{72}$ with maximum level cardinality 6 and all the $\Gamma_k(\mu_6)$ with level cardinalities $2,3,4$ by Theorem 6.

$$\mu_6(x) = \begin{cases} 1 & \text{if } x \in < 0 > \\ 1/2 & \text{if } x \in < 36 > \setminus < 0 > \\ 1/3 & \text{if } x \in < 12 > \setminus < 36 > \\ 1/4 & \text{if } x \in < 4 > \setminus < 12 > \\ 1/5 & \text{if } x \in < 2 > \setminus < 4 > \\ 1/6 & \text{if } x \in < 1 > \setminus < 2 > \end{cases}$$

and

Cases	$k=0$	$gcd(k,4)=2$	$gcd(k,4)=1$
Number of homo-morphisms	1	$\phi(2)=1$	$\phi(4)=2$
Homo-morphisms	Γ_0	Γ_2	Γ_1,Γ_3
Fuzzy module	$\Gamma_0(\mu_6)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 0 & \text{if } y\in<1>\setminus<0>\end{cases}$	$\Gamma_2(\mu_6)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/6 & \text{if } y\in<10>\setminus<0>\\ 0 & \text{if } y\in<1>\setminus<10>\end{cases}$	$\Gamma_1(\mu_6)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/5 & \text{if } y\in<10>\setminus<0>\\ 1/6 & \text{if } y\in<5>\setminus<10>\\ 0 & \text{if } y\in<1>\setminus<5>\end{cases}$

Next, we are considering a fuzzy module μ_5 on $\mathbb{Z}_{72}$ with level cardinality 5 and all the $\Gamma_k(\mu_5)$ with level cardinalities $2,3,4$ by Theorem 6.

$$\mu_5(x)=\begin{cases}1 & \text{if } x\in<24>\\ 1/2 & \text{if } x\in<8>\setminus<24>\\ 1/3 & \text{if } x\in<4>\setminus<8>\\ 1/4 & \text{if } x\in<2>\setminus<4>\\ 1/5 & \text{if } x\in<1>\setminus<2>\end{cases}$$

and

Cases	$k=0$	$gcd(k,4)=2$	$gcd(k,4)=1$
Number of homo-morphisms	1	$\phi(2)=1$	$\phi(4)=2$
Homo-morphisms	Γ_0	Γ_2	Γ_1,Γ_3
Fuzzy module	$\Gamma_0(\mu_5)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 0 & \text{if } y\in<1>\setminus<0>\end{cases}$	$\Gamma_2(\mu_5)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/5 & \text{if } y\in<10>\setminus<0>\\ 0 & \text{if } y\in<1>\setminus<10>\end{cases}$	$\Gamma_1(\mu_5)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/4 & \text{if } y\in<10>\setminus<0>\\ 1/5 & \text{if } y\in<5>\setminus<10>\\ 0 & \text{if } y\in<1>\setminus<5>\end{cases}$

So level cardinalities of the fuzzy modules $\Gamma_k(\mu_i)$, $k=0,1,2,3$, $i=5,6$ on $\mathbb{Z}_{20}$ is atmost 4.

*Note 1.*If we consider some fuzzy module μ on $\mathbb{Z}_{72}$ of level cardinality between 3 and 6 (including 3 and 6), we also get the level cardinality of $\Gamma_k(\mu)$ the maximum 4, as we describe the process in Theorem 6.

*Remark.*In Theorem 6 we have proved that the level cardinality of the fuzzy module $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost 4, where $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ is a $\mathbb{Z}$-module homomorphism with $gcd(m,n)=p^2$ and μ is an arbitrary fuzzy module on $\mathbb{Z}_m$. In the next theorem, we are considering the $\mathbb{Z}$-module homomorphism $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ where $gcd(m,n)=p^s$, $s\in\mathbb{Z}^+$.

Theorem 7.*Let $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n)=p^s$, p is a prime, $s\in\mathbb{Z}^+$ and let μ be a fuzzy module on $\mathbb{Z}_m$ then the level cardinality of $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost $s+2$.*

*Proof.*Here as the $gcd(m,n)=p^s$, the possible $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_m$ into $\mathbb{Z}_n$ are $\Gamma_k(x)=\frac{n}{p^s}kx \pmod n$ where $k=0,1,2,\ldots,p^s-1$. $\Gamma_k(\mu)$ is a fuzzy module on $\mathbb{Z}_n$ for an arbitrary fuzzy module μ on $\mathbb{Z}_m$. Now the $\mathbb{Z}$-module homomorphisms are divided into **s+1** cases according to the values of $gcd(p^s,k)$. Let r be the highest power of p in the prime factorisation of m.

Case 1 : When $k=0$ (i.e $gcd(p^s,k)=0$).
$\Gamma_0(x)=0$ for all $x\in\mathbb{Z}_m$
The fuzzy module $\Gamma_0(\mu)$ is same as in **Case 1 :** of Theorem 6, for all fuzzy modules μ on $\mathbb{Z}_m$ and hence its level cardinality is 2.
Case 2 : When $gcd(p^s,k)=p^{s-1}$.
The $\mathbb{Z}$-module homomorphisms, $\Gamma_k(x)=\frac{n}{p^s}kx \pmod n$ can be written as $\Gamma_k(x)=\frac{n}{p}k'x \pmod n$ where $k=k'p^{s-1}$ and $gcd(k',p)=1$. Hence there are $\phi(p)=p-1$ values for k' and hence $p-1$ such $\mathbb{Z}$-module homomorphisms. In these homomorphisms $\Gamma_k(x)=0$ if and only if x is a multiple p or 0. Hence the submodule $<p>=\{0,p,2p,\ldots,(\frac{m}{p}-1)p\}$ of $\mathbb{Z}_m$ of order $\frac{m}{p}$ is mapped to 0 in $\mathbb{Z}_n$ under these Γ_ks. So under these $\Gamma_k's$ the fuzzy module $\Gamma_k(\mu)$ is same as that of **Case 2** $gcd(p^2,k)=p$ in Theorem 6 and its level cardinality is atmost 3.
Case 3 : When $gcd(p^s,k)=p^{s-2}$.
The $\mathbb{Z}$-module homomorphisms $\Gamma_k(x)=\frac{n}{p^s}kx \pmod n$ can be written as $\Gamma_k(x)=\frac{n}{p^2}k'x \pmod n$ where $k=k'p^{s-2}$ and $gcd(p^2,k)=1$. So there are $\phi(p^2)=p(p-1)$ values for k' and hence $p(p-1)$ such $\mathbb{Z}$-module homomorphisms.

As $\Gamma_k(x) = \frac{n}{p^2}k'x$ (mod n) with $gcd(p^2,k) = 1$, under these $\Gamma_k's$ the fuzzy module $\Gamma_k(\mu)$ is same as that of **Case 3** $gcd(p^2,k) = 1$ in Theorem 6 and its level cardinality is atmost 4.

Hence the homomorphic image $\Gamma_k(\mu)$ of an arbitrary fuzzy module μ has maximum level cardinality 4 when $gcd(p^s,k) = p^{s-2}$.

The cases $gcd(p^s,k) = p^{s-3}$, $gcd(p^s,k) = p^{s-4}$, ..., $gcd(p^s,k) = p^{s-(s-2)}$, $gcd(p^s,k) = p^{s-(s-1)}$ are similar.

Case s+1 : when $gcd(p^s,k) = 1$
Consider the $\mathbb{Z}$-module homomorphisms $\Gamma_k(x) = \frac{n}{p^s}kx$ (mod n) where $gcd(p^s,k) = 1$, there are $\phi(p^s) = p^{s-1}(p-1)$ values for k and hence $p^{s-1}(p-1)$ such $\mathbb{Z}$-module homomorphisms. In these $\mathbb{Z}$-module homomorphisms $\Gamma_k(x) = 0$ if and only if x is a multiple p^s or 0. Hence under these Γ_ks, the submodule $< p^s >= \{0, p^s, 2p^s, \ldots, (\frac{m}{p^s}-1)p^s\}$ of $\mathbb{Z}_m$ of order $\frac{m}{p^s}$ is mapped to 0 in $\mathbb{Z}_n$. So the elements in $\mathbb{Z}_m$ of orders which are not factors of $\frac{m}{p^s}$ under these Γ_ks will have non zero images *i.e* the elements of $\mathbb{Z}_m$ under these Γ_ks having non zero images will have orders lp^{r-s+i} with $l \mid (\frac{m}{p^r})$, $1 \le i \le s$, $l \in \mathbb{Z}^+$. Also by Theorem 1 order of $\Gamma_k(h)$ divides both n and order of h for all $h \in \mathbb{Z}_m$, hence $|f(h)|$ divides $gcd(m,n) = p^s$. So the possible orders of $\Gamma_k(x) = \frac{n}{p^s}kx($ mod n) in $\mathbb{Z}_n$ are $1, p, p^2, \ldots, p^s$ for all $x \in \mathbb{Z}_m$. So the elements in $\mathbb{Z}_n$ of orders $1, p, p^2, \ldots, p^s$ can only have preimages under these Γ_ks *i.e*, the submodule $< \frac{n}{p^s} >= \{0, \frac{n}{p^s}, 2\frac{n}{p^s}, \ldots, (p^s-1)\frac{n}{p^s}\}$ of $\mathbb{Z}_n$ of order p^s only have preimage in $\mathbb{Z}_m$ under Γ_k. So if $x \in < p^s >$ *i.e* $|x| = \frac{m}{p^s}$ or its divisors, then $\frac{n}{p^s}kx($ mod n) $= 0 \in \mathbb{Z}_n$, if $x \in < p^{s-1} > \setminus < p^s > \subset \mathbb{Z}_m$ or $|x| = lp^{r-s+1}$ with $l \mid (\frac{m}{p^r})$ then $\frac{n}{p^s}kx($ mod n) $= a_1$, where a_1 is any element in $< \frac{n}{p} > \setminus < 0 >$ in $\mathbb{Z}_n$, if $x \in < p^{s-2} > \setminus < p^{s-1} > \subset \mathbb{Z}_m$ or $|x| = lp^{r-s+2}$ with $l \mid (\frac{n}{p^r})$ then $\frac{n}{p^s}kx($ mod n) $= a_2$, where a_2 is any element in $< \frac{n}{p}^2 > \setminus < \frac{n}{p} >$ in $\mathbb{Z}_n$, ..., if $x \in < p > \setminus < p^2 > \subset \mathbb{Z}_m$ or $|x| = lp^{r-s+(s-1)}$ with $l \mid (\frac{m}{p^r})$ then $\frac{n}{p^s}kx($ mod n) $= a_{s-1}$, where a_{s-1} is any element in $< \frac{n}{p^{s-1}} > \setminus < \frac{n}{p^{s-2}} >$ in $\mathbb{Z}_n$ and if $x \in < 1 > \setminus < p > \subset \mathbb{Z}_m$ or $|x| = lp^{r-s+s}$ with $l \mid (\frac{m}{p^r})$ then $\frac{n}{p^s}kx($ mod n) $= a_s$ where a_s is any element in $< \frac{n}{p^s} > \setminus < \frac{n}{p^{s-1}} >$ in $\mathbb{Z}_n$

$$\Gamma_k(x) = \begin{cases} 0 & \text{if } |x| = \frac{m}{p^s} \text{ or its divisors} \\ a_1 \in < \frac{n}{p} > \setminus < 0 > \subset \mathbb{Z}_n & \text{if } |x| = lp^{r-s+1} \text{ with } l \mid (\frac{m}{p^r}) \\ a_2 \in < \frac{n}{p^2} > \setminus < \frac{n}{p} > \subset \mathbb{Z}_n & \text{if } |x| = lp^{r-s+2} \text{ with } l \mid (\frac{m}{p^r}) \\ \vdots & \\ a_{s-1} \in < \frac{n}{p^{s-1}} > \setminus < \frac{n}{p^{s-2}} > \subset \mathbb{Z}_n & \text{if } |x| = lp^{r-1} \text{ with } l \mid (\frac{m}{p^r}) \\ a_s \in < \frac{n}{p^s} > \setminus < \frac{n}{p^{s-1}} > \subset \mathbb{Z}_n & \text{if } |x| = lp^r \text{ with } l \mid (\frac{m}{p^r}) \end{cases}$$

Now let μ be an arbitrary fuzzy module on $\mathbb{Z}_m$ then $\Gamma_k(\mu)(y) = 0$ for all $y \in < 1 > \setminus < \frac{n}{p^s} > \subset \mathbb{Z}_n$, if $n \neq p^s$ and the fuzzy module $\Gamma_k(\mu)$ with maximum level cardinality is defined as follows. Let $t_1 = \vee\{\mu(x_1) \ / \ x_1 \in \mathbb{Z}_m,\ |x_1| = lp^{r-s+1},\ l \mid (\frac{m}{p^r})\}$, $t_2 = \vee\{\mu(x_2) \ / \ x_2 \in \mathbb{Z}_m,\ |x_2| = lp^{r-s+2},\ l \mid (\frac{m}{p^r})\}$, ... , $t_s = \vee\{\mu(x_s) \ / \ x_s \in \mathbb{Z}_m,\ |x_s| = lp^r,\ l \mid (\frac{m}{p^r})\}$ and $\mu(0) > t_1 > t_2 > \cdots > t_s > 0$ then $\Gamma_k(\mu)$ have level submodules of orders 1, p, p^2, ..., p^s and n (if $n \neq p^s$). Also $\Gamma_k(\mu)$ on $\mathbb{Z}_n$ is

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in < \frac{n}{p} > \setminus < 0 > \\ t_2 & \text{if } y \in < \frac{n}{p^2} > \setminus < \frac{m}{p} > \\ \vdots & \\ t_s & \text{if } y \in < \frac{n}{p^s} > \setminus < \frac{n}{p^{s-1}} > \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^s} > \end{cases}$$

So level cardinality of $\Gamma_k(\mu)$ is $s+2$.

Hence, if $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ is a $\mathbb{Z}$-module homomorphism and μ an arbitrary fuzzy module on $\mathbb{Z}_m$, the maxima for the level cardinality of its homomorphic image $\Gamma(\mu)$ on $\mathbb{Z}_n$ is $s+2$ when $gcd(p^s,k) = 1$.

*Example 2.*Here we take the $\mathbb{Z}$-module homomorphism between $\mathbb{Z}_{640}$ and $\mathbb{Z}_{576}$. Since $gcd(640,576) = 64$, the $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_{640}$ into $\mathbb{Z}_{576}$ are $\Gamma_k(x) = 5kx$ (mod 576) $\forall x \in \mathbb{Z}_{640}$, $k = 0,1,2,\ldots,63$. By Theorem 2, any fuzzy module on $\mathbb{Z}_{640}$ have level cardinality atmost 9. If we consider some fuzzy module μ on $\mathbb{Z}_{640}$ of level cardinality between 7 and 9, we also get the level cardinality of $\Gamma_k(\mu)$ the maximum 8, as we discussed in Example 1. Below we are considering

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

a fuzzy module μ on $\mathbb{Z}_{640}$ with level cardinality 7 and all the $\Gamma_k(\mu)'s$ with level cardinalities $2,3,\dots 8$.

$$Let\ \ \mu(x)=\begin{cases}1 & \text{if } x\in<128>\\ 1/2 & \text{if } x\in<32>\setminus<128>\\ 1/3 & \text{if } x\in<16>\setminus<32>\\ 1/4 & \text{if } x\in<8>\setminus<16>\\ 1/5 & \text{if } x\in<4>\setminus<8>\\ 1/6 & \text{if } x\in<2>\setminus<4>\\ 1/7 & \text{if } x\in<1>\setminus<2>\end{cases}$$

Cases	Number of homomorphisms	Homomorphisms	Fuzzy module
$k=0$	1	Γ_0	$\Gamma_0(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 0 & \text{if } y\in<1>\setminus<0>\end{cases}$
$gcd(k,64)=32$	$\phi(2)=1$	Γ_{32}	$\Gamma_{32}(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/7 & \text{if } y\in<288>\setminus<0>\\ 0 & \text{if } y\in<1>\setminus<288>\end{cases}$
$gcd(k,64)=16$	$\phi(4)=2$	Γ_{16},Γ_{48}	$\Gamma_{16}(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/6 & \text{if } y\in<288>\setminus<0>\\ 1/7 & \text{if } y\in<144>\setminus<288>\\ 0 & \text{if } y\in<1>\setminus<144>\end{cases}$
$gcd(k,64)=8$	$\phi(8)=4$	$\Gamma_8,\Gamma_{24},\Gamma_{40},\Gamma_{56}$	$\Gamma_8(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/5 & \text{if } y\in<288>\setminus<0>\\ 1/6 & \text{if } y\in<144>\setminus<288>\\ 1/7 & \text{if } y\in<72>\setminus<144>\\ 0 & \text{if } y\in<1>\setminus<72>\end{cases}$
$gcd(k,64)=4$	$\phi(16)=8$	$\Gamma_4,\Gamma_{12},\Gamma_{20},\Gamma_{28},\Gamma_{36},\Gamma_{44},\Gamma_{52},\Gamma_{60}$	$\Gamma_4(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/4 & \text{if } y\in<288>\setminus<0>\\ 1/5 & \text{if } y\in<144>\setminus<288>\\ 1/6 & \text{if } y\in<72>\setminus<144>\\ 1/7 & \text{if } y\in<36>\setminus<72>\\ 0 & \text{if } y\in<1>\setminus<36>\end{cases}$
$gcd(k,64)=2$	$\phi(32)=16$	$\Gamma_2,\Gamma_6,\Gamma_{10},\Gamma_{14},\Gamma_{18},\Gamma_{22},\Gamma_{26},\Gamma_{30},\Gamma_{34},\Gamma_{38},\Gamma_{42},\Gamma_{46},\Gamma_{50},\Gamma_{54},\Gamma_{58},\Gamma_{62}$	$\Gamma_2(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/3 & \text{if } y\in<288>\setminus<0>\\ 1/4 & \text{if } y\in<144>\setminus<288>\\ 1/5 & \text{if } y\in<72>\setminus<144>\\ 1/6 & \text{if } y\in<36>\setminus<72>\\ 1/7 & \text{if } y\in<18>\setminus<36>\\ 0 & \text{if } y\in<1>\setminus<18>\end{cases}$
$gcd(k,64)=1$	$\phi(64)=32$	$\Gamma_1,\Gamma_3,\Gamma_5,\Gamma_7,\Gamma_9,\Gamma_{11},\Gamma_{13},\Gamma_{15},\Gamma_{17},\Gamma_{19},\Gamma_{21},\Gamma_{23},\Gamma_{25},\Gamma_{27},\Gamma_{29},\Gamma_{31},\Gamma_{33},\Gamma_{35},\Gamma_{37},\Gamma_{39},\Gamma_{41},\Gamma_{43},\Gamma_{45},\Gamma_{47},\Gamma_{49},\Gamma_{51},\Gamma_{53},\Gamma_{55},\Gamma_{57},\Gamma_{59},\Gamma_{61},\Gamma_{63}$	$\Gamma_1(\mu)(y)=\begin{cases}1 & \text{if } y\in<0>\\ 1/2 & \text{if } y\in<288>\setminus<0>\\ 1/3 & \text{if } y\in<144>\setminus<288>\\ 1/4 & \text{if } y\in<72>\setminus<144>\\ 1/5 & \text{if } y\in<36>\setminus<72>\\ 1/6 & \text{if } y\in<18>\setminus<36>\\ 1/7 & \text{if } y\in<9>\setminus<18>\\ 0 & \text{if } y\in<1>\setminus<9>\end{cases}$

Hence the fuzzy module $\Gamma_1(\mu)$ has maximum level cardinality 8.

Remark. In Theorem 7 we proved that level cardinality of the homomorphic image $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost $s+2$, where $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ is a $\mathbb{Z}$-module homomorphism with $gcd(m,n)=p^s$ and μ is an arbitrary fuzzy module on $\mathbb{Z}_m$. In the next theorem, we consider the $\mathbb{Z}$-module homomorphism $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ where $gcd(m,n)=p^rq^s$, p, q are distinct primes and $r,s\in\mathbb{Z}^+$.

Theorem 8. *Let $\Gamma:\mathbb{Z}_m\to\mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n)=p^rq^s$, where p and q are distinct primes and $r,s\in\mathbb{Z}^+$ and let μ be a fuzzy module on $\mathbb{Z}_m$ then level cardinality of $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost $r+s+2$.*

Proof. Here as $gcd(m,n)=p^rq^s$, the possible $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_m$ into $\mathbb{Z}_n$ are $\Gamma_k(x)=\frac{n}{p^rq^s}kx \pmod{n}$ where $k=0,1,2,\dots,p^rq^s-1$. $\Gamma_k(\mu)$ is a fuzzy module on $\mathbb{Z}_n$ for any arbitrary fuzzy module μ on $\mathbb{Z}_m$. Now the

$\mathbb{Z}$-module homomorphisms are divided into $(r+1)(s+1)$ cases according to the values of $gcd(p^r q^s, k)$ and the $(r+1)(s+1)$ cases are $k = 0$(i.e $gcd(p^2, k) = 0$), $gcd(p^r q^s, k) = p^i q^j$ where $0 \le i + j \le r + s - 1$, $0 \le i \le r$ and $0 \le j \le s$. Let r_1 be the highest power of p and s_1 be the highest power of q in the prime factorisation of m.

Here we are considering the case when $gcd(p^r q^s, k) = 1$, for which the fuzzy module $\Gamma_k(\mu)$ have maximum level cardinality. In the $\mathbb{Z}$-module homomorphism $\Gamma_k(x) = \frac{n}{p^r q^s} kx \ (mod \ n)$, with $gcd(p^r q^s, k) = 1$ there are $\phi(p^r q^s) = p^{r-1} q^{s-1}(p-1)(q-1)$ values for k and hence $p^{r-1} q^{s-1}(p-1)(q-1)$ such homomorphisms. The fuzzy module Γ_k on $\mathbb{Z}_m$ is defined as follows.

$$\Gamma_k(x) = \begin{cases} 0 & \text{if } |x| == l p^{r_1 - r} q^{s_1 - s} \text{ where } l \mid (\frac{m}{p^{r_1} q^{s_1}}) \\ a_{ij} & \text{if } |x| = l p^{r_1 - r + i} q^{s_1 - s + j} \text{ where } l \mid (\frac{m}{p^{r_1} q^{s_1}}) \end{cases}$$

where a_{ij} is any element in $< \frac{n}{p^i q^j} > \setminus (< \frac{n}{p^{i-1} q^j} > \cup < \frac{n}{p^i q^{j-1}} >) \subset \mathbb{Z}_n$, $0 \le i \le r$,$0 \le j \le s$, when $i = 0$ or $j = 0$, then take $< \frac{n}{p^i q^{j-1}} > = < \frac{n}{p^{i-1} q^j} > = < 0 >$.
Now let μ be an arbitrary fuzzy module on $\mathbb{Z}_m$ then $\Gamma_k(\mu)(y) = 0$ for all $y \in < 1 > \setminus < \frac{n}{p^r q^s} > \subset \mathbb{Z}_n$, if $n \ne p^r q^s$ and the fuzzy module $\Gamma_k(\mu)$ with maximum level cardinality is defined as follows. Let $t_1 = \vee\{\mu(x_1) \ / \ x_1 \in \mathbb{Z}_m, |x_1| = l p^{r_1 - r + 1} q^{s_1 - s}$ or $|x_1| = l p^{r_1 - r} q^{s_1 - s + 1}$, where $l \mid (\frac{n}{p^{r_1} q^{s_1}})\}$, $t_2 = \vee\{\mu(x_2) \ / \ x_2 \in \mathbb{Z}_m, |x_2| = l p^{r_1 - r + 2} q^{s_1 - s}$ or $|x_2| = l p^{r_1 - r + 1} q^{s_1 - s + 1}$ or $|x_2| = l p^{r_1 - r} q^{s_1 - s + 2}$, where $l \mid (\frac{m}{p^{r_1} q^{s_1}})\}$, ... , $t_{r+s} = \vee\{\mu(x_{r+s}) \ / \ x_{r+s} \in \mathbb{Z}_m, |x_{r+s}| = l p^{r_1} q^{s_1}$, where $l \mid (\frac{m}{p^{r_1} q^{s_1}})\}$ and $\mu(0) > t_1 > t_2 > \cdots > t_{r+s} > 0$ then $\Gamma_k(\mu)$ with maximum level cardinality will be as follows. Also, when $n \ne p^r q^s$, we have

$$\Gamma_k(\mu)(y) = 0 \text{ if } y \in < 1 > \setminus < \frac{n}{p^r q^s} >$$

$$\Gamma_k(\mu)(y) = \begin{cases} 1 & \text{if } y \in < 0 > \\ t_1 & \text{if } y \in (< \frac{n}{p} > \cup < \frac{n}{q} >) \setminus < 0 > \\ t_2 & \text{if } y \in (< \frac{n}{p^2} > \cup < \frac{n}{pq} > \cup < \frac{n}{q^2} >) \setminus (< \frac{n}{p} > \cup < \frac{n}{q} >) \\ \vdots & \\ t_{r+s} & \text{if } y \in < \frac{n}{p^r q^s} > \setminus (< \frac{n}{p^{r-1} q^s} > \cup < \frac{n}{p^r q^{s-1}} >) \\ 0 & \text{if } y \in < 1 > \setminus < \frac{n}{p^r q^s} > \end{cases}$$

Hence, if $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ is a $\mathbb{Z}$-module homomorphism and μ an arbitrary fuzzy module on $\mathbb{Z}_m$, the maxima for the level cardinality of its homomorphic image $\Gamma(\mu)$ on $\mathbb{Z}_n$ is $r + s + 2$ when $gcd(p^r q^s, k) = 1$.

*Example 3.*Here we take $\mathbb{Z}$-module homomorphisms between $\mathbb{Z}_{4320}$ and $\mathbb{Z}_{4032}$. Since $gcd(4320, 4032) = 288 = 2^5 3^2$, the $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}_{4320}$ into $\mathbb{Z}_{4032}$ are $\Gamma_k(x) = 14kx \ (\bmod 4032) \ \forall x \in \mathbb{Z}_{4320}, \ \ k = 0, 1, 2, \ldots, 287$. By Theorem 2, the fuzzy module on $\mathbb{Z}_{4320}$ have level cardinality atmost 10. Let μ be such a fuzzy module on $\mathbb{Z}_{4320}$. Maximum level cardinality of the fuzzy module $\Gamma_k(\mu)$ on $\mathbb{Z}_{4032}$ for $k = 0, 1, \ldots, 287$ is the one with $gcd(k, 288) = 1$. So there are $\phi(288) = 24$ such $\Gamma_k(\mu)'s$ on $\mathbb{Z}_{4032}$. Below we are considering a fuzzy module μ on $\mathbb{Z}_{4320}$ with maximum cardinality 10 and one of the $\Gamma_k(\mu)$ $(k = 1)$ with maximum level cardinality 9.

$$\mu(x) = \begin{cases} 1 & \text{if } x \in < 0 > \\ 1/2 & \text{if } x \in < 2160 > \setminus < 0 > \\ 1/3 & \text{if } x \in < 720 > \setminus < 2160 > \\ 1/4 & \text{if } x \in < 144 > \setminus < 720 > \\ 1/5 & \text{if } x \in < 72 > \setminus < 144 > \\ 1/6 & \text{if } x \in < 24 > \setminus < 72 > \\ 1/7 & \text{if } x \in < 12 > \setminus < 24 > \\ 1/8 & \text{if } x \in < 6 > \setminus < 12 > \\ 1/9 & \text{if } x \in < 2 > \setminus < 6 > \\ 1/10 & \text{if } x \in < 1 > \setminus < 2 > \end{cases}$$

and

$$\Gamma_1(\mu)(y) = \begin{cases} 1 & \text{if } y \in <0> \\ 1/2 & \text{if } y \in <2016> \setminus <0> \\ 1/5 & \text{if } y \in <1008> \setminus <2016> \\ 1/6 & \text{if } y \in <336> \setminus <1008> \\ 1/7 & \text{if } y \in <168> \setminus <336> \\ 1/8 & \text{if } y \in <84> \setminus <168> \\ 1/9 & \text{if } y \in <28> \setminus <84> \\ 1/10 & \text{if } y \in <14> \setminus <28> \\ 0 & \text{if } y \in <1> \setminus <14> \end{cases}$$

Theorem 9. *Let $\Gamma : \mathbb{Z}_m \to \mathbb{Z}_n$ be a $\mathbb{Z}$-module homomorphism with $gcd(m,n) = d = p_1^{s_1}.p_2^{s_2}\dots p_k^{s_t}$, where $p_i's$ are distinct primes, $s_i \in \mathbb{Z}^+$, $i = 1,2,\dots t$ and let μ be a fuzzy module on $\mathbb{Z}_m$ then level cardinality of $\Gamma(\mu)$ on $\mathbb{Z}_n$ is atmost $s+2$ where $s = s_1 + s_2 + \cdots + s_t$.*

Proof. Let r_i be the highest power of p_i in the prime factorisation of m for $i = 1,2,\dots t$ where $gcd(m,n) = d = p_1^{s_1}.p_2^{s_2}\dots p_t^{s_t}$, $s_i \le r_i$, $i = 1,2,\dots t$. The $\mathbb{Z}$-module homomorphism of $\mathbb{Z}_m$ into $\mathbb{Z}_n$ are $\Gamma_\alpha(x) = \frac{n}{d}\alpha x$ (mod n) where $\alpha = 0,1,2,\dots d-1$. Then the cases are $gcd(\alpha,d) = 0, p_1^{m_1}.p_2^{m_2}\dots p_t^{m_t}$ where $0 \le m_i \le s_i$, $i = 1,2,\dots,t$ and not all $m_i = s_i$.
When $gcd(\alpha,d) = p_1^{m_1}.p_2^{m_2}\dots p_t^{m_t}$ where $0 \le m_i \le s_i$ the homomorphism is $\Gamma_\alpha(x) = \frac{n}{d}\alpha x$ (mod n) and it has order $p_1^{s_1-m_1}.p_2^{s_2-m_2}\dots p_t^{s_t-m_t}$ and for any fuzzy module μ on $\mathbb{Z}_m$, then the maximum level cardinality of the homomorphic image of the fuzzy module μ on $\mathbb{Z}_m$, $\Gamma_\alpha(\mu)$ on $\mathbb{Z}_n$ is $(s_1 - m_1) + (s_2 - m_2) + \cdots + (s_t - m_t) + 2$.

From the above theorems, we can see that the maximum level cardinality of homomorphic image of fuzzy module is attained when $gcd(\alpha,d) = 1$, then the module homomorphism Γ_α maps $\mathbb{Z}_m$ to the submodule $< \frac{n}{d} >$ of $\mathbb{Z}_n$ and its order is d. By Theorem 2, maximum level cardinality of the submodule $< \frac{n}{d} >$ of $\mathbb{Z}_n$ is $d+1$ and also $\Gamma_\alpha(\mu)(y) = 0$ $\forall y \in <1> \setminus < \frac{n}{d} >$. Thus level cardinality of $\Gamma_\alpha(\mu)$ is atmost $d+2$.

4 Conclusion

Earlier studies on the level cardinality of fuzzy modules under $\mathbb{Z}$-module homomorphisms were restricted to cases where $gcd(m,n)$ is a prime or the product of two distinct primes, yielding fixed upper bounds on the number of membership levels in the image. In this work, we are extending the analysis to the more general cases $gcd(m,n)$ is a prime power and, further, a product of prime powers. The level cardinality of fuzzy submodules and its images under fuzzy module homomorphism places an vital role in fuzzy Noetherian module theory [3]. The level cardinality of fuzzy Noetherian submodules is important when we consider the ascending or desceding chain in fuzzy Noetherian module theory. In future, we study on the properties of the generalisation of Noetherian module theory to fuzzy Noetherian module theory by using the properties of these level cardinalities.

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