Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/19.1.10

Reachability Of Linear Systems Over The Symmetrized Max-Plus Algebra

Suroto[1,*], *Diah Junia Eksi Palupi*[2] *and Ari Suparwanto*[3]

[1] Department of Mathematics, Universitas Jenderal Soedirman
[2] Department of Mathematics, Universitas Gadjah Mada
[3] Department of Mathematics, Universitas Gadjah Mada

Received: Oct. 12, 2024 Accepted: Sept. 30,2025

Abstract: The criteria of reachability [resp. observability] of max-plus algebraic linear system is determined using asticity of columns or rows of reachability [resp. observability] matrix. In this paper, we determine the criteria of reachability [resp. observability] of symmetrized max-plus algebraic linear system. The determination of these criteria is carried out using rank of reachability [resp. observability] matrix. The linear system is reachable [resp. observable] when the reachability [resp. observability] matrix is full row or column rank in balance sense.

Keywords: Linear Systems; Observability; Reachability; Rank; Symmetrized Max-Plus Algebra

2010 Mathematics Subject Classification. 93C05; 15A03; 15A80.

1 Introduction

A linear system is a mathematical model of a system that relies on the application of a linear operator in conventional algebra and systems theory. Compared to the nonlinear example, linear systems usually have features and attributes that are more simpler. Linear systems, as a mathematical abstraction, have significant applications in signal processing, telecommunication, and automatic control theory. For instance, linear systems are frequently used to describe the propagation medium in wireless communication systems. The discussion of conventional algebraic linear systems can be referred to [11][13], and more specifically discrete linear systems can be referred to [8].

The max-plus algebra, denoted by $\mathbb{R}_{\max}$, is a semiring $\mathbb{R} \cup \{-\infty\}$, with addition defined as "maximum" and multiplication defined as "plus". In this algebra, the zero element is $-\infty$ and the unity element is 0. Unlike in linear algebra, there is no additive inverse for elements in the max-plus algebra, except for the zero element [2][4][5][9].

The discussion of linear systems has been carried out in max-plus algebra. Several discussions about max-plus algebraic linear systems have been carried out, including [1][7]. Furthermore, a discussion regarding the controllability of max-plus algebraic linear systems has also been carried out [6]. Meanwhile, discussion of the reachability [resp. observability] of max-plus algebraic linear systems has been discussed [10]. The determination of criteria of reachability [resp. observability] is done using asticity of columns or rows of reachability [resp. observability] matrix. The result obtained is the weak reachability [resp. observability] of max-plus algebraic linear system. This is caused by the absence of additive inverse of non zero element in max-plus algebra. This is different from determining the reachability [resp. observability] criteria for a linear system in conventional algebra which is carried out using a rank of matrix.

The max-plus algebra can be extended to a larger set using a process called symmetrization. The symmetrization results of $\mathbb{R}_{\max}$ is called the symmetrized max-plus algebra, denoted by $\mathbb{S}$ [3][4]. Furthermore, $\mathbb{R}_{\max}$ can be considered as the positive or zero part of $\mathbb{S}$. The construction is similar to the extension of natural numbers to integers in conventional algebra, but complications arise due to the idempotency of the "max" operator. The generalization of the notion of an equation is called a balance, denoted by ∇. Since $\mathbb{R}_{\max}$ can be considered as the positive or zero part of $\mathbb{S}$, $\mathbb{R}_{\max}$ is a subset of $\mathbb{S}$. In [3], rank of matrix over $\mathbb{S}$ is defined using minor rank. Meanwhile, discussion about linear independently is used

* Corresponding author e-mail: suroto@unsoed.ac.id

to characterize of rank of matrix over $\mathbb{S}$ as in conventional algebra was discussed in [12] and solution of systems of linear balances over $\mathbb{S}$ was discussed in [14] which has a role like a linear equation system in conventional algebra [15].

This paper studies the reachability [resp. observability] of symmetrized max-plus algebraic linear system. The criteria of reachability [resp. observability] is determined using the rank of reachability [resp. observability] matrix. Since $\mathbb{R}_{\max}$ can be considered as the positive or zero part of $\mathbb{S}$, the results of this paper potentially can be used to solve and improve the criteria of weak reachability [resp. observability] of the max-plus algebraic linear system in [10].

The paper is organized as follows. Section 1 gives an introduction and Section 2 gives the symmetrized max-plus algebra. We discuss the result in Section 3 and Section 4, i.e the existence solution of the linear system and reachability [resp. observability] of symmetrized max-plus algebraic linear system in balance sense.

2 The Symmetrized Max-Plus Algebra

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. The operation on $\mathbb{R}_{\max}$ is defined as follows

$$p_1 \oplus p_2 = \max(p_1, p_2)$$
$$p_1 \otimes p_2 = p_1 + p_2$$

where $\max(p_1, -\infty) = p_1$, for all $p_1, p_2 \in \mathbb{R}_{\max}$. The algebraic structure of $\mathbb{R}_{\max}$ is an idempotent commutative semiring and it is called max-plus algebra. The zero and unity element in $\mathbb{R}_{\max}$ is $\varepsilon = -\infty$ and $e = 0$, respectively. The symmetrization process is carried out to solve the problem of the absence of an additive inverse of non zero element in $\mathbb{R}_{\max}$. Let $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ which equipped addition and multiplication operations as follows:

$$(p_1, p_2) \oplus (q_1, q_2) = (p_1 \oplus q_1, p_2 \oplus q_2)$$
$$(p_1, p_2) \otimes (q_1, q_2) = (p_1 \otimes q_1 \oplus p_2 \otimes q_2, p_1 \otimes q_2 \oplus p_2 \otimes q_1)$$

for all $(p_1, p_2), (q_1, q_2) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. The algebraic structure of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ is an idempotent commutative semi ring. The zero and unity element in $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ is $(\varepsilon, \varepsilon)$ and $(0, \varepsilon)$, respectively.

Definition 1([4]). *Let* $(p_1, p_2), (q_1, q_2) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. *The balance relation* ∇ *is defined as follows:*

$$(p_1, p_2) \nabla (q_1, q_2) \textit{ iff } p_1 \oplus q_2 = p_2 \oplus q_1$$

If (p_1, p_2) is not balance with (q_1, q_2), then it is denoted by $(p_1, p_2)\tilde{\nabla}(q_1, q_2)$. The reflexive and symmetric properties apply to the balance relation, but not to the transitive. So this relation is not an equivalence relation. Consequently, we cannot obtain the quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by ∇.

Definition 2([4]). *Let* $(p_1, p_2), (q_1, q_2) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. *The relation B is defined as follows:*

$$(p_1, p_2) B (q_1, q_2) = \begin{cases} (p_1, p_2) \nabla (q_1, q_2) & ; p_1 \neq q_1 \text{ and } p_2 \neq q_2 \\ (p_1, p_2) = (q_1, q_2) & ; p_1 = q_1 \text{ or } p_2 = q_2 \end{cases}$$

Since B is an equivalence relation then the quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by B can be obtained, and this results three types equivalence classes as follows

$$\overline{(p, -\infty)} = \{(p, x) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max} | x < p\} \text{ is called a max-positive or zero,}$$
$$\overline{(-\infty, p)} = \{(x, p) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max} | x < p\} \text{ is called a max-negative or zero,}$$
$$\overline{(p, p)} = \{(p, p) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}\} \text{ is called a balanced class.}$$

The quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by B is called the symmetrized max-plus algebra and denoted by $\mathbb{S}$. The algebraic structure of $\mathbb{S}$ is an idempotent commutative semiring, where the zero element and unity element is $\overline{\varepsilon} = \overline{(\varepsilon, \varepsilon)}$ and $\overline{e} = \overline{(0, \varepsilon)}$, respectively.

In the following section, $\overline{(w, -\infty)}$, $\overline{(-\infty, w)}$ and $\overline{(w, w)}$ is simply written as w, $\ominus w$ and $w^\bullet$, respectively. Then, the set of all classes of max-positive or zero class, max-negative or zero, balanced and signed elements is denoted by $\mathbb{S}^\oplus$, $\mathbb{S}^\ominus$, $\mathbb{S}^\bullet$ and $\mathbb{S}^\vee$, respectively. Note that, $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$, $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet = \mathbb{S}$, $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \overline{\varepsilon}$ and $\mathbb{S}^\vee_\star = \mathbb{S}^\vee \setminus \mathbb{S}^\bullet$ is the set of all unity elements.

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

Theorem 1([3]). *Let $p_1, p_2 \in \mathbb{R}_{\max}$, then*

$$p_1 \oplus (\ominus p_2) = \begin{cases} p_1 & ; p_1 > p_2 \\ \ominus p_2 & ; p_1 > p_2 \\ p_1^{\bullet} & ; p_1 = p_2 \end{cases}$$

Theorem 2([3]). *For all $p_1, p_2, p_3 \in \mathbb{S}$, $p_1 \ominus p_3 \nabla p_2$ if and only if $p_1 \nabla p_2 \oplus p_3$. Furthermore, for all $p_1, p_2 \in \mathbb{S}^{\vee}$, if $p_1 \nabla p_2$ then $p_1 = p_2$.*

Theorem 3(Weak Substitution[4]). *For all $p_1, p_2, p_3 \in \mathbb{S}$ and $x \in \mathbb{S}^{\vee}$, if $x \nabla p_1$ and $p_3 \otimes x \nabla p_2$ then $p_3 \otimes p_1 \nabla p_2$.*

Definition 3([3]). *Let $M \in \mathbb{S}^{m \times n}$. The minor rank of M is the maximum of dimension of square submatrix which its determinant is not balanced.*

In the discussion of rank, minor rank of M can be characterized using linear independently as in conventional algebra [12].

3 Linear Systems in the Symmetrized Max-Plus Algebra

This section discusses the reachability [resp. observability] of symmetrized max-plus algebraic linear system. We addopt max-plus algebraic linear system [10], as follows:

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1) \tag{1}$$

$$y(k) = C \otimes x(k) \tag{2}$$

where k is index of event, $x(k)$, $y(k)$ and $u(k)$ is completion times vector of the kth event of size $n \times 1$, system output times of size $p \times 1$ and part arrival times vector of size $m \times 1$, respectively. Then, A, B and C are matrices of size $n \times n$, $n \times m$ and $n \times p$, respectively, are max-plus algebraic matrix of functions of the system service and transportation times. The equation in (1) and (2) is modified into symmetrized max-plus algebraic linear system in balance sense as follows:

$$x(k+1) \nabla A \otimes x(k) \oplus B \otimes u(k+1 \tag{3}$$

$$y(k) \nabla C \otimes x(k) \tag{4}$$

which all etries of vectors and matrices ranging over $\mathbb{S}$. The relation "=" in (1) and (2) is replaced by "∇" in (3) and (4), respectively, in solving the additive inverse problem in $\mathbb{R}_{\max}$.

Definition 4(Reachable State). *Let $x(0) \in \mathbb{S}^n$. A state $x_q \in \mathbb{S}^n$ is reachable in q-steps from $x(0)$ if there exists a control sequence $u(1), u(2), \cdots, u(q) \in \mathbb{S}^m$ which steers state $x(0)$ to reach state $x(q) \nabla x_q$.*

The set of all state $x \in \mathbb{S}^n$ which can be reached from $x(0)$ in q-steps is denoted by $\mathfrak{R}_{(x(0),q)}$. By performing the recursion process and considering the weak substitution in (3), we get state at event index n as:

$$x(n) \nabla (A^n \otimes x(0)) \oplus (A^{n-1} \otimes B \otimes u(1)) \oplus (A^{n-2} \otimes B \otimes u(2)) \oplus \cdots \oplus (A \otimes B \otimes u(n-1)) \oplus (B \otimes u(n)) \tag{5}$$

Consider the matrix

$$M_{\mathfrak{R}} = \begin{bmatrix} B & A \otimes B & A^2 \otimes B \ldots A^{n-2} \otimes B & A^{n-1} \otimes B \end{bmatrix} \in \mathbb{S}^{n \times nm}$$

and the vector

$$U_{\mathfrak{R}} = \begin{bmatrix} [u(n) & u(n-1) & u(n-2) \ldots u(2) & u(1)] \end{bmatrix}^T \in \mathbb{S}^{nm \times 1}$$

From (5), it is obtained

$$x(n) \nabla (A^n \otimes x(0)) \oplus M_{\mathfrak{R}} \otimes U_{\mathfrak{R}} \tag{6}$$

Since $(A^n \otimes x(0)) \in \mathbb{S}^n$ then there is a minus of $(A^n \otimes x(0))$ in (1) and (6) such that

$$x(n) \ominus (A^n \otimes x(0)) \nabla M_{\mathfrak{R}} \otimes U_{\mathfrak{R}} \tag{7}$$

The balance (7) shows that there is a role for $(A^n \otimes x(0))$ which this role cannot be maximally performed in [10].

Reachability refers to the problem of steering a system from the origin to a specified state using the input. The following definition shows reachability of symmetrized max-plus algebraic linear system.

Definition 5(Reachable System). *A symmetrized max-plus algebraic linear system is reachable if all states $x \in \mathbb{S}^{n\times 1}$ can be reached from state $x(0)$.*

Since each of $(A^n \otimes x(0))$ and $M_{\mathfrak{R}}$ are known then we must determine $U_{\mathfrak{R}}$ that satisfies (7) for all $x(n) \in \mathbb{S}^{n\times 1}$ in solving the reachability problem in symmetrized max-plus algebraic linear system. Therefore, the problem of determining $U_{\mathfrak{R}}$ is analogues to determine solution of the system of linear balances (7).

Theorem 4. *A symmetrized max-plus algebraic linear system is reachable if and only if rank of $M_{\mathfrak{R}}$ is n.*

Proof.$(\leftarrow)$ Since rank of $M_{\mathfrak{R}}$ is n, $M_{\mathfrak{R}}$ is full-row rank. Thus, the $n\times n$ square submatrix A_1 can be determined in which the columns of A_1 are selected from the columns of $M_{\mathfrak{R}}$ such that $\det(A_1)\tilde{\nabla}\varepsilon$. Matrix $M_{\mathfrak{R}}$ is partitioned into $M_{\mathfrak{R}} = [A_1 \ \ A_2] \otimes Q^T$, where Q is an $nm\times nm$ permutation matrix and $F = A_1{}^{-1} \otimes A_2$. If $U_{\mathfrak{R}}$ is partitioned into $\begin{bmatrix}(U_{\mathfrak{R}})_1\\(U_{\mathfrak{R}})_2\end{bmatrix}$ then

$$\begin{bmatrix}(U_{\mathfrak{R}})_1\\(U_{\mathfrak{R}})_2\end{bmatrix} = Q\otimes \begin{bmatrix}A_1{}^{-1}\otimes x(n)\ominus(A^n\otimes x(0))\\ \varepsilon\end{bmatrix} \oplus Q\otimes\begin{bmatrix}\ominus F\\ I_{nm-n}\end{bmatrix}\otimes h$$

for $h\in\mathbb{S}^{nm-n}$ satisfies (7). Therefore, we can always find the input $(U_{\mathfrak{R}})$ that satisfies (7). This means it is always possible to carry any initial state to any destination state with at most n-step, so that the linear system is reachable.
$(\rightarrow)$ Assume that the linear system is reachable. Then for any state $x(n)$ there is an input $U_{\mathfrak{R}}$ such that

$$x(n)\ominus(A^n\otimes x(0))\nabla M_{\mathfrak{R}}\otimes U_{\mathfrak{R}}. \tag{8}$$

Suppose that rank $M_{\mathfrak{R}} = r < n$. Look at the linear balance system (8). Since rank $M_{\mathfrak{R}} = r$, then we can determine the $r\times r$ square submatrix A_1 whose columns are selected from colums of $M_{\mathfrak{R}}$ such that $\det(A_1)\tilde{\nabla}\varepsilon$. Next, $M_{\mathfrak{R}}$ is partitioned into

$$M_{\mathfrak{R}} = P^T\otimes\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\otimes Q^T$$

where P is an $n\times n$ permutation matrix which is partitioned into $\begin{bmatrix}P_1\\P_2\end{bmatrix}$, Q is an $nm\times nm$ permutation matrix, $F = A_1{}^{-}1\otimes A_2$ and $G = A_3\otimes A_!{}^{-}1$. If we select the destination state $x(k)$ such that

$$(P_2\otimes x(k))\ominus(A^k\otimes x(0))\tilde{\nabla}(G\otimes P_1)\otimes(x(k)\ominus A^k\otimes x(0)) \tag{9}$$

then the linear balance system (8) has no solution. Therefore, there is no an input $U_{\mathfrak{R}}$ such that satisfies (8). Consequently, the linear system is not reachable and it is a contradiction. So, rank of $M_{\mathfrak{R}}$ must be n. □

Theorem 4 can be used as a tool to characterize the reachability of symmetrized max-plus algebraic linear system. The following example discusses about reachability of linear system.

Example 1. Consider the symmetrized max-plus algebraic linear system

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\end{bmatrix}\nabla\begin{bmatrix}0 & 0^\bullet\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}\oplus\begin{bmatrix}0\\ \varepsilon\end{bmatrix}\otimes u(k+1)$$

$$\begin{bmatrix}y_1(k)\\y_2(k)\\y_3(k)\end{bmatrix}\nabla\begin{bmatrix}1 & 0^\bullet\\ \varepsilon & 0\\ 0 & \ominus 1\end{bmatrix}\otimes\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}$$

where $\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. The reachability matrix of the linear systems is $M_{\mathfrak{R}} = \begin{bmatrix}0 & 0^\bullet\\ 0 & \varepsilon\end{bmatrix}$. Since rank of $M_{\mathfrak{R}}$ is 2, then the linear system is reachable. This means all states $x\in\mathbb{S}^{2\times 1}$ can be reached from state $x(0) = \begin{bmatrix}0\\0\end{bmatrix}$. For example, a state $\begin{bmatrix}\ominus 2\\1\end{bmatrix}$, we can find input $u(1) = 1$, $u(2) = \ominus 2$ which drive the initial state $\begin{bmatrix}0\\0\end{bmatrix}$ reach to $\begin{bmatrix}\ominus 2\\1\end{bmatrix}$. □

The following definition discusses the observability of symmetrized max-plus algebraic linear system.

Definition 6. *A symmetrized max-plus algebraic linear system is observable if the initial state $x(0)$ can always be known by observing a finite number of output sequences*

Look at the balance (4). By the weak substitution properties, we substitute (5) into (4) and it is obtained

$$y(n-1)\nabla(C\otimes A^{n-1}\otimes x(0))\oplus(C\otimes A^{n-2}\otimes B\otimes u(1))\oplus\cdots\oplus(C\otimes A\otimes B\otimes u(n-2))\oplus(C\otimes B\otimes u(n-1)). \quad (10)$$

The balance (10) can be expressed as

$$Y\nabla M_O\otimes x(0)\oplus H\otimes U. \quad (11)$$

where $Y=\begin{bmatrix} y(0)\\ y(1)\\ y(2)\\ \vdots\\ y(n-1)\end{bmatrix}\in\mathbb{S}^{np\times 1}$, $M_O=\begin{bmatrix} C\\ C\otimes A\\ C\otimes A^2\\ \vdots\\ C\otimes A^{n-1}\end{bmatrix}\in\mathbb{S}^{np\times n}$, $H=\begin{bmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon\\ C\otimes B & \varepsilon & \cdots & \varepsilon\\ C\otimes A\otimes B & C\otimes B & \cdots & \varepsilon\\ \vdots & \vdots & \ddots & \vdots\\ C\otimes A^{n-2}\otimes B & C\otimes A^{n-3}\otimes B & \cdots & C\otimes B\end{bmatrix}\in\mathbb{S}^{np\times nm}$,

and $U=\begin{bmatrix} u(1)\\ u(2)\\ u(3)\\ \vdots\\ u(n-1)\end{bmatrix}\in\mathbb{S}^{nm\times 1}$.

The observability of symmetrized max-plus algebraic linear system is problem to determine the initial state by observing a finite number of output sequences. According to (11), it is obtained

$$M_O\otimes x(0)\nabla Y\ominus H\otimes U \quad (12)$$

Since $H\otimes U$ is known and Y is observed then we must determine $x(0)$ that satisfies (12) in solving the observability problem in symmetrized max-plus algebraic linear system. Therefore, the problem of determining $x(0)$ is analogues to determine solution of the system of linear balance (12).

Theorem 5. *A symmetrized max-plus algebraic linear system is observable if and only if rank of M_O is n*

Proof.($\leftarrow$) Since rank M_O is n, M_O is full-column rank. Thus, the $n\times n$ square submatrix A_1 can be determined in which the rows of A_1 are selected from the rows of M_O such that $\det(A_1)\tilde{\nabla}\varepsilon$. The matrix M_O is partitioned into $M_O=P^T\otimes\begin{bmatrix}A_1\\A_2\end{bmatrix}$, where P is an $pn\times pn$ permutation matrix which be partitioned into $\begin{bmatrix}P_1\\P_2\end{bmatrix}$ for $P_1\in\mathbb{S}^{n\times pn}$, and $A_2\otimes A_1^{-1}=G$ is element of $\mathbb{S}^{(pn-n)\times n}$. It is obtained that

$$x(0)={A_1}^{-1}\otimes P_1\otimes(Y\ominus H\otimes U) \quad (13)$$

where $P_2\otimes(Y\ominus H\otimes U)\nabla G\otimes P_1\otimes(Y\ominus H\otimes U)$ satisfies the linear balance system (12). If we give an output Y then it can be determined state $x(0)$. Therefore, the linear system is observable.
($\rightarrow$) Since the linear system is observable, if we give an output Y then it can be determined state $x(0)$ such that

$$M_O\otimes x(0)\nabla Y\ominus H\otimes U.$$

Suppose that rank $M_O=r<n$. Since rank $M_O=r$ then the $r\times r$ square submatrix A_1 can be determined in which the rows of A_1 are selected from the rows of M_O such that $\det(A_1)\tilde{\nabla}\varepsilon$. Matrix M_O is partitioned into $M_O=P^T\otimes\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}\otimes Q^T$ where P is an $pn\times pn$ permutation with partition $P=\begin{bmatrix}P_1\\P_2\end{bmatrix}$, Q is an $n\times n$ permutation matrix, $F={A_1}^{-1}\otimes A_2$ and $G=A_3\otimes {A_1}^{-1}$. If we select output Y' such that

$$P_2\otimes(Y'\ominus H\otimes U)\tilde{\nabla}G\otimes P_1\otimes(Y'\ominus H\otimes U). \quad (14)$$

then the linear balance system (12) has no solution. Therefore, we don't get state $x(0)$ by observabing output Y' which satisfies (14) and the linear system is not observable and it is contradiction. So, rank of M_O is n. □

Theorem 5 can be used as a tool to characterize the observability of symmetrized max-plus algebraic linear system. The following example discussed about observability of linear system.

Example 2. Consider the symmetrized max-plus algebraic linear system in Example 1. The observability matrix of the linear systems is

$$M_O = \begin{bmatrix} 1 & 0^{\bullet} \\ \varepsilon & 0 \\ 0 & \ominus 1 \\ 1 & 1^{\bullet} \\ 0 & 1 \\ \ominus 1 & \ominus 2 \end{bmatrix}.$$

Since rank of M_O is 2, then the linear system is observable. This means the initial state $x(0)$ can always be known by observing a finite number of output sequences. For example, if we observe output sequences $\begin{bmatrix} 1 \\ 0 \\ \ominus 1 \end{bmatrix}$, $\begin{bmatrix} 1^{\bullet} \\ 1 \\ \ominus 2 \end{bmatrix}$, then we can know the initial state $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by using input $u(1) = 1$, $u(2) = \ominus 2$. □

4 Conclusion

The symmetrized max-plus algebraic linear system can be formed from max-plus algebraic linear system. The criteria of reachability [resp. observability] of symmetrized max-plus algebraic linear system can be determined using rank of reachability [resp. observability] matrix. The symmetrized max-plus algebraic linear systems are reachable [resp. observable] if reachability [resp. observability] matrix is full rank. The future research can be done in development of computational programme to compute solution of symmetrized max-plus algebraic linear system.

Declarations

Competing interests: The authors declare that they have no conflict of interest.
Authors' contributions:All the authors have equally contributed to complete the manuscript
Funding: This research has received no specific grant from any funding agency
Availability of data and materials:No data is used
Acknowledgments: The authors would like to thank the editor and two anonymous referees who kindly reviewed the earlier version of this manuscript and provided valuable suggestions and comments.

References

[1] B. De Schutter, T. Van den Boom, J. Xu and S.S. Farahani. Analysis and Control of Max-Plus Linear Discrete Event Systems: An Introduction.*Discrete Event Dynamic Systems*,**30(1)**(2020), 25–54.
[2] B. De Schutter and T. Van den Boom. Max-Plus Algebra and Max-Plus Linear Discrete Event Systems: An introduction.*9th International Workshop on Discrete Event Systems*, (2008), 36–42.
[3] B. De Schutter.*Max-Algebraic System Theory for Discrete Event Systems*, Ph.D.Thesis Katholieke Universiteit, Leuven, (1996).
[4] F. Baccelli, G. Cohen, G.J. Olsder and J.P. Quadrat, *Synchronization and Linearity*, New York: John Wiley Sons, (2001).
[5] J. Komenda, S. Lahaye, J.L. Boimond and T. Van den Boom. Max-Plus Algebra in the History of Discrete Event Systems. *Annual Reviews in Control*,**45**(2018), 240–249.
[6] J.M. Prou and E. Wagneur. Controllability in the Max-Algebra. *Kybernetika*,**35(1)**(1999), 13–24.
[7] J.P. Quadrat. Max-Plus Algebra and Applications to System Theory and Optimal Control.*Proceedings of the International Congress of Mathematicians*, (1995), 1511–1522.
[8] K. Ogata, *Discrete Time Control Systems*, New Jersey: Prentice-Hall International, Inc, (1995).
[9] L. Hogben, *Handbook of Linear Algebra*, Chapman Hall/CRC, (2007).
[10] M.J. Gazarik and E.W. Kamen. Reachability and Observability of Linear Systems over Max-Plus. *Kybernetika*,**35(1)**(1999), 2–12.
[11] P.J. Antsaklis and A.N. Michel, *A Linear Systems Primer*, Birkhäuser: Springer, (2007).
[12] Suroto, D.J.E. Palupi and A. Suparwanto. Characterization of Rank of a Matrix over the Symmetrized Max-Plus Algebra. *Jordan Journal of Mathematics and Statistics*,**15(4A)**(2022), 843–856.
[13] W.J. Rugh, *Linear System Theory (2nd ed)*, Prentice Hall Inc., (1996).
[14] Suroto, A. Wardayani and N. Istikaanah. Solution of System of Linear Balances Using Minor Rank in the Symmetrized Max-Plus Algebra. *IAENG International Journal of Applied Mathematics*,**55(7)**(2025), 1931–1939.

[15] A.D. Al-Rabtah. A New Iterative Method for Solving Linear Systems of Equations. *Jordan Journal of Mathematics and Statistics*,**5(3)**(2012), 169–184.