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A New Extension Of b-Metric Spaces In $\mathbb{BC}$ Frameworks With An Application To Integral Equations

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Abstract: In this paper, the term ”Bicomplex valued extended b-metric spaces” or briefly ”$\mathbb{BC}$-valued extended b-metric spaces” refers to a new extension of bicomplex valued b-metric spaces that we introduce in this study. Using this novel class of metric spaces, we extend numerous findings from literature and establish fixed point theorems with the aid of illustrative examples. Moreover, as an application of our results, we show that a unique solution of the Urysohn integral exists.

Keywords: $\mathbb{BC}$ numbers; Metric spaces; $\mathbb{BC}$- b-metric spaces; Fixed point.

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1 Introduction and Preliminaries

The theory of metric spaces (MS) has developed considerably over time, transitive from intuitive geometric ideas in a formal mathematical structure. This growth has had profound implications across mathematics, influencing various branches and applications (see, for instance [1,2] and reference therein). In particular, In [3,4] the notion of b- MS and its applications have been proposed. Later, Kamran et al. [5] generalized b-MS under the name extended b-MS. On the other hand, the concept of complex-valued ($\mathbb{C}$-valued) MS and its properties were introduced by Azam et al. in [6]. While, Choi et al. [7] offered bicomplex-valued ($\mathbb{BC}$-valued) MS as a generalization of $\mathbb{C}$-valued MS. Recently, various results on ($\mathbb{BC}$-valued) MS have been archived by Abdalla et al. [8,9]. Inspired by foregoing endeavors, we give a new extension of ($\mathbb{BC}$-valued b-MS. Upon using this new expansion of MS, several findings in the previous works are generalizable. Also, we create outcomes on common fixed points of two self-mappings that satisfy a contrastive condition and present illustrative instances. In addition, using our findings, we demonstrate that there is a unique solution to the Urysohn integral equation.

1.1 Bicomplex Numbers $\mathbb{BC}$

Following [10], let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{BC}$ be the sets of real and complex bicomplex numbers respectively.

$$\mathbb{BC} := \{u = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3 \ : \ x_k \in \mathbb{R}, k = 0,1,2,3\}.$$

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Also, the set can be expressed $\mathbb{BC}$ as

$$\mathbb{BC} = \{u = u_1 + i_2u_2 : u_1, u_2 \in \mathbb{C}\}, \tag{1}$$

where $u_1 = x_0 + i_1x_1$, $u_2 = x_2 + i_1x_3$ and i_1, i_2 are independent imaginary units with the property $i_1^2 = -1 = i_2^2$ and their product $j = i_1i_2$ is another unit such that $j^2 = 1$. The units i_1, i_2, and j satisfy

$$i_1i_2 = j, \quad i_1j = -i_2, \quad i_2j = -i_1.$$

If $u = u_1 + i_2u_2$ and $u_1^2 + u_2^2 \neq 0$, then the inverse u^{-1} of u exists and is defined as

$$u^{-1} = \frac{1}{u} = \frac{u_1 - i_2u_2}{u_1^2 + u_2^2}.$$

The set $\mathbb{BC}$ is a commutative ring with the operations $+$ and . defined in the natural way.

The subsets of $\mathbb{BC}$ are

$$\mathbb{C}(i_k) := \{s_1 + s_2i_k;\ s_1, s_2 \in \mathbb{R}\},\ k = 1,2, \tag{2}$$

and the well-known set of hyperbolic numbers

$$\mathbb{D} := \{s_1 + s_2j;\ s_1, s_2 \in \mathbb{R}\}. \tag{3}$$

Now, we define three types of conjugation on $\mathbb{BC}$ as (see [10]):

$$u^{\ddagger_1} := \bar{z}_1 + \bar{z}_2i_2, \quad u^{\ddagger_2} := u_1 - u_2\, i_2, \quad u^{\ddagger_3} := \bar{u}_1 - \bar{u}_2\, i_2. \tag{4}$$

Therefore, three modulus can be defined on $\mathbb{BC}$:

$$|u|_{i_1}^2 := u.u^{\ddagger_2} = u_1^2 + u_2^2 \in \mathbb{C}(i_1)$$

$$|u|_{i_2}^2 := u.u^{\ddagger_1} = (|u_1|^2 - |u_2|^2) + 2Re(u_1\bar{u}_2)i_2 \in \mathbb{C}(i_2)$$

$$|u|_j^2 := u.w^{\ddagger_3} = (|u_1|^2 + |u_2|^2) - 2Im(u_1\bar{u}_2)j \in \mathbb{D}.$$

The norm of a bicomplex number $u = u_1 + i_2u_2$ is defined by

$$\|u\| = \|u_1 + i_2u_2\| = (|u_1|^2 + |u_2|^2)^{\frac{1}{2}};$$

alternatively if $u = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3$, where $x_k \in \mathbb{R}, k = 0,1,2,3$, then

$$\|u\| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}.$$

The partial order relation $\preceq_{i_2}$ on $\mathbb{BC}$ is defined as follows:

If $u = u_1 + i_2u_2$, $v = v_1 + i_2v_2 \in \mathbb{BC}$, then we say $u \preceq_{i_2} v \Longleftrightarrow u_1 \preceq_{i_1} v_1$ and $u_2 \preceq_{i_1} v_2$. In other words, $u \preceq_{i_2} v$ if one of the conditions below is satisfied:

(a_1) $u_1 = v_1$ and $u_2 = v_2$,
(a_2) $u_1 \prec_{i_1} v_1$ and $u_2 = v_2$,
(a_3) $u_1 = v_1$ and $u_2 \prec_{i_1} v_2$,
(a_4) $u_1 \prec_{i_1} v_1$ and $u_2 \prec_{i_1} v_2$,

1.2 Bicomplex valued metric spaces ($\mathbb{BC}$-valued metric spaces)

Definition 1.*A functional $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \longrightarrow \mathbb{BC}$ is called a $\mathbb{BC}$-valued metric on set Υ if for any $\mu, \tau, r \in \Upsilon$ the following holds:*

(m_1) $0 \preceq_{i_2} \varpi_{\mathbb{BC}}(\mu,\tau)$,
(m_2) $\varpi_{\mathbb{BC}}(\mu,\tau) = 0 \iff \mu = \tau$,
(m_3) $\varpi_{\mathbb{BC}}(\mu,\tau) = \varpi_{\mathbb{BC}}(\tau,\mu)$,
(m_4) $\varpi_{\mathbb{BC}}(\mu,r) \preceq_{i_2} \varpi_{\mathbb{BC}}(\mu,\tau) + \varpi_{\mathbb{BC}}(\tau,r)$.

In this case, we call the ordered pair $(\Upsilon, \varpi_{\mathbb{BC}})$ a bicomplex-valued metric space ($\mathbb{BC}$-valued metric spaces).

Definition 2.*[11]. Let $s \geq 1$. A functional $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \longrightarrow \mathbb{BC}$ is called a **bicomplex valued b-metric** ($\mathbb{BC}$-valued b-metric) on a non-empty set Υ if for any $\mu, \tau, r \in \Upsilon$ the following conditions are satisfied:*

(m_1) $0 \preceq_{i_2} \varpi_{\mathbb{BC}}(\mu,\tau)$,
(m_2) $\varpi_{\mathbb{BC}}(\mu,\tau) = 0 \iff \mu = \tau$,
(m_3) $\varpi_{\mathbb{BC}}(\mu,\tau) = \varpi_{\mathbb{BC}}(\tau,\mu)$,
(m_4) $\varpi_{\mathbb{BC}}(\mu,r) \preceq_{i_2} s\big[\varpi_{\mathbb{BC}}(\mu,\tau) + \varpi_{\mathbb{BC}}(\tau,r)\big]$.

*Then $(\Upsilon, \varpi_{\mathbb{BC}})$ is called a **$\mathbb{BC}$-valued b-metric space**.*

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2 $\mathbb{BC}$-valued extended b-metric spaces

Here, we introduce the $\mathbb{BC}$-valued extended b-MS or briefly $\mathbb{BC}$-valued extended b-MS.

Definition 3.*Suppose Υ is a set and $\xi : \Upsilon \times \Upsilon \to [1,\infty)$. A $\mathbb{BC}$-**valued extended** b**-metric** is a function $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \to \mathbb{BC}$ with $\mu,\tau,r \in \Upsilon$. the following criteria apply:* (1) $\varpi_{\mathbb{BC}})(\mu,\tau) = 0 \Longleftrightarrow \mu = \tau$;
(2) $\varpi_{\mathbb{BC}}(\mu,\tau) = \varpi_{\mathbb{BC}}(\tau,\mu)$;
(3) $\varpi_{\mathbb{BC}}(\mu,\tau) \preceq_{i_2} \xi(\mu,t)\left[\varpi_{\mathbb{BC}}(\mu,r) + \varpi_{\mathbb{BC}}(r,\tau)\right]$,
*for all $\mu,\tau,r \in \Upsilon$. The pair $(\Upsilon,\varpi_{\mathbb{BC}})$ is called a $\mathbb{BC}$-**valued extended** b**-MS**.*

*Example 1.*Let $\Upsilon = [0,+\infty), \xi : \Upsilon \times \Upsilon \to [1,+\infty)$ be a function defined by $\xi(\mu,\tau) = 1+\mu+\tau$ and $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \to \mathbb{BC}$ be given as $\varpi_{\mathbb{BC}}(\mu,\tau) = 0$ if $\mu = \tau$ and $\varpi_{\mathbb{BC}}(\mu,\tau) = i_2|v-t|$ if $\mu \neq \tau$ where $|.|$ denotes the usual real modulus. Then $(\Upsilon,\varpi_{\mathbb{BC}})$ is a $\mathbb{BC}$-valued extended b-MS.

*Example 2.*Let $\Upsilon = [0,1], \xi : \Upsilon \times \Upsilon \to [1,+\infty)$ be a function defined by $\xi(\mu,\tau) = 1+\tau v$ and $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \to \mathbb{BC}$ be given as $\varpi_{\mathbb{BC}}(\mu,\tau) = (i_1+i_2)|\mu-\tau|$.
Then $(\Upsilon,\varpi_{\mathbb{BC}})$ is a $\mathbb{BC}$-valued extended b-MS.

Definition 4.*Let (u_n) be a sequence in a $\mathbb{BC}$-valued extended b-MS $(\Upsilon,\varpi_{\mathbb{BC}})$. The sequence (u_n) is said to converge to a limit $u \in \Upsilon$ if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, then a natural number N_ε exists such that $\varpi_{\mathbb{BC}}(u_n,u) \prec_{i_2} \varepsilon$ for all natural numbers $n > N_\varepsilon$. In this case we write $u_n \longrightarrow u$ or $\lim\limits_{n\longrightarrow\infty} u_n = u$.*

Definition 5.*A sequence (u_n) in a $\mathbb{BC}$-valued extended b-MS is called a Cauchy sequence if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, then a natural number N_ε exists such that $\varpi_{\mathbb{BC}}(u_n,u_m) \prec_{i_2} \varepsilon$ for any natural numbers $n,m > N_\varepsilon$*

Definition 6.*A $\mathbb{BC}$-valued extended b-MS $(\Upsilon,\varpi_{\mathbb{BC}})$ is called complete if every Cauchy sequence in Υ converges to a limit in Υ.*

Definition 7.*Suppose $(\Upsilon,\varpi_{\mathbb{BC}})$ is a $\mathbb{BC}$-valued extended b-MS. Let $g \in \Upsilon$ and $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$.*

(i)We call the set $B(g,\varepsilon) := \{h \in \Upsilon, \varpi_{\mathbb{BC}}(g,h) \prec_{i_2} \varepsilon\}$ the open ε-ball of g.
(ii)We say a self-mapping Ξ on Υ is continuous at $g \in \Upsilon$ if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, there exists $0 \prec_{i_2} \delta \in \mathbb{BC}$ such that $\Xi(B(g,\delta)) \subseteq B(\Xi g,\varepsilon)$.

Clearly, if a mapping Ξ is continuous at u in the $\mathbb{BC}$-valued extended b-metric type space $(\Upsilon,\varpi_{\mathbb{BC}})$, then $u_n \to u$ implies that $\Xi u_n \to \Xi u$ as $n \to +\infty$.

Definition 8.*The max function for the partial order $\preceq_{i_2}$ is defined as follows:*

(1)$\max\{u,v\} = v \Leftrightarrow u \preceq_{i_2} v$.
(2)$u \preceq_{i_2} \max\{v,t\} \Rightarrow u \preceq_{i_2} v$ *or* $u \preceq_{i_2} t$
(3)$\max\{u,v\} = v \Leftrightarrow u \preceq_{i_2} v$ *or* $\| u \| \leq \| v \|$.

Definition 9.*[12]. We say two families of self-mappings $(\Xi_i)_{i=1}^m$ and $(\Lambda_i)_{i=1}^n$ are pairwise-commuting if*

(a)$\Xi_{\kappa_1}\Xi_{\kappa_2} = \Xi_{\kappa_2}\Xi_{\kappa_1}, \kappa_1,\kappa_2 \in \{1,2,\cdots,m\}$.
(b)$\Lambda_{\kappa_1}\Lambda_{\kappa_2} = \Lambda_{\kappa_2}\Lambda_{\kappa_1}, \kappa_1,\kappa_2 \in \{1,2,\cdots,n\}$.
(c)$\Xi_{\kappa_1}\Lambda_{\kappa_2} = \Lambda_{\kappa_2}\Xi_{\kappa_1}, \kappa_1 \in \{1,2,\cdots,m\},\ \kappa_2 \in \{1,2,\cdots,n\}$.

Lemma 1.*[13]. Let $(\Upsilon,\varpi_{\mathbb{BC}})$ be a $\mathbb{BC}$-valued MS.*
A sequence (u_n) in Υ converges to $u \in \Upsilon$ if and only if $\|\varpi_{\mathbb{BC}}(u_n,u)\| \longrightarrow 0$ as $n \longrightarrow +\infty$.

Lemma 2.*[13]. Let $(\Upsilon,\varpi_{\mathbb{BC}})$ be a $\mathbb{BC}$-valued MS and (u_n) be a sequence in Υ such that $\lim\limits_{n\longrightarrow+\infty} u_n = u$.*
Then for any $a \in \Upsilon$, $\lim\limits_{n\longrightarrow+\infty} \|\varpi_{\mathbb{BC}}(u_n,a)\| = \|\varpi_{\mathbb{BC}}(u,a)\|$.

3 Main results

Now we state and demonstrate the main results of the paper.

Theorem 1. *Let $(\Upsilon, \varpi_{\mathbb{BC}})$ be a $\mathbb{BC}$-valued extended b-MS with degenerate $\big(1+\varpi_{\mathbb{BC}}(u,v)\big)$ for all $u,v \in \Upsilon$ and $\Xi, \Lambda : \Upsilon \to \Upsilon$ be self-maps satisfying the following condition:*

$$\varpi_{\mathbb{BC}}(\Xi u, \Lambda v) \preceq_{i_2} \alpha.\max\Big\{\varpi_{\mathbb{BC}}(u,v), \frac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(v,\Lambda v)}{1+\varpi_{\mathbb{BC}}(\Xi u,\Lambda v)}\Big\} \tag{5}$$

for all $u,v \in \Upsilon$, where $1+\varpi_{\mathbb{BC}}(\Xi u, \Lambda v) \neq 0$ and α is a real with $0<\alpha<1$ be such that for each $u_0 \in \Upsilon$, $\lim\limits_{n,m \to +\infty} \xi(u_n,u_m) < \dfrac{1}{\alpha}$, where
$u_{2n+1} = \Xi u_{2n},\ u_{2n+2} = \Lambda u_{2n+1}, n = 0,1,2,\ldots.$
Then the mappings Ξ and Λ have a unique common fixed point.

Proof. Let $u_0 \in \Upsilon$ be arbitrary. We define a sequence (u_n) in Υ as

$$u_{2k+1} = \Xi u_{2k},\ u_{2k+2} = \Lambda u_{2k+1}, k = 0,1,2,\ldots$$

Then

$$\begin{aligned}
\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2}) &= \varpi_{\mathbb{BC}}(\Xi u_{2k}, \Lambda u_{2k+1}) \\
&\preceq_{i_2} \alpha.\max\Big\{\varpi_{\mathbb{BC}}(u_{2k},u_{2k+1}), \tfrac{\varpi_{\mathbb{BC}}(u_{2k},\Xi u_{2k})\varpi_{\mathbb{BC}}(u_{2k+1},\Lambda u_{2k+1})}{1+\varpi_{\mathbb{BC}}(\Xi u_{2k},\Lambda u_{2k+1})}\Big\} \\
&= \alpha.\max\Big\{\varpi_{\mathbb{BC}}(u_{2k},u_{2k+1}), \tfrac{\varpi_{\mathbb{BC}}(u_{2k},u_{2k+1})\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2})}{1+\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2})}\Big\} \\
&\preceq_{i_2} \alpha.\varpi_{\mathbb{BC}}(u_{2k},u_{2k+1}).
\end{aligned}$$

Thus

$$\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2}) \preceq_{i_2} \alpha.\varpi_{\mathbb{BC}}(u_{2k},u_{2k+1}). \tag{6}$$

Similarly

$$\begin{aligned}
\varpi_{\mathbb{BC}}(u_{2k+2},u_{2k+3}) &= \varpi_{\mathbb{BC}}(u_{2k+3},u_{2k+2}) \\
&= \varpi_{\mathbb{BC}}(\Xi u_{2k+2}, \Lambda u_{2k+1}) \\
&\preceq_{i_2} \alpha.\max\Big\{\varpi_{\mathbb{BC}}(u_{2k+2},u_{2k+1}), \\
&\qquad \tfrac{\varpi_{\mathbb{BC}}(u_{2k+2},\Xi u_{2k+2})\varpi_{\mathbb{BC}}(u_{2k+1},\Lambda u_{2k+1})}{1+\varpi_{\mathbb{BC}}(\Xi u_{2k+2},\Lambda u_{2k+1})}\Big\} \\
&= \alpha.\max\Big\{\varpi_{\mathbb{BC}}(u_{2k+2},u_{2k+1}), \\
&\qquad \tfrac{\varpi_{\mathbb{BC}}(u_{2k+2},u_{2k+3})\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2})}{1+\varpi_{\mathbb{BC}}(u_{2k+3},u_{2k+2})}\Big\} \\
&\preceq_{i_2} \alpha.\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2}).
\end{aligned}$$

Hence

$$\varpi_{\mathbb{BC}}(u_{2k+2},u_{2k+3}) \preceq_{i_2} \alpha.\varpi_{\mathbb{BC}}(u_{2k+1},u_{2k+2}). \tag{7}$$

Therefore from (6) and (7) for $m,n \in \mathbb{N}$, we have

$$\begin{aligned}
&\varpi_{\mathbb{BC}}(u_n,u_{n+1}) \preceq_{i_2} \alpha.\varpi_{\mathbb{BC}}(u_{n-1},u_n) \\
&\preceq_{i_2} \alpha^2.\varpi_{\mathbb{BC}}(u_{n-2},u_{n-1}) \preceq_{i_2} \cdots \preceq_{i_2} \alpha^n \varpi_{\mathbb{BC}}(u_0,u_1).
\end{aligned}$$

For $m,n \in \mathbb{N}$,

$$\begin{aligned}
\varpi_{\mathbb{BC}}(u_n,u_m) &\preceq_{i_2} \xi(u_n,u_m)\big[\varpi_{\mathbb{BC}}(u_n,u_{n+1}) + \varpi_{\mathbb{BC}}(u_{n+1},u_m)\big] \\
&\preceq_{i_2} \xi(u_n,u_m)\alpha^n\varpi_{\mathbb{BC}}(u_0,u_1) + \xi(u_n,u_m)\varpi_{\mathbb{BC}}(u_{n+1},u_m) \\
&\preceq_{i_2} \xi(u_n,u_m)\alpha^n\varpi_{\mathbb{BC}}(u_0,u_1) \\
&\quad + \xi(u_n,u_m)\xi(u_{n+1},u_m)\alpha^{n+1}\varpi_{\mathbb{BC}}(u_0,u_1) \\
&\quad + \cdots + \xi(u_n,u_m)\xi(u_{n+1},u_m)\xi(u_{n+2},u_m) \\
&\quad \cdots \xi(u_{m-2},u_m)\xi(u_{m-1},u_m)\alpha^{m-1}\varpi_{\mathbb{BC}}(u_0,u_1)
\end{aligned}$$

Then

$$\begin{aligned}
\varpi_{\mathbb{BC}}(u_n,u_m) &\preceq_{i_2} \varpi_{\mathbb{BC}}(u_0,u_1)\big[\xi(u_n,u_m)\alpha^n + \xi(u_n,u_m)\xi(u_{n+1},u_m)\alpha^{n+1} \\
&\quad + \cdots + \xi(u_n,u_m)\xi(u_{n+1},u_m)\xi(u_{n+2},u_m) \\
&\quad \cdots \xi(u_{m-2},u_m)\xi(u_{m-1},u_m)\alpha^{m-1}\big]
\end{aligned}$$

That is,

$$\begin{aligned}
\|\varpi_{\mathbb{BC}}(u_n,u_m)\| &\leq \|\varpi_{\mathbb{BC}}(u_0,u_1)\|\big[\xi(u_n,u_m)\alpha^n + \xi(u_n,u_m)\xi(u_{n+1},u_m)\alpha^{n+1} \\
&\quad + \cdots + \xi(u_n,u_m)\xi(u_{n+1},u_m)\xi(u_{n+2},u_m) \\
&\quad \cdots \xi(u_{m-2},u_m)\xi(u_{m-1},u_m)\alpha^{m-1}\big]
\end{aligned}$$

Since, $\lim\limits_{n,m\to+\infty}\xi(u_{n+1},u_m)\alpha<1$ so that the series $\sum\limits_{n=1}^{+\infty}\alpha^n\prod\limits_{i=1}^{n}\xi(u_i,u_m)$ converges by ratio test for each $m\in\mathbb{N}$. Let:

$$S=\sum_{n=1}^{+\infty}\alpha^n\prod_{i=1}^{n}\xi(u_i,u_m),\ S_n=\sum_{k=1}^{n}\alpha^k\prod_{i=1}^{k}\xi(u_i,u_m)$$

Thus for $m>n$, the above inequality implies

$$\|\varpi_{\mathbb{BC}}(u_n,u_m)\|\ \leq\ \|\varpi_{\mathbb{BC}}(u_0,u_1)\|\left[S_{m-1}-S_n\right]$$

Letting $n\to+\infty$, we conclude that (u_n) is a Cauchy sequence. Since $(\Upsilon,\varpi_{\mathbb{BC}})$ is complete, let $u_n\to u\in\Upsilon$.Thus

$$\lim_{n\to+\infty}\Xi u_{2n}=\lim_{n\to+\infty}\Lambda u_{2n+1}=u \tag{8}$$

Now from the given condition we have:

$$\begin{aligned}\varpi_{\mathbb{BC}}(\Xi u,u)\ &\preceq_{i_2}\ \xi(\Xi u,u)\left[\varpi_{\mathbb{BC}}(\Xi u,\Lambda u_{2n+1})+\varpi_{\mathbb{BC}}(\Lambda u_{2n+1},u)\right]\\ &\preceq_{i_2}\ \xi(\Xi u,u)\alpha.\max\left\{\varpi_{\mathbb{BC}}(u,u_{2n+1}),\tfrac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(u_{2n+1},\Lambda u_{2n+1})}{1+\varpi_{\mathbb{BC}}(\Xi u,\Lambda u_{2n+1})}\right\}\\ &\quad+\xi(\Xi u,u)\varpi_{\mathbb{BC}}(\Lambda u_{2n+1},u)\\ &\preceq_{i_2}\ \xi(\Xi u,u)\alpha.\max\left\{\varpi_{\mathbb{BC}}(u,u_{2n+1}),\tfrac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(u_{2n+1},u_{2n+2})}{1+\varpi_{\mathbb{BC}}(\Xi u,u_{2n+2})}\right\}\\ &\quad+\xi(\Xi u,u)\varpi_{\mathbb{BC}}(u_{2n+2},u)\end{aligned}$$

$\to 0$ as $n\to+\infty$. Thus $\varpi_{\mathbb{BC}}(\Xi u,u)\preceq_{i_2}0$. Thus $\|\ \varpi_{\mathbb{BC}}(\Xi u,u)\ \|\leq 0$ and hence $\Xi u=u$. Again

$$\begin{aligned}\varpi_{\mathbb{BC}}(u,\Lambda u)=\varpi_{\mathbb{BC}}(\Xi u,\Lambda u)\ &\preceq_{i_2}\ \alpha.\max\left\{\varpi_{\mathbb{BC}}(u,u),\tfrac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(u,\Lambda u)}{1+\varpi_{\mathbb{BC}}(\Xi u,\Lambda u)}\right\}\\ &=\ 0\end{aligned}$$

Hence $\Lambda u=u$.
Now for the uniqueness part, let us suppose that $\Xi v=\Lambda v=v$ for some $v\in\Upsilon$.Then

$$\begin{aligned}\varpi_{\mathbb{BC}}(u,v)=\varpi_{\mathbb{BC}}(\Xi u,\Lambda v)\ &\preceq_{i_2}\ \alpha.\max\left\{\varpi_{\mathbb{BC}}(u,v),\tfrac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(v,\Lambda v)}{1+\varpi_{\mathbb{BC}}(\Xi u,\Lambda v)}\right\}\\ &=\ \alpha\varpi_{\mathbb{BC}}(u,v)\end{aligned}$$

This implies $(1-\alpha)\ \|\ \varpi_{\mathbb{BC}}(u,v)\ \|\leq 0$.

Since $0<\alpha<1$, we must have $u=v$ and this completes the proof.

By setting $\Xi=\Lambda$ in Theorem 1, one deduces the following:

Corollary 1.*Let $(\Upsilon,\varpi_{\mathbb{BC}})$ be a $\mathbb{BC}$-valued extended b-MS with degenerate $\big(1+\varpi_{\mathbb{BC}}(u,v)\big)$ for all $u,v\in\Upsilon$ and $\Xi:\Upsilon\to\Upsilon$ be self-map satisfying the following condition:*

$$\varpi_{\mathbb{BC}}(\Xi u,\Xi v)\preceq_{i_2}\alpha.\max\left\{\varpi_{\mathbb{BC}}(u,v),\frac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(v,\Xi v)}{1+\varpi_{\mathbb{BC}}(\Xi u,\Xi v)}\right\} \tag{9}$$

for all $u,v\in\Upsilon$, where $1+\varpi_{\mathbb{BC}}(\Xi u,\Xi v)\neq 0$ and α is a real with $0<\alpha<1$ be such that for each $u_0\in\Upsilon$, $\lim\limits_{n,m\to+\infty}\xi(u_n,u_m)<\dfrac{1}{\alpha}$, where
$u_{n+1}=\Xi u_n,\ n=0,1,2,\ldots.$

Then Ξ has a unique fixed point.

Now we give an application of Theorem 1.

Theorem 2.*If two pairwise commuting finite families of self-mapping $(\Xi_i)_{i=1}^{m}$ and $(\Lambda_i)_{i=1}^{n}$ defined on a complete $\mathbb{BC}$-valued extended b-MS $(\Upsilon,\varpi_{\mathbb{BC}})$ with mappings Ξ and Λ (with $\Xi=\Xi_1\Xi_2\cdots\Xi_m$ and $\Lambda=\Lambda_1\Lambda_2\cdots\Lambda_n$) satisfy case (5), then $(\Xi_i)_{i=1}^{m}$ and $(\Lambda_i)_{i=1}^{n}$ have a unique common fixed point.*

*Proof.*By using the same approach in [11], we arrive at the proof.

Taking $\Xi_1=\Xi_2=\cdots=\Xi_m=T$ and $\Lambda_1=\Lambda_2=\cdots=\Lambda_n=S$, in Theorem 2, we have

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Corollary 2.*If T and S are two commuting self-mappings defined on a complete $\mathbb{BC}$-valued extended b-MS $(\Upsilon, \varpi_{\mathbb{BC}})$ satisfying the condition*

$$\varpi_{\mathbb{BC}}(T^m u, S^n v) \preceq_{i_2} \alpha.\max\left\{\varpi_{\mathbb{BC}}(u,v), \frac{\varpi_{\mathbb{BC}}(u,T^m u)\varpi_{\mathbb{BC}}(v,S^n v)}{1+\varpi_{\mathbb{BC}}(T^m u, S^n v)}\right\} \tag{10}$$

for all $u,v \in \Upsilon$, where $1+\varpi_{\mathbb{BC}}(T^m u, S^n v) \neq 0$ and α is a real with $0<\alpha<1$ such that for each $u_0 \in \Upsilon$, $\lim\limits_{k,l\to+\infty} \xi(u_k,u_l) < \dfrac{1}{\alpha}$, where
$u_{2k+1} = T^m u_{2k}$, $u_{2k+2} = S^n u_{2k+1}, k = 0,1,2,\ldots$.
Then T and S have a unique common fixed point.

By setting $m=n$ and $T=S$ in Corollary 2, we deduce the following corollary.

Corollary 3.*If $T:\Upsilon\to\Upsilon$ is a mapping defined on a complete $\mathbb{BC}$-valued extended b-metric spaces $(\Upsilon, \varpi_{\mathbb{BC}})$ satisfying the condition*

$$\varpi_{\mathbb{BC}}(T^n u, T^n v) \preceq_{i_2} \alpha.\max\left\{\varpi_{\mathbb{BC}}(u,v), \frac{\varpi_{\mathbb{BC}}(u,T^n u)\varpi_{\mathbb{BC}}(v,T^n v)}{1+\varpi_{\mathbb{BC}}(T^n u, T^n v)}\right\} \tag{11}$$

for all $u,v \in \Upsilon$, where $1+\varpi_{\mathbb{BC}}(T^n u, T^n v) \neq 0$ and α is a real with $0<\alpha<1$ be such that for each $u_0 \in \Upsilon$, $\lim\limits_{k,l\to+\infty} \xi(u_k,u_l) < \dfrac{1}{\alpha}$, where
$u_{k+1} = T^n u_k, k = 0,1,2,\ldots$.
Then T has a unique fixed point.

Example 3.. Let $\Upsilon = [0,1]$. Define $\varpi_{\mathbb{BC}} : \Upsilon\times\Upsilon \to \mathbb{BC}$ and $\xi : \Upsilon\times\Upsilon \to [1,+\infty)$ as:

$$\varpi_{\mathbb{BC}}(u,v) = i_2 \mid u-v \mid$$

and $\xi(u,v) = u+v+1$ Then $\varpi_{\mathbb{BC}}$ is a complete $\mathbb{BC}$-valued extended b-metric on Υ. Define $\Xi : \Upsilon\to\Upsilon$ by $\Xi u = \frac{u}{2}$. We have:

$$\varpi_{\mathbb{BC}}(\Xi u, \Xi v) = \frac{1}{2} i_2 \mid u-v \mid$$

and

$$\max\left\{\varpi_{\mathbb{BC}}(u,v), \frac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(v,\Xi v)}{1+\varpi_{\mathbb{BC}}(\Xi u, \Xi v)}\right\} = \varpi_{\mathbb{BC}}(u,v)$$

because,

$$\frac{\varpi_{\mathbb{BC}}(u,\Xi u)\varpi_{\mathbb{BC}}(v,\Xi v)}{1+\varpi_{\mathbb{BC}}(\Xi u, \Xi v)} = \frac{-uv}{4+\mid u-v\mid^2} + i_2 \frac{\frac{1}{2}uv \mid u-v \mid}{4+\mid u-v\mid^2} \preceq_2 i_2 \mid u-v \mid$$

Then

$$\varpi_{\mathbb{BC}}(\Xi u, \Xi v) = \frac{1}{2} i_2 \mid u-v \mid \preceq_2 \frac{1}{2}\varpi_{\mathbb{BC}}(u,v) = \alpha.\varpi_{\mathbb{BC}}(u,v).$$

Note that for each $u\in\Upsilon : u_n = \frac{u}{2^n}$. Thus we obtain:

$$\lim_{n,m\to+\infty} \xi(u_n,u_m) = \lim_{n,m\to+\infty} \left(\frac{u}{2^n} + \frac{u}{2^m} + 1\right) < 2 = \frac{1}{\alpha}.$$

Therefore, all conditions of Corollary 1 hold and hence Ξ has a unique fixed point $u^* = 0$.

4 Application to integral equations

Consider the Urysohn integral equations:

$$\begin{cases} \kappa(\rho) = g(\rho) + \displaystyle\int_\theta^\lambda H_1(\rho,\tau,\kappa(\tau))\, d\tau \\ \kappa(\rho) = g(\rho) + \displaystyle\int_\theta^\lambda H_2(\rho,\tau,\kappa(\tau))\, d\tau \end{cases} \tag{12}$$

where

(i)$\kappa(\rho)$ is unknown variable for each $\rho \in [\theta,\lambda], \theta > 0$,
(ii)$g(\rho)$ is the deterministic free term defined for $\rho \in [\theta,\lambda]$,
(iii)$H_1(\rho,\tau,.)$ and $H_2(\rho,\tau,.)$ are deterministic kernels defined for $\rho,\tau \in [\theta,\lambda]$.

Let $\Upsilon = (C[\theta,\lambda],\mathbb{R}^n)$ and $\varpi_{\mathbb{BC}} : \Upsilon \times \Upsilon \to \mathbb{BC}$ defined by

$$\varpi_{\mathbb{BC}}(\kappa_1(\rho),\kappa_2(\rho)) = \max_{\rho\in[\theta,\lambda]} \parallel \kappa_1(\rho) - \kappa_2(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta}$$

for all $\kappa_1,\kappa_2 \in \Upsilon$.
Obviously, $(\Upsilon,\varpi_{\mathbb{BC}})$ is a complete $\mathbb{BC}$-valued extended b-metric space, where $\xi : \Upsilon \times \Upsilon \to [1,\infty)$ is defined as $\xi(\kappa_1,\kappa_2) = 2$.
Put

$$\varphi_{\kappa_1}(\rho) = \int_\theta^\lambda H_1(\rho,\tau,\kappa_1(\tau))\, d\tau,\ \psi_{\kappa_2}(\rho) = \int_\theta^\lambda H_2(\rho,\tau,\kappa_2(\tau))\, d\tau.$$

Further let us consider a Urysohn type integral system as (12) under the following conditions:

(1)$g \in \Upsilon$,
(2)$H_1,H_2 : [\theta,\lambda] \times [\theta,\lambda] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.
(3)There exist $0 < \alpha \leq \frac{1}{3}$ such that the inequality:

$$\Delta(\kappa_1,\kappa_2)(\rho) \preceq_{i_2} \alpha.L(\kappa_1,\kappa_2)(\rho), \tag{13}$$

where,

$$L(\kappa_1,\kappa_2)(\rho) = \max\left\{\Delta_1(\kappa_1,\kappa_2)(\rho), \frac{\Delta_2(\kappa_1,\kappa_2)(\rho)\Delta_3(\kappa_1,\kappa_2)(\rho)}{1+\Delta_4(\kappa_1,\kappa_2)(\rho)}\right\},$$

$$\Delta(\kappa_1,\kappa_2)(\rho) = \parallel \varphi_{\kappa_1}(\rho) - \psi_{\kappa_2}(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta}$$

$$\Delta_1(\kappa_1,\kappa_2)(\rho) = \parallel \kappa_1(\rho) - \kappa_2(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta},$$

$$\Delta_2(\kappa_1,\kappa_2)(\rho) = \parallel \kappa_1(\rho) - \varphi_{\kappa_1}(\rho) - g(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta},$$

$$\Delta_3(\kappa_1,\kappa_2)(\rho) = \parallel \kappa_2(\rho) - \psi_{\kappa_2}(\rho) - g(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta},$$

$$\Delta_4(\kappa_1,\kappa_2)(\rho) = \parallel \varphi_{\kappa_1}(\rho) - \psi_{\kappa_2}(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta},$$

holds for all $\kappa_1,\kappa_2 \in \Upsilon$.

Theorem 3.*The system (12) under the conditions (1) and (2) has a unique common solution.*

*Proof.*For $\kappa_1,\kappa_2 \in (C[\theta,\lambda],\mathbb{R})$ and $\rho \in [\theta,\lambda]$, we define the continuous mappings $\Xi,\Lambda : \Upsilon \to \Upsilon$ by

$$\Xi\kappa_1 = \varphi_{\kappa_1} + g$$

$$\Lambda\kappa_2 = \psi_{\kappa_2} + g$$

then we have

$$\varpi_{\mathbb{BC}}(\kappa_1,\kappa_2) = \max_{\rho\in[\theta,\lambda]} \parallel \kappa_1(\rho) - \kappa_2(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta} = \max_{\rho\in[\theta,\lambda]} \Delta_1(\kappa_1,\kappa_2)(\rho),$$

$$\varpi_{\mathbb{BC}}(\kappa_1,\Xi\kappa_1) = \max_{\rho\in[\theta,\lambda]} \parallel \kappa_1(t) - \varphi_{\kappa_1}(\rho) - g(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta} = \max_{\rho\in[\theta,\lambda]} \Delta_2(\kappa_1,\kappa_2)(\rho),$$

$$\varpi_{\mathbb{BC}}(\kappa_2,\Lambda\kappa_2) = \max_{\rho\in[\theta,\lambda]} \parallel \kappa_2(\rho) - \psi_{\kappa_2}(\rho) - g(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta} = \max_{\rho\in[\theta,\lambda]} \Delta_3(\kappa_1,\kappa_2)(\rho),$$

$$\varpi_{\mathbb{BC}}(\Xi\kappa_1,\Lambda\kappa_2) = \max_{\rho\in[\theta,\lambda]} \parallel \varphi_{\kappa_1}(\rho) - \psi_{\kappa_2}(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta} = \max_{\rho\in[\theta,\lambda]} \Delta_4(\kappa_1,\kappa_2)(\rho).$$

From assumption (13), for each $\rho \in [\theta,\lambda]$ we have:

$$\begin{aligned}\Delta(\kappa_1,\kappa_2)(\rho) &= \parallel \varphi_{\kappa_1}(\rho) - \psi_{\kappa_2}(\rho) \parallel_\infty \sqrt{1+\theta^2} e^{i_2 \tan^{-1}\theta} \\ &\preceq_{i_2} \alpha.L(\kappa_1,\kappa_2)(\rho) \\ &= \alpha.\max\left\{\Delta_1(\kappa_1,\kappa_2)(\rho), \tfrac{\Delta_2(\kappa_1,\kappa_2)(\rho)\Delta_3(\kappa_1,\kappa_2)(\rho)}{1+\Delta_4(\kappa_1,\kappa_2)(\rho)}\right\}\end{aligned}$$

which implies that

$$\begin{aligned}\max_{\rho\in[\theta,\lambda]} \varDelta(\kappa_1,\kappa_2)(\rho) &\preceq_{i_2} \alpha.\max_{\rho\in[\theta,\lambda]} \max\Big\{\varDelta_1(\kappa_1,\kappa_2)(\rho), \frac{\varDelta_2(\kappa_1,\kappa_2)(\rho)\varDelta_3(\kappa_1,\kappa_2)(\rho)}{1+\varDelta_4(\kappa_1,\kappa_2)(\rho)}\Big\}\\ &\preceq_{i_2} \alpha.\max\Big\{\max_{\rho\in[\theta,\lambda]} \varDelta_1(\kappa_1,\kappa_2)(\rho),\\ &\qquad \frac{\max_{\rho\in[\theta,\lambda]} \varDelta_2(\kappa_1,\kappa_2)(\rho)\max_{\rho\in[\theta,\lambda]} \varDelta_3(\kappa_1,\kappa_2)(\rho)}{1+\max_{\rho\in[\theta,\lambda]} \varDelta_4(\kappa_1,\kappa_2)(\rho)}\Big\}\end{aligned}$$

Therefore,

$$\varpi_{\mathbb{BC}}(\varXi\kappa_1,\varLambda\kappa_2) \preceq_{i_2} \alpha.\max\Big\{\varpi_{\mathbb{BC}}(\kappa_1,\kappa_2), \frac{\varpi_{\mathbb{BC}}(\kappa_1,\varXi\kappa_1)\varpi_{\mathbb{BC}}(\kappa_2,\varLambda\kappa_2)}{1+\varpi_{\mathbb{BC}}(\varXi\kappa_1,\varLambda\kappa_2)}\Big\}.$$

Also, $\alpha\in(0,\frac{1}{3}]\subset(0,1)$, and for each $u_0\in\varUpsilon$, we have $\lim_{n,m\to+\infty}\xi(u_n,u_m)=2<\dfrac{1}{\alpha}$ since $\frac{1}{\alpha}\geq 3$. Thus all the conditions of Theorem 1 are satisfied. Therefore, the system of Urysohn integral equations has a unique common solution in $\varUpsilon$.

5 Conclusion

It is well established that the Banach contraction principle, along with its generalizations in various topological settings, can be effectively applied to obtain fixed point results and analytical solutions for different types of contractions, including those of integral type. In the first part of this paper, we established common fixed point theorems within the framework of $\mathbb{BC}$-valued extended b-MS. In the application section, we utilized the obtained results to solve Urysohn-type integral equations within the same $\mathbb{BC}$-valued extended b-MS setting. An open problem remains: further exploration of fixed point results for multi-valued contractions in the context of $\mathbb{BC}$-valued extended b-MS.

Declarations

Competing interests: The authors declare that there are no conflict of interest.
Authors' contributions: The study conception and design done by [ZM and MA]. Material preparation, data collection and analysis were performed by [HA]. The first draft of the manuscript was written by [HA and DAJ] and all authors commented on previous versions of the manuscript. The authors read and approved the final manuscript.
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