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Analyzing New Estimations to The Best Approximation

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Abstract: A method to compute the best approximation is the main purpose of this paper. The realisation of approximation theory is critical in analysis. Our papers primary benefit is that it is built on Egoroffs principle, which we investigate in our theorems. We presented the findings of research on approximation using Remez algorithm. This algorithm is an effective tool for analysing the approximations used in our research.

Keywords: Best approximation; Modulus of smoothness; Jackson type estimates.

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1 Introduction and Literature Review

The area of best approximation for functions in continuous spaces plays a central and foundational role in establishing rigorous results across many branches of mathematics, particularly in approximation theory, and is also of significant intrinsic interest. In addition to its theoretical importance, it supports a broad spectrum of influential applications in areas such as functional analysis, differential equations, dynamical systems, mathematical physics, probability theory, mathematical statistics, and image processing, etc.

In fact, if the word best approximation is used, this term refers to more divisions in mathematical analysis, which was developed as a result of the works of Chebyshev (1853), Weierstrass (1885), Lebesgue (1898), Bernstein (1952, 1937), Nikolskij (1945), Kolmogorov (1985) and was followed by many of their followers. They have been to explore that approach in developing the best approximation which stipulates "**approximates a function f which is defined on a finite interval $[a,b]$ with preserving certain intrinsic shape properties and known as convexity or monotonicity of the functions**".

The work of Chebyshev (18211894) is widely regarded as the foundation of the concept of best approximation. He discovered the concept of the best uniform approximation of functions by polynomials. Chebyshevs interest in this problem was partly inspired by mechanics, as understanding optimal approximations was crucial to the design of steam engines that powered the Russian industries. The motion of these engines, which is approximately linear, follows a bow-shaped path because of the opposing movements of the cranks on either side of the machinery during operation.

Our discussion of the study's motivation and the related literature will take place here.

In 2014, we have investigated estimate for a sequence of positive linear operators by Korovkin type theorem defined on the space of all real valued continuous. We gave an establish between Riesz's representation theory and Lebesgue Stieltjes integral-i, by using Riesz's functional supremum formula via statistical limit, [1]. Kopotun, Leviatan and Shevchuk, in (2021), establish best possible pointwise estimates for approximation on $[a,b]$, [2]. They started "the question" of how

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well a function can be approximated (continuous or continuously differentiable) employing algebraic polynomials, given that the polynomials must also interpolate these functions.

Al-Muhja, Akhadkulov and Ahmad, in (2021a) and (2021b), Al-Muhja, Almyaly, in (2022), Kareem, Kamel and Hussain, in (2022), Abd Al–Sada, in (2022), Al-Muhja, Almyaly and Dheyab, in (2025), Mthawer and Abd Al-Sada, in (2025), were successful in investigating some types best weighted approximation. They developed approximation on the interval $[-1,1]$ by piecewise polynomial. Furthermore, they examined the estimation of the best weighted approximation and convex and coconvex functions with varying degrees of polynomial order. Then, contributed some estimates to open problems of direct theorem like Jackson type estimates. They have significant impact on the convex and coconvex multi-approximation. However, they discussed inverse inequality for the coconvex approximation of the function f using multi algebraic polynomials, (see [3,4,5,6,7,8,9]). In the sequel, we have collected the principal benefit of the study "best approximation" conducted in 2024. This benefit is derived from the utilisation of a separation method based on two-best-approximations, which are determined by convex polynomials. The study focuses on analysing the pupil deviation of the eye, as referenced in [10]. Leviatan and Shevchuk concluded to the Jackson type estimates of shape preserving approximation, [11].

The basic definitions and results related to best approximation are following, (see [12], [13]).

Let us first review essential definitions and concepts. Let $\mathbb{C}([a,b])$ represent the space of all continuous functions defined on the interval $[a,b]$, endowed with the uniform norm, specifically,

$$\|f\|_{\mathbb{C}} = \max_{x\in[a,b]} |f(x)|.$$

Subsequently, we revisit the concept of the best approximation.

Definition 1.*[13] Let $\mathbb{C}([a,b])$ be the space of all continuous functions, and let π_n represent the space of all algebraic polynomials of degree not exceeding n. The fact that $\pi_n \subset \mathbb{C}([a,b])$ is readily apparent. The polynomial $p^* \in \pi$ is considered the most accurate (best approximation) to a function $f \in \mathbb{C}([a,b])$ on the π if*

$$\|f - p^*\|_{\mathbb{C}} \leq \|f - p\|_{\mathbb{C}} \tag{1}$$

for all $p \in \pi$. As follows, the degree of the best approximation for a given $f \in \mathbb{C}([a,b])$ is defined:

$$E_n(f) = \inf_{p_n\in\pi_n} \|f - p_n\|_{\mathbb{C}}.$$

For infimum over π_n, the inequality (1) is true since it holds for every $p_n \in \pi_n$. Hence, the following is another way to define the best approximation. A polynomial $p_n^* \in \pi_n$ is considered the best approximation to $f \in \mathbb{C}([a,b])$ if

$$\|f - p_n^*\|_{\mathbb{C}} \leq E_n(f).$$

We now present the Chebyshev approximation theorem, an important result in the field of approximation theory.

Theorem 1.*[14] If f is a continuous function on a segment $[a,b]$, then, for any $n = 0,1,2,\cdots$, the subspace π_n of polynomials p_n of degree $\leq n$ of the form*

$$p_n(x) = \sum_{j=0}^{n} a_j x^{n-j},$$

contains a polynomial (i.e., an element of π_n) of best approximation to the function f.

The work of Chebyshev focused on the precise calculation of the degree $E_n(f)$ for a certain n, which included making the matching well-described polynomial p_n to control how much $E_n(f)$ decreases.

When Jackson first introduced the modulus of continuity in 1921, he determined the degree of approximation [15]. Ditzian and Totik [16] subsequently expanded the modulus of continuity through the symmetric difference of the function f.

Here we present a formalization of the symmetric difference.

Definition 2.*[16] The λth symmetric difference of f is given by*

$$\Delta_h^\lambda(f,x) = \begin{cases} \sum_{i=0}^{\lambda}\binom{\lambda}{i}(-1)^{\lambda-i} f(x + (\frac{2i-\lambda}{2})h)\,; & x \pm \frac{\lambda h}{2} \in [a,b]\,, \\ 0\,; & otherwise. \end{cases}$$

The definition of the modulus of smoothness was provided to enhance the understanding of the approximation rate.

Definition 3.[17] *The weighted Ditzian-Totik modulus of smoothness (DTMS) of a function $f \in L_p([a,b])$, when $0 < p \leq \infty$, is defined by*

$$\omega^{\phi}_{\lambda,r}(f,t)_p = \sup\{\|\phi(x)^r \Delta^{\lambda}_{h\phi}(f,x)\|_p : 0 < h \leq t\}$$

where $\phi(x) = \sqrt{1-x^2}$. If $r = 0$, then

$$\omega^{\phi}_{\lambda}(f,t)_p = \omega^{\phi}_{\lambda,0}(f,t)_p = \sup\{\|\Delta^{\lambda}_{h\phi}(f,x)\|_p : 0 < h \leq t\}$$

is the usual DTMS. Also, note that $\omega^{\phi}_{0,r}(f,t)_p = \|\phi^r f\|_p$.

The better measure of the approximation rate is the modulus of smoothness. While finding the best approximation to a function, it assesses the approximation error.

Despite the fact that this strategy was successful in establishing the best approximation using $\mathbb{C}([a,b])$. There is still a need to define and investigate more about estimates of the best approximation of particular functions. The goal of this paper presents new Jackson-type estimates for the best polynomial approximation by linking the best approximation to the DitzianTotik modulus of smoothness, a well-established measure of function regularity. Furthermore, this paper offers a novel application of Egoroffs principle to approximation theory, which is not common in the literature and could open up new theoretical insights.

The combination of classical tools (Chebyshev polynomials, modulus of smoothness) with a measure-theoretic principle (Egoroffs) and computational tools (Remez algorithm) adds a fresh perspective to approximation theory. The numerical examples and their alignment with theoretical estimates enhance the credibility and utility of the work.

The paper also includes MATLAB code to reproduce the approximation results, highlighting its potential application in fields such as functional analysis, numerical analysis, and applied mathematics.

2 Main Results

Our contributions to the ongoing discussion on Jackson type estimates, building on Egoroff's principle, are introduced in this section.

Theorem 2.(Egoroff type theorem) *Assume $H, H_k \subseteq \mathbb{R}$ and $f \in \mathbb{C}^r([a,b])$ has measurable function on H, H_k such that I is an index set, $k \in I$ and $r = 1,2$. If $p_n \in \Pi_n$ satisfies $p_n^{(j-1)}(x_j) = f^{(j-1)}(x_j)$, for all $0 \leq j \leq k-1$. We then obtain,*

$$E_n(f)_{\mathbb{C}(H_k)} \leq C\omega^{\phi}_{\lambda}(f, \frac{1}{n})_{\mathbb{C}(H)}.$$

Theorem 3.(Egoroff-Korovkin type theorem) *Assume that $H, H_k \subseteq \mathbb{R}$, and H is a compact set to $\mathbb{R}$. For all $f \in \mathbb{C}([a,b])$. We then obtain,*

$$E_n(f)_{\mathbb{C}(H_k)} \leq C\omega^{\phi}_{\lambda}(f, \frac{1}{n})_{\mathbb{C}(H)} \tag{2}$$

uniformly in n, if and only if

$$E_n(f_\zeta)_{\mathbb{C}(H_k)} \leq C\omega^{\phi}_{\lambda}(f_\zeta, \frac{1}{n})_{\mathbb{C}(H)}\ , \quad \zeta = 1,2,3, \tag{3}$$

uniformly in n, where $f_1(x) = 1$, $f_2(x) = x$ and $f_3(x) = x^2$, for all $x \in \mathbb{R}$.

*Proof.*Since f_ζ $(\zeta = 1,2,3)$ belong to $\mathbb{C}([a,b])$, implies (2) to (3) are obvious. Now, assume that (3) holds, $H, H_k \subseteq \mathbb{R}$, and H is a compact set to $\mathbb{R}$. For the function $f_\zeta \in \mathbb{C}([a,b])$ given in (3), where $\zeta = 1,2,3$. Then,

$$\begin{aligned}
E_n(f)_{\mathbb{C}(H_k)} &= \inf_{p_n \in \Pi_n} \|f - p_n\|_{\mathbb{C}(H_k)} \\
&\leq \|f - p_n\|_{\mathbb{C}(H_k)}\ , \quad p_n \in \Pi_n \\
&\leq C_\circ \sup \|\phi \Delta^{\lambda}(f,x)\|_{\mathbb{C}(H_k)} \\
&\leq C_\circ \lim_{k\to\infty} \sup \|\phi \Delta^{\lambda}(f,x)\|_{\mathbb{C}(H_k)}
\end{aligned}$$

$$\leq C_\circ \sup \lim_{k\to\infty} \|\phi\Delta^\lambda(f,x)\|_{\mathbb{C}(H_k)}$$

$$\leq C_\circ \sup \|\phi\Delta^\lambda(f,x)\|_{\mathbb{C}(H_\circ)}\ , \quad such\ that\ H_\circ = \cup_{k=1}^\infty H_k$$

$$\leq C_\circ \sup \max_{x\in H_\circ} |\phi\Delta^\lambda(f,x)|$$

$$\leq C_\circ \sup\{\max_{x\in H_1} |\phi\Delta^\lambda(f,x)| + \max_{x\in H_2} |\phi\Delta^\lambda(f,x)|$$

$$+\cdots+ \max_{x\in H_K} |\phi\Delta^\lambda(f,x)| + \dots\}.$$

Since H_k is a compact set to $\mathbb{R}$, we can easily show that $H \subseteq H_1 \cup H_2 \cup \cdots \cup H_k \cup \dots$, implies

$$E_n(f)_{\mathbb{C}(H_k)} \leq C \sup \|\phi\Delta^\lambda(f,x)\|_{\mathbb{C}(H)}$$

$$\leq C\omega_\lambda^\phi(f,\frac{1}{n})_{\mathbb{C}(H)}.$$

Now, by virtue of Korovkin type theorem and Eq. (2), we extension of Egoroff's principle.

Proposition 4*If H, H_k, r and f are defined in Theorems 2 and 3. For $1 \leq n \leq r$, $\hat{T}_n = \{t_j\}_{j=0}^{n=2}$ denotes the Chebyshev partition. We then obtain,*

$$E_n(f)_{\mathbb{C}(H_k)} \leq C\omega_\lambda^\phi(f,\frac{1}{n})_{\mathbb{C}(H)}.$$

*Proof.*We prove our outcome by virtue of Chebyshev polynomial and partition. Assume that $\hat{T}_n(x_j) = \{t_j\}_{j=0}^{n=2} = \{-\cos(o), -\cos(\frac{\pi}{2}), -\cos(\pi)\} = \{-1,0,1\}$. Let $f \in \mathbb{C}([a,b])$ be a continuous function and $[a,b]$ be a closed interval and has $\{-1,0,1\}$.

Suppose that $x_\circ = 0$ and $\max_{x\in[a,b]}\{R_n\} \leq \frac{K_{n+1}}{(n+1)!}\alpha^{n+1}$, where

$$K_{n+1} = \max_{\zeta\in[a,b]} |\hat{T}_n(\zeta)| = \max_{\zeta\in[a,b]} \{|-\cos(o)|, |-\cos(\frac{\pi}{2})|, |-\cos(\pi)|\}.$$

So,

$$\alpha = \max\{x_\circ - 0, b - x_\circ\} = \max\{0-0, b-0\} = b.$$

Therefore,

$$\frac{K_{n+1}}{(n+1)!}\alpha^{n+1} \leq \frac{2}{(n+1)!}b^{n+1} \leq \frac{b^3}{3} \leq e^1.$$

If $p_n(x)$, $p_n^{(j-1)}(x)$ and j are defined in Theorem 2. Then $q_n(x,f)$ denotes the restriction of $\hat{T}_n(x)$ to $[x_j, x_{j-1}]$, (if $x_j = \cos\frac{j\pi}{n}$).

The polynomial $p_n(x;f) = q_n(x,f) + \sum_{j=0}^n \hat{T}_n(x_j)$, as $n = 2$ to obtain

$$||f - p_n||_{\mathbb{C}(H_k)} = \max_{x\in H_k} |f(x) - p_n(x;f)|$$

$$= \max_{x\in H_k} |f(x) - (q_2(x;f) + \sum_{j=0}^{2} \hat{T}_2(x_j))|$$

$$= \lim_{k\to\infty} \max_{x\in H_k} |f(x) - q_2(x;f) - \hat{T}_2(x_\circ) - \hat{T}_2(x_1) - \hat{T}_2(x_2)|$$

$$= \lim_{k\to\infty} \max_{x\in H_k} |f(x) - q_2(x;f)|$$

$$= C(e^1) \lim_{k\to\infty} \sup ||\phi\Delta_\lambda(f,x)||_{\mathbb{C}(H_k)}.$$

By virtue of proof of Theorem 3, we then obtain the proof.

n	**Points**	**Errors**	**polynomials**	C	$\omega_1^{\phi}(f,\frac{1}{n})$
1	$\frac{3\pi}{7}$	0.3899	$p_1(x) = -0.025678x + 1.1742$	16.81	0.0232
	$\frac{3\pi}{2}$	4.5074		194.3	0.0232
	$\frac{5\pi}{2}$	8.1962		353.3	0.0232

Table 1: Best approximation of the function $f(x) = 1 + \frac{\cos(x)}{x}$ with degree of polynomial $n = 1$, $h = 0.23$ and the interval $I = [\frac{\pi}{6}, \frac{23\pi}{9}]$ by Remez algorithm and (C is a constant), (see Proposition 4).

n	**Errors**	**polynomials**	C	ω_2^{ϕ}
1	0.000011	$p_1(x) = 0.033259x + 0.17919$	0.000009	1.2
2	0.000049	$p_2(x) = 1.1918x^2 + 0.62826x + 0.34856$	0.000041	1.2
3	0.000037	$p_3(x) = 1.4004x^3 + 0.61152x^2 + 0.15325x - 0.15459$	0.000032	1.2
4	0.000064	$p_4(x) = 0.94101x^4 + 0.95173x^3 - 0.13009x^2 + 0.61319x - 0.01333$	0.000055	1.2
5	0.000044	$p_5(x) = 1.2655x^5 + 0.49672x^4 + 0.70171x^3 - 0.37154x^2 - 0.082284x + 0.07723$	0.000038	1.2

Table 2: Best approximation of the function $f(x) = 1 + \frac{\cos(x)}{x}$ with degree of polynomials $n = 1,2,3,4$, $h = 0.23$, (and $n = 5$, $h = 0.2$), such that the interval $I = [\frac{\pi}{4}, 2\pi]$ by Remez algorithm and (C is a constant).

2.1 Proof of Theorem 2.

Using the estimates from Theorem 3 and Proposition 4, we will now show that the following proof of Theorem 2 is correct.

*Proof.*Let I be an index set, and $H, H_k \in \mathbb{R}$, where $k = n$, consequently, this lead to

$$x_\circ, x_1, x_2, \ldots, x_n \in H_k. \tag{4}$$

Since $f \in \mathbb{C}^r([a,b])$, $r = 1,2$. As a result, this generated $p_n \in \mathbb{C}^r([a,b])$, such that $p_n^{(j-1)}(x_j) = f^{(j-1)}(x_j)$, $0 \le j \le k-1$.

Claim 1. If f is an achieve Korovkin type theorem. By virtue of the Theorem 3, we then obtain the proof.

Claim 2. If f is hold (4), because it is defined that way (Chebyshev polynomial). Simply by virtue of the Proposition 4, again, we then obtain the proof.

Claim 3. Should none of the aforementioned exist. Since f is a measurable function, therefore, f Lebesgue Stieltjes integrable-i was produce. By virtue of ([12] Definitions 1.1.4, 1.1.5, pp. 8, and Theorem 4.2.2, pp. 91), we then obtain the proof, too.

The analysis of Theorems 2, 3, and Proposition 4 are discussed in the next section. One possible application of our results are the Remez algorithm.

3 Analysis of the Study

We demonstrate the implementation of our findings (Theorems 2, 3, and Proposition 4), which are displayed in the tables below, in this section.

Function approximations for points in H_k, where functions have $2-$monotone are shown in Table 1. The value of the polynomial and the value of the error are found using the Remez algorithm for each degree of the polynomial $n = 1$.

Table 2 displays the acquired outcomes. Table 2 presents the values for the constant in Theorems 2, 3, and Proposition 4, as well as the best approximation and modulus of smoothness. The error is obtained using the Remez algorithm.

Table 3 includes several columns pertaining to the best approximation of function $f(x) = \frac{1}{x+1}$. It contains the degrees of approximation-related polynomials, such as $n = 1,2,3,4$, and 5 degrees. In order to assess accuracy, errors are also taken into account. We get the error by Remez algorithm. Specifically, the difference between the function's actual and

n	Points	Errors	polynomials	C	ω_2^ϕ
1	$x=0.5$ $x=0.6$ $x=0.8$	0.0000004	$p_1(x)=0.56289x+0.10352$	0.00000041	0.111
2	$x=0.5$ $x=0.6$ $x=0.8$	0.0000371	$p_2(x)=0.55097x^2+0.28849x-0.11404$	0.0004	0.111
3	$x=0.5$ $x=0.6$ $x=0.8$	0.000001	$p_3(x)=0.4065x^3+0.17983x^2+0.04312x-0.043755$	0.000018	0.111
4	$x=0.5$ $x=0.6$ $x=0.8$	0.000006	$p_4(x)=0.60721x^4+0.17499x^3+0.13604x^2-0.44231x-0.1081$	0.00006	0.111
5	$x=0.5$ $x=0.6$ $x=0.8$	0.0000277	$p_5(x)=0.49899x^5+0.36786x^4-0.11408x^3+0.1986x^2-0.72204x-0.038361$	0.000027	0.111

Table 3: Best approximation of the function $f(x)=\frac{1}{x+1}$ with degree of polynomials $n=1,2,3,4$, $h=0.23$, (and $n=5$, $h=0.2$), such that intervals $H_k=[0.5,0.8]$ and $H=[0,1]$ by Remez algorithm and (C is a constant).

approximate values (polynomial) is estimated. For this purpose, polynomial functions are utilised, and constants, which represent fixed values, are also utilised in the process of approximation. Furthermore, the modulus of smoothness are a measurement that determines the degree to which the function deviates from its curve of polynomial. This provides an understanding of the degree to which the approximation is accurate.

4 Conclusions

This investigation has been devoted to the investigation of methods for computing the "best approximation," a fundamental component of approximation theory that is indispensable for functional analysis. We have not only established a rigorous theoretical framework but also investigated its implications through various theorems by employing Egoroff's principle as a foundational element. In our research, the Remez algorithm has been instrumental in the analysis and refinement of the approximations under investigation, providing a potent computational tool. Additionally, our results emphasise the critical role of approximation theory in the advancement of functional analysis, providing both theoretical insights and practical applications. The synergy between theoretical principles and computational techniques, as shown in this study emphasises the effectiveness of such approaches in addressing complex problems across various disciplines. In the future, further exploration and refinement of these methodologies is expected to deepen our comprehension of the best approximations and improve their application in scientific and engineering fields.

5 Appendix

Matlab Codes:

The Matlab code for utilising the Remez algorithm to compute approximations is provided here.

An application of the Remez algorithm is used in the following code to compute the best polynomial approximation of the functions:

1. $f(x)=1+\frac{cos(x)}{x}$ at points $\frac{3\pi}{7}$, $\frac{3\pi}{2}$, and $\frac{5\pi}{2}$ that are provided, with $n=1$, (see Table 1).

2. $f(x)=1+\frac{cos(x)}{x}$ at points $\frac{\pi}{4}$, and 2π that are provided, with $n=1,2,3,4$, $h=0.23$, (also, $n=5$, $h=0.2$), (see Table 2).

3. $f(x)=\frac{1}{x+1}$ at points $0.5, 0.6$, and 0.8 that are provided, with $n=1,2,3,4$, $h=0.23$, (also, $n=5$, $h=0.2$), (see Table 3).

Algorithm 1.
f = @(x) 1 + cos(x)./x;
x_points = [3*pi/7, 3*pi/2, 5*pi/2];
degree = 1;
A = zeros(length(x_points), degree + 1);
b = zeros(length(x_points), 1);
for i = 1:length(x_points)
x = x_points(i);
for j = 0:degree
$A(i, j+1) = x^j$;
end
$b(i) = f(x)$;
end
coefficients $= A \setminus b$;
for i = 1:length(x_points)
x = x_points(i);
fprintf('polynomial $x = \%.4f : \%s \setminus n', x$, poly2str(flip(coefficients'), 'x'));
fprintf('Real-valued-function $x = \%.4f : \%.8f \setminus n', x, f(x)$);
fprintf('Estimation $x = \%.4f : \%.8f \setminus, n', x$ polyval(coefficients, x));
fprintf('Error: $\%.8f \setminus n \setminus n'$, abs(f(x) - polyval(coefficients, x)));
end

Table 4: Computing code for the Remez algorithm result in Table 1.

Algorithm 2. Table 2
f = @(x) 1 + cos(x)./x;
x_points = [pi/4, 2 pi];
degree = 1, 2, 3, 4, or 5;
for i = 1:length(x_points)
x = x_points(i);
$g = @(a)(f(x) - polyval(a, x))^2$;
a = fminsearch(g, zeros(degree+1, 1));
fprintf('polynomial $x = \%9.12f : \%s \setminus n', x$, poly2str(flip(a'), 'x'));
fprintf('Error: $\%.6f \setminus n'$, abs(f(x) - polyval(a, x)));
end

Table 5: The Remez algorithm's computation code is shown in Table 2.

Algorithm 3. Table 3
f = @(x) 1./(x+1);
x_points = [0.5, 0.6, 0.8];
degree = 1,2,3,4, or 5;
for i = 1:length(x_points)
x = x_points(i);
$g = @(a)(f(x) - polyval(a, x))^2$;
a = fminsearch(g, zeros(degree+1, 1));
fprintf('polynomial $x = \%.1f : \%s \setminus n', x$, poly2str(flip(a'), 'x'));
fprintf('Error: $\%9.14f \setminus n'$, abs(f(x) - polyval(a, x)));
end

Table 6: The computation code for the Remez algorithm is displayed in Table 3.

Declarations

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