



# A Note On The Distribution Of $k$ -Full Integers In Arithmetic Progressions

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**Abstract:** A  $k$ -full integer ( $k \geq 2$ ) is a positive integer  $n$  such that  $p^k$  divides  $n$  whenever  $p$  is a prime divisor of  $n$ . Let  $N_k$  be the set of such integers. For a real  $x \geq 1$ , we present an asymptotic formula for the number of natural numbers  $\{n \leq x, n \in N_k\}$  such that the sum of their digits,  $s_g(n)$ , in base  $g \geq 2$  satisfies  $s_g(n) \equiv a \pmod{b}$ , where  $a \in \mathbb{Z}$  and  $b \geq 2$ .

**Keywords:** Exponential sums,  $K$ -full integers, Sum of digits function.

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## 1 Introduction

Let  $k \geq 2$  be an integer. A positive integer  $n$  is said to be  $k$ -full if either  $n = 1$  or for each prime divisor  $p$  of  $n$ ,  $p$  has multiplicity greater than or equal to  $k$ . These numbers are referred to as powerful or square full if  $k = 2$ . The number of such numbers  $\leq x$  is represented by  $N_k(x)$  for  $x \geq 1$ . The study of special sequences of integers has long been a cornerstone of analytic and combinatorial number theory (see for instance [3, 10, 15]). Erdős and Szekeres [8] investigated the asymptotic behavior for the counting function of  $k$ -full numbers for the first time. In fact, they showed

$$N_k(x) = x^{\frac{1}{k}} \prod_{p: \text{prime}} \left( 1 + \sum_{m=k+1}^{2k-1} p^{-\frac{m}{k}} \right) + O\left(x^{\frac{1}{k+1}}\right). \quad (1)$$

Since then, numerous peoples have examined the  $k$ -full numbers and obtained a lot of relevant results (see for instance [4, 5, 13]). Actually, [11, Chapter 14.4] provides a helpful survey for these asymptotic formulae.

Let  $g \geq 2$  be an integer. Then we can write every positive integer  $n$  in a unique way as

$$n = \sum_{j \geq 0} \varepsilon_j(n) g^j, \quad (2)$$

We call this representation  $g$ -ary digital expansion of  $n$  with respect to base  $g$  and the digits are  $\varepsilon_j(n) \in \{0, 1, \dots, g-1\}$ . In accordance with (2), we define the  $g$ -ary sum of digits function of a non negative integer  $n$  as

$$s_g(n) = \sum_{j \geq 0} \varepsilon_j(n).$$

The asymptotic growth of this function has been widely addressed in the literature (see, for example, [6, 9, 12, 14]). The following asymptotic formula was obtained by Gelfond [9] after he examined the distribution of the sum of digits function

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along  $k$ -free integers, that is, the integers whose prime factors are all of multiplicity  $< k$  and got the following asymptotic formula: For  $g, b, k > 1$  and  $a$  be integers, then we have

$$\#\{n \leq x, n \text{ is } k\text{-free}, s_g(n) \equiv a \pmod{b}\} = \frac{x}{b\zeta(k)} + O_g(x^{\lambda_1}), \tag{3}$$

where  $\lambda_1 = \frac{1+(k-1)\lambda_0}{k}$ ,  $\lambda_0 = \frac{1}{2\log g} \log\left(\frac{g \sin(\pi/2b)}{\sin(\pi/2bg)}\right) < 1$  and  $\zeta(\cdot)$  is the Riemann zeta function.

Moreover, (3) has been generalized in [2] on strongly  $g$ -additive functions, notably the functions  $f : \mathbb{N} \rightarrow \mathbb{R}$  that satisfy  $f(ag^i + b) = f(a) + f(b)$ , for all  $(a, i, b) \in \mathbb{N}^2 \times \mathbb{N}_0$  and  $0 \leq b < g^i$ .

The goal of this work is to consider a similar situation to (3) for  $k$ -full integers. Indeed, we shall prove the following

**Theorem 1.** *Let  $k, b \geq 2$  be integers,  $g$  be a sufficiently large prime number. Then, for every integer  $a$ , there is a constant  $\theta_{g,k,b} > 0$  such that*

$$\#\{n \leq x, n \in N_k; s_g(n) \equiv a \pmod{b}\} = \frac{D}{b} N_k(x; a, D) + O(x^{\frac{1}{k} - \theta_{g,k,b}})$$

where  $D = \gcd(g - 1, b)$  and  $N_k(x; a, D) = \#\{n \leq x, n \in N_k : n \equiv a \pmod{D}\}$ .

Additionally, the sequence  $(\beta s_g(n))_{n \in N_k}$  is uniformly distributed modulo 1 if and only if  $\beta$  is an irrational number.

Finally, our paper is structured as follows: Let  $\alpha \in \mathbb{R}$ , the following exponential sum  $\sum_{n \leq x, n \in N_k} e(\alpha s_g(n))$  is essential to the proof of our result, and we attempt to give an upper bound for it in Section 2. Next, we provide a proof for Theorem 1.1 in Section 3. Along the paper, we set  $e(x) = \exp(2\pi ix)$ .

## 2 An estimate of exponential sum over $k$ -full integers

**Proposition 1.** *Let  $k \geq 2$  be an integer and  $g$  be a sufficiently large prime. If  $(g - 1)\alpha \in \mathbb{R} \setminus \mathbb{Z}$ , then there is a constant  $\theta_{g,k,\alpha} > 0$  such that*

$$\sum_{n \leq x, n \in N_k} e(\alpha s_g(n)) \ll x^{\frac{1}{k} - \theta_{g,k,\alpha}},$$

and the implied constant depends on  $g, k$  and  $\alpha$ .

*Proof.* Let

$$R_k(x) = \sum_{n \leq x, n \in N_k} e(\alpha s_g(n)).$$

Since, each  $k$ -full number  $n$  can be expressed in a unique way as  $n = d_1^k d_2^{k+1} \dots d_k^{2k-1}$  where  $d_2 \dots d_k$  is square free. Then, if we let  $\mu^2$  be the characteristic function of square free numbers, we have

$$\begin{aligned} |R_k(x)| &= \left| \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \mu^2(d_2 \dots d_k) \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})) \right| \\ &\leq \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \mu^2(d_2 \dots d_k) \left| \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})) \right| \\ &\leq \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \left| \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})) \right| \end{aligned} \tag{4}$$

Now, we shall use the following result proved by Drmota et al [6] which assert that: For every integer valued function  $P$  of degree  $m \geq 2$  with  $P(\mathbb{N}) \subset \mathbb{N}$  and a real number  $\alpha$  with  $(g - 1)\alpha \in \mathbb{R} \setminus \mathbb{Z}$ , there is a constant  $\eta_{g,\alpha,m} > 0$  such that

$$\sum_{n \leq x} e(\alpha s_g(P(n))) \ll x^{1 - \eta_{g,\alpha,m}}, \tag{5}$$

where  $g$  is a sufficiently large prime number and the leading coefficient of  $P$  is coprime to  $g$ .

In order to apply (5) with  $P(d_1) = d_1^k d_2^{k+1} \dots d_k^{2k-1}$ , we divided the sum in (4) into parts according to  $\gcd(d_2^{k+1} \dots d_k^{2k-1}, g) = 1$  or not. So, we write

$$\begin{aligned}
 |R_k(x)| &\leq \sum_{\substack{d_2^{k+1} \dots d_k^{2k-1} \leq x \\ (d_2^{k+1} \dots d_k^{2k-1}, g) = 1}} \left| \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e\left(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})\right) \right| \\
 &+ \sum_{\substack{d_2^{k+1} \dots d_k^{2k-1} \leq x \\ (d_2^{k+1} \dots d_k^{2k-1}, g) \neq 1}} \left| \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e\left(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})\right) \right| \\
 &= R_k^{(1)}(x) + R_k^{(2)}(x).
 \end{aligned}$$

For  $R_k^{(1)}$ , we have  $\gcd(d_2^{k+1} \dots d_k^{2k-1}, g) = 1$ . So, conditions of (5) are satisfied, then we obtain

$$\left| \sum_{d_1 \leq (x/d_2^{k+1} \dots d_k^{2k-1})^{1/k}} e\left(\alpha s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1})\right) \right| \ll \left( \frac{x}{d_2^{k+1} \dots d_k^{2k-1}} \right)^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)}.$$

Furthermore, we write

$$\begin{aligned}
 R_k^{(1)}(x) &\ll \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \left( \frac{x}{d_2^{k+1} \dots d_k^{2k-1}} \right)^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \\
 &= x^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \left( \frac{1}{d_2^{k+1} \dots d_k^{2k-1}} \right)^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \\
 &= x^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \sum_{d_2 < x^{\frac{1}{k+1}}} \dots \sum_{d_k < \left(\frac{x}{d_2^{k+1} \dots d_{k-1}^{2k-2}}\right)^{(1/2k-1)}} \prod_{i=2}^k \frac{1}{d_i^{\frac{(k-1+i)(1-\eta_{g,\alpha,k})}{k}}} \\
 &\ll x^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \sum_{d_2 < x^{\frac{1}{k+1}}} d_2^{-\frac{(k+1)(1-\eta_{g,\alpha,k})}{k}} \\
 &= x^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} \sum_{d_2 < x^{\frac{1}{k+1}}} \frac{d_2^{\eta_{g,\alpha,k}}}{d_2^{1+\frac{1-\eta_{g,\alpha,k}}{k}}} \\
 &\ll x^{\left(\frac{1-\eta_{g,\alpha,k}}{k}\right)} x^{\frac{\eta_{g,\alpha,k}}{k+1}} = x^{\frac{1}{k} - \frac{\eta_{g,\alpha,k}}{k(k+1)}}.
 \end{aligned}$$

Consequently, we show that there exists a constant  $\theta_{g,k,\alpha} > 0$  such that

$$R_k^{(1)}(x) \ll x^{\frac{1}{k} - \theta_{g,k,\alpha}},$$

For  $R_k^{(2)}$ , we let the prime factorization of  $d_2^{k+1} \dots d_k^{2k-1}$  as  $\prod_{j=1}^r p_j^{\delta_j}$ . Since  $g$  is a prime, there exists  $i \in \{1, \dots, r\}$  such that  $p_i = g$ . Hence the condition  $\gcd(d_2^{k+1} \dots d_k^{2k-1}, g) \neq 1$  is equivalent to  $\gcd(d_2^{k+1} \dots d_k^{2k-1}, g) = g$  which is equivalent to  $\gcd\left(\prod_{\substack{j=1 \\ i \neq j}}^r p_j^{\delta_j}, g\right) = 1$ .

In addition, using the fact that  $s_g$  is strongly  $g$ -additive function (see [1, Chapter 3]). We get

$$s_g(d_1^k d_2^{k+1} \dots d_k^{2k-1}) = s_g\left(d_1^k \prod_{j=1}^r p_j^{\delta_j}\right) = s_g\left(d_1^k \prod_{\substack{j=1 \\ i \neq j}}^r p_j^{\delta_j}\right)$$

So, we can apply again (5) in this case and since  $\prod_{\substack{j=1 \\ i \neq j}}^r p_j^{\delta_j} \leq d_2^{k+1} \dots d_k^{2k-1}$ , we get

$$R_k^{(2)}(n) \ll \sum_{d_2^{k+1} \dots d_k^{2k-1} \leq x} \left( \frac{x}{d_2^{k+1} \dots d_k^{2k-1}} \right)^{\left( \frac{1-\eta_{g,\alpha,k}}{k} \right)}.$$

By the same arguments as in  $R_k^{(1)}$ , we found

$$R_k^{(2)}(x) \ll x^{\frac{1}{k} - \frac{\eta_{g,\alpha,k}}{k(k+1)}}.$$

Assembling what we have proved leads to,

$$R_k(x) \ll x^{\frac{1}{k} - \theta_{g,k,\alpha}}.$$

### 3 Proof of Theorem 1

By the classical orthogonality formula, we have

$$\#\{n \leq x, n \in N_k; s_g(n) \equiv a \pmod{b}\} = \sum_{n \leq x, n \in N_k} \frac{1}{b} \sum_{i=0}^{b-1} e\left(\frac{i}{b}(s_g(n) - a)\right).$$

We let

$$\begin{cases} D = \gcd(b, g-1), & b' = b/D, \\ I = \{tb'; 0 \leq t < D\}, \\ I' = \{0, \dots, b-1\} \setminus I = \{tb' + r, 0 \leq t < D, 1 \leq r < b'\}. \end{cases}$$

Since,  $s_g(n) \equiv n \pmod{D}$  for all divisors  $D \mid (g-1)$ . Hence, if  $i = tb' \in I$ , then

$$e\left(\frac{i}{b}s_g(n)\right) = e\left(\frac{t}{D}s_g(n)\right) = e\left(\frac{t}{D}n\right).$$

and consequently,

$$\begin{aligned} \sum_{n \leq x, n \in N_k} \frac{1}{b} \sum_{i \in I} e\left(\frac{i}{b}(s_g(n) - a)\right) &= \sum_{n \leq x, n \in N_k} \frac{1}{b} \sum_{t=1}^D e\left(\frac{t}{D}(n - a)\right) \\ &= \frac{D}{b} \sum_{\substack{n \leq x, n \in N_k \\ n \equiv a \pmod{D}}} 1 \\ &= \frac{D}{b} N_k(x; a, D). \end{aligned}$$

Thus,

$$\begin{aligned} \#\{n \leq x, n \in N_k; s_g(n) \equiv a \pmod{b}\} &= \frac{D}{b} N_k(x; a, D) \\ &+ \frac{1}{b} \sum_{i \in I'} e\left(\frac{-i}{b}a\right) \sum_{n \leq x, n \in N_k} e\left(\frac{i}{b}s_g(n)\right). \end{aligned}$$

If  $I' = \emptyset$ ; which is in accordance with the degenerate case  $b \mid g-1$  then we're finished. If  $I' \neq \emptyset$ ; then we consider  $g' = (g-1)/D$  so that  $(g', b') = 1$ . Moreover, for  $i = tb' + r \in I'$ , we obtain

$$\frac{(g-1)i}{b} = \frac{Dg'(tb' + r)}{Db'} = g't + \frac{g'r}{b'} \notin \mathbb{Z}.$$

So, by Proposition 1 there is a constant  $\theta_{g,k,b,i} > 0$  verifying

$$\sum_{n \leq x, n \in N_k} e\left(\frac{i}{b}s_g(n)\right) \ll x^{\frac{1}{k} - \theta_{g,k,b,i}}.$$

If we put  $\theta_{g,k,b} = \min_{i \in I'} \theta_{g,k,b,i}$ , then this implies

$$\#\{n \leq x, n \in N_k; s_g(n) \equiv a \pmod{b}\} = \frac{D}{b} N_k(x; a, D) + O(x^{\frac{1}{k} - \theta_{g,k,b}}).$$

Now, If  $\beta \in \mathbb{Q}$  then  $(\beta s_g(n))_{(n \in N_k)}$  attains only finitely many values modulo 1. Hence, the sequence  $(\beta s_g(n))_{(n \in N_k)}$  is not uniformly distributed modulo 1.

On the contrary, if  $\beta \notin \mathbb{Q}$  then  $(g-1)h\beta \notin \mathbb{Z}$ . Thus, we can apply Proposition 1 here we replace  $\beta$  by  $h\beta$  and in view of (1), we observe that

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in N_k}} e(h\beta s_g(n)) \ll x^{-\theta_{g,k,b}}.$$

Therefore, by Weyl's criterion [7]  $(\beta s_g(n))_{(n \in N_k)}$  is uniformly distributed modulo 1.

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