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Moments of kth Lower Record Values From Inverse-Rayleigh Distribution And Characterization

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Abstract: This paper focuses on the inverse Rayleigh distribution, a widely used model for lifetime data. We derive expressions for the moments of k^{th} lower record values and establish recursive relations between these moments. Moreover, we present characterization results based on these recursive relations and conditional moments. We also compute key statistical properties, including mean, variance, skewness, kurtosis, and product moments. Finally, we demonstrate the application of a characterization theorem using both simulated data from this distribution and a real data set.

Keywords: Lower record values; Single moment; Product moments; Inverse Rayleigh distribution; Characterization.

2010 Mathematics Subject Classification. 62G30; 62E10; 60E05.

1 Introduction

Trayer [44] proposed a new distribution for modelling lifetime data, called the inverse Rayleigh distribution (IRD). Later, Voda [45] studied its properties in detail and obtained maximum likelihood estimator of the scale parameter of this distribution. The IRD has been extensively used in a variety of situations, particularly those involving life testing, survival analysis and industrial reliability problems. The instantaneous failure rate of the IRD shows both increasing and decreasing failure rate patterns for lifetime data.

Gharraph [21] conducted a comprehensive study on the IRD, deriving five key measures of location—mean, harmonic mean, geometric mean, mode, and median and explored several estimation techniques for its parameter. Mukherjee and Maiti [31] introduced a percentile estimator for the scale parameter of the one-parameter IRD and examined its asymptotic efficiency. Howlader et al. [23] developed Bayesian prediction bounds for both Rayleigh and inverse Rayleigh lifetime models, further demonstrating that the IRD can serve as a suitable alternative to the log-normal model in analyzing survival times for certain diseases. Soliman et al. [42] investigated both Bayesian and non-Bayesian approaches to parameter estimation within the inverse Rayleigh framework. Expanding the model, Almarashi et al. [4] proposed a two-parameter extension of the IRD using the half-logistic transformation, enhancing its ability to capture moderately right-skewed and near-symmetric lifetime data. Chiodo and Noia [14] highlighted the applicability of the IRD for modeling extreme wind speeds, an essential factor in wind power generation and mechanical safety evaluations. Moreover, Chiodo et al. [15] introduced the compound IRD, specifically designed for extreme wind speed modeling in wind turbine safety and energy production. Their work provides a practical framework for real-world applications, supported by a novel Bayesian estimation procedure, extensive numerical simulations, and robustness assessments.

Definition 1. *A random variable T is said to follow the inverse Rayleigh distribution if its probability density function (PDF) has the following form:*

$$g(t) = \left(\frac{2\beta}{t^3}\right)\exp\left(-\frac{\beta}{t^2}\right), \;\; x > 0, \;\; \beta > 0 \tag{1}$$

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and the associated distribution function (DF) is given by:

$$G(t) = \exp\left(-\frac{\beta}{t^2}\right), \;\; t > 0, \;\; \beta > 0. \tag{2}$$

Now, the relation between DF and PDF can be obtained in view of (1) and (2) as below:

$$G(t) = \left(\frac{t^3}{2\beta}\right) g(t). \tag{3}$$

Record values have become a crucial statistical model in various real-world scenarios, including weather forecasting, economic analysis, and sports. Predicting future record values is valuable in many practical applications. For instance, forecasting the amount of water a dam will capture or discharge from rainfall aids in effective planning. Similarly, estimating the intensity of the next major earthquake is vital for disaster management strategies, while predicting new records in athletic events assists in preparing athletes for rigorous training.

Chandler [13] was a pioneer in the statistical study of record values within a sequence of independent and identically distributed (iid) continuous random variables. Later, Dziubdziela and Kopociński [17] expanded the Chandler's concept by introducing k^{th} record values. When $k = 1$, we obtain the upper (lower) record values. Ahsanullah [1] and Kamps [25], in their studies, have shown various applications of k^{th} record values.

A considerable research has been devoted to the study of moment properties of record values and the characterization of probability distributions. Early contributions by Nagaraja [32,33,34] established fundamental results on the characterization of continuous distributions through regressions of adjacent order statistics and record values. Subsequent works extended these ideas by deriving recurrence relations and expressions for single and product moments of record values from a wide range of lifetime and continuous distributions, including the exponential, generalized Pareto, Lomax, Gumbel, Weibull, Burr, and exponential–Weibull families, see Balakrishnan et al. [11], Balakrishnan and Ahsanullah [8,9,10], Pawlas and Szynal [35,36,37], Khan et al. [26,27]. Further, generalizations have been explored through the study of k^{th} record values, with emphasis on recurrence relations, characterization and associated statistical inference, see Grudzien and Szynal [22], Bienik and Szynal [12]. Moreover, perspectives on distributional characterizations via conditional expectations and regression properties of record values were provided by Franco and Ruiz [18,19] and Athar et. al.[7]. Collectively, these studies form a comprehensive framework for understanding the structural properties of distributions through record values, offering both theoretical insights and practical applications in reliability analysis and lifetime data modeling. For more detailed survey one may refer to Raqab [38], Mohsin and Ahmad [30], Sultan [43], Ahsanullah and Nevzorov [3], , Makouei et al. [28], Alam et al.[2], Athar and Fawzy [6], Rychlik and Szymkowiak [39] and references therein.

Definition 2. *Let $\{T_n,\ n \geq 1\}$ be a sequence of identically and independently distributed (iid) random variables with DF $G(t)$ and PDF $g(t)$. For a fixed positive integer k , the sequences $\{L_k(n),\ n \geq 1\}$ of k^{th} lower record times for the sequence $\{T_n,\ n \geq 1\}$ is defined as below:*

$$L_k(1) = 1$$

$$L_k(n+1) = min\{j > L_k(n) : T_{k:L_k(n)+k-1} > T_{k:j+k-1}\},\ n \in N.$$

The sequence $\{R_n^{(k)},\ n \geq 1\}$, where $R_n^{(k)} = T_{k:L_k(n)+k-1},\ n = 1,2,...,$ is called the sequence of k^{th} lower record values of $\{T_n,\ n \geq 1\}$. Note that for $k = 1$, we have $R_n^{(1)} = T_{L(n)},\ n \geq 1$, which are the lower record values of $\{T_n,\ n \geq 1\}$. For convenience, we shall take $R_0^{(k)} = 0$.

The PDF of $R_n^{(k)}$ is given by

$$g_{R_n^{(k)}}(t) = \frac{k^n}{\Gamma(n)}\,[-\ln G(t)]^{n-1}\,[G(t)]^{k-1}\,g(t), \;\; n \geq 1. \tag{4}$$

The joint PDF of $R_m^{(k)}$ and $R_n^{(k)}$ is given by [see Pawlas and Syznal (1998)]

$$g_{R_m^{(k)},R_n^{(k)}}(t,u) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}[-\ln G(u)+\ln G(t)]^{n-m-1}$$
$$\times [G(u)]^{k-1}\,g(u), \;\; u < t,\ 1 \leq m < n,\ n \geq 2. \tag{5}$$

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The conditional PDF $R_n^{(k)}$ given $R_m^{(k)} = t$ is given by

$$g_{R_n^{(k)}|R_m^{(k)}}(u \mid t) = \frac{k^{n-m}}{\Gamma(n-m)}[-\ln G(u) + \ln G(t)]^{n-m-1} \times \left(\frac{G(u)}{G(t)}\right)^{k-1} \frac{g(u)}{G(t)} du, \quad u < t. \tag{6}$$

1.1 Purpose and Motivation of the Study

In this paper, we focus on the study of k^{th} lower record values arising from the IRD. The primary purpose of the study is to explore and derive analytical expressions for the moments of lower record values associated with the IRD, which is widely used in reliability analysis and life-testing studies. The motivation behind this work stems from the growing interest in record value theory due to its practical applications in areas such as meteorology, sports, and industrial quality control, where only extreme observations are available or of interest. Specifically, studying lower record values provides insights into early failures or rare minimum events in lifetime data. By focusing on the IRD, the paper aims to fill a gap in statistical literature, offering new theoretical results that can be instrumental in better understanding the behavior of systems modeled by this distribution and enhancing statistical inference in reliability and survival analysis contexts.

The paper is organized as follows. Section 2 contains the explicit expression, recurrence relations, and the related numerical computations for the single moments of k^{th} lower record values. Section 3 contains explicit expressions, recurrence relations, and the related numerical computations for product moments. Moreover, Section 4 contains characterization theorems and the associated numerical computations. Finally, Section 5 includes the conclusion regarding the work presented in this article.

2 Single moment

In this section, we will derive explicit expressions and recurrence relations for the single moments of the k^{th} lower record values from the IRD.

Theorem 1. *For the IRD as given in (1), let k be a fixed positive integer such that $k \geq 1$, and let $n \geq 1$. Then, for $j = 1, 2, \ldots$ with $n > (j/2)$*

$$E(R_n^{(k)})^j = (\beta k)^{j/2} \frac{\Gamma(n-(j/2))}{\Gamma(n)}. \tag{7}$$

Proof. In view of (4), we have

$$E(R_n^{(k)})^j = \frac{k^n}{\Gamma(n)} \int_0^\infty t^j [-\ln G(t)]^{n-1} \, [G(t)]^{k-1} \, g(t) dt. \tag{8}$$

By setting $z = G(t)$ in (8), we get

$$E(R_n^{(k)})^j = \beta^{j/2} \frac{k^n}{\Gamma(n)} \int_0^1 [-\ln z]^{n-(j/2)-1} \, z^{k-1} \, dt. \tag{9}$$

We have the result from Gradshteyn and Ryzhik [[20], p-557] as:

$$\int_0^1 [-\ln x]^{\mu-1} \, x^{\nu-1} \, dx = \frac{\Gamma \mu}{\nu^\mu}, \quad \mu > 0, \;\; \nu > 0. \tag{10}$$

Thus, (7) can be obtained by using (10) in (9).

Remark. If $k = 1$ in (7), then we get the explicit expressions for single moments of lower record values from IRD as:

$$E(X_{L(n)}^j) = \beta^{j/2} \frac{\Gamma(n-(j/2))}{\Gamma(n)}. \tag{11}$$

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Theorem 2. *For the IRD as given in (1) and conditions as stated in Theorem 1.*

$$E(R_n^{(k)})^j = E(R_{n-1}^{(k)})^j - \frac{j}{2\beta k} E(R_n^{(k)})^{j+2}. \tag{12}$$

Proof. Bieniek and Syznal (2002) shown that,

$$E(R_n^{(k)})^j - E(R_{n-1}^{(k)})^j = -\frac{jk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} t^{j-1}[-\ln G(t)]^{n-1}[G(t)]^k dt. \tag{13}$$

Now, on using (3) in (13), we get the result given in (12).

Remark. At $k = 1$ in (12), the recurrence relations for the single moments of lower record values from IRD has the form

$$E(X_{L(n)})^j = E(X_{L(n-1)})^j - \frac{j}{2\beta} E(X_{L(n)})^{j+2} \tag{14}$$

Remark. The statistical properties of the k^{th} lower record values based on Theorem 1 are presented in Tables 2.1 and 2.2 below:

Table 2.1

	$\beta = 0.5, k = 1$				$\beta = 1.5, k = 1$			
n	Mean	Variance	Skewness	Kurtosis	Mean	Variance	Skewness	Kurtosis
1	1.2533	-	-	-	4.8541	-	-	-
2	0.6267	0.1073	25.8912	-	2.427	0.3219	25.8896	-
3	0.4699	0.0291	5.6524	18.2177	1.8203	0.0872	5.6923	18.2146
4	0.3917	0.0133	3.0330	9.9592	1.5169	0.0399	2.9568	10.0138
5	0.3427	0.0076	2.0569	7.4548	1.3272	0.0226	2.0887	7.3949
6	0.3084	0.0049	1.5527	6.2643	1.1946	0.0146	1.5581	6.0444
7	0.2827	0.0034	1.2457	5.573	1.0951	0.0102	1.3393	4.7401
8	0.2625	0.0025	1.0396	5.1224	1.0168	0.0075	0.8198	7.2043
9	0.2461	0.0019	0.8921	4.8046	0.9532	0.0058	1.0583	3.4673
10	0.2324	0.0015	0.7809	4.5701	0.9003	0.0046	0.7536	4.8578
	$\beta = 2, k = 1$				$\beta = 2.5, k = 1$			
n	Mean	Variance	Skewness	kurtosis	Mean	Variance	Skewness	kurtosis
1	2.5067	-	-	-	2.8025	-	-	-
2	1.2533	0.4292	25.8845	-	1.4012	0.5366	25.8655	-
3	0.9399	0.1166	5.6039	18.1946	1.0509	0.1456	5.6383	18.1996
4	0.7833	0.0531	3.0014	9.9569	0.8758	0.0663	3.0732	9.9236
5	0.6854	0.0302	2.0723	7.305	0.7663	0.0378	2.0650	7.3558
6	0.6169	0.0194	1.6327	5.9598	0.6897	0.0243	1.6106	6.0124
7	0.5655	0.0135	1.3721	5.2157	0.6322	0.0170	1.2051	5.4184
8	0.5251	0.0099	1.2398	3.5274	0.587	0.0125	1.0995	4.5327
9	0.4922	0.0077	0.8605	3.6209	0.5503	0.0097	0.8732	3.9358
10	0.4649	0.0061	0.9356	4.5578	0.5198	0.0076	0.7914	4.0635

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Table 2.2

	$\beta=0.5,k=3$				$\beta=1.5,k=3$			
n	Mean	Variance	Skewness	kurtosis	Mean	Variance	Skewness	kurtosis
1	2.1708	-	-	-	3.7599	-	-	-
2	1.0854	0.3219	25.89	-	1.8799	0.9659	25.8633	-
3	0.8141	0.0872	5.6923	18.2146	1.4099	0.2622	5.6225	18.2041
4	0.6784	0.0398	3.0423	9.9998	1.1749	0.1196	2.9867	9.9882
5	0.5935	0.0228	1.8934	7.9416	1.0281	0.0680	2.0581	7.4193
6	0.5342	0.0146	1.5581	6.0444	0.9253	0.0438	1.5545	6.2593
7	0.4897	0.0102	1.3393	4.7401	0.8482	0.0306	1.2257	5.8208
8	0.4547	0.0075	0.8198	7.2043	0.7876	0.0226	0.9881	5.1313
9	0.4263	0.0058	1.0583	3.4673	0.7384	0.0173	0.9001	4.9637
10	0.4026	0.0046	0.4504	11.1991	0.6973	0.0138	0.6899	5.052
	$\beta=2,k=3$				$\beta=2.5,k=3$			
n	Mean	Variance	Skewness	kurtosis	Mean	Variance	Skewness	kurtosis
1	4.3416	-	-	-	4.8541	-	-	-
2	2.1708	1.2876	25.8896	-	2.427	1.6097	25.8816	-
3	1.6281	0.3493	5.6512	18.2178	1.8203	0.4365	5.6602	18.2208
4	1.3568	0.1591	3.0533	9.9569	1.5169	0.1990	3.0354	9.9567
5	1.1872	0.0906	2.0781	7.4528	1.3272	0.1135	2.0158	7.4788
6	1.0684	0.0585	1.5230	6.2939	1.1946	0.0729	1.5796	6.2469
7	0.9794	0.0408	1.2292	5.6826	1.0951	0.0508	1.3042	5.5882
8	0.9094	0.0301	1.0397	4.9467	1.0168	0.0376	1.0664	5.0669
9	0.8526	0.0231	0.8895	4.8256	0.9532	0.0289	0.8238	5.2053
10	0.8052	0.0184	0.6585	5.2057	0.9003	0.0228	0.8678	3.9948

Table 2.1 and Table 2.2 present numerical characteristics, that is, mean, variance, skewness, and kurtosis of lower record values for the IRD for the arbitrary chosen values of the shape parameter $\beta(0.5,1.5,2.0,2.5)$ for $k=1$, $k=3$ and number of records $n=1(1)10$. It has been observed that for all values of β, there is a decrease in means and variances as n increases. For fixed n, as β increases, the mean increases, which indicates that the higher values of β shift the distribution rightward, resulting in larger expected values of lower record values. Further, the distribution of lower records is extremely right skewed for small n and become more symmetric as n increases. Kurtosis is very high for a small n and decreases with large n. High kurtosis indicates heavy tails and outliers and lower kurtosis shows data becoming less peaked and normally distributed. The above tables effectively illustrate how the statistical properties of lower record values evolve with both n and shape parameter β, which is valuable for modeling and inference involving rare or extreme events.

3 Product moments

In this section, we will derive the exact expression and recurrence relations for the product moments of k^{th} lower record values.

Theorem 3. *For the distribution given in (1), fix a positive integer $k\geq 1$. Then, for $1\leq m\leq n-1$, $n\geq 2$ and $i,j=1,2,\ldots$,*

$$E[(R_m^{(k)})^i(R_n^{(k)})^j]=\frac{(\beta k)^{(i+j)/2}}{\Gamma(m)\Gamma(n-m)}\sum_{l=0}^{n-m-1}(-1)^{n-m-l-1}\binom{n-m-1}{l}\frac{\Gamma(n-(i+j)/2)}{[n-(i/2)-l-1]}. \tag{15}$$

Proof. Using (5), we get

$$E[(R_m^{(k)})^i(R_n^{(k)})^j]=\frac{k^n}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_u^\infty t^iu^j[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}$$
$$\times[-\ln G(u)+\ln G(t)]^{n-m-1}[G(u)]^{k-1}\,g(u)dtdu. \tag{16}$$

Now expanding $[-\ln G(u)+\ln G(t)]^{n-m-1}$ in (16) binomially, we get

$$E[(R_m^{(k)})^i(R_n^{(k)})^j]=\frac{k^n}{\Gamma(m)\Gamma(n-m)}\sum_{l=0}^{n-m-1}(-1)^{n-m-l-1}\binom{n-m-1}{l}$$

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$$\times \int_0^\infty u^j \; [-\ln G(u)]^l [G(u)]^{k-1} g(u) \; I(u) du, \tag{17}$$

where

$$I(u) = \int_u^\infty t^i [-\ln G(t)]^{n-u-2} \frac{g(t)}{G(t)} \, dt. \tag{18}$$

Consider $s = -\ln G(t)$ in (18). Then, we have

$$\begin{aligned} I(u) &= \beta^{i/2} \int_0^{-\ln G(u)} s^{n-(i/2)-u-2} \, ds \\ &= \frac{\beta^{i/2} [-\ln G(u)]^{n-(i/2)-l-1}}{[n-(i/2)-l-1]}. \end{aligned}$$

Now, using the value of $I(u)$ in (17) and simplifying, the resulting equation yields

$$\begin{aligned} E[(R_m^{(k)})^i (R_n^{(k)})^j] &= \frac{\beta^{i/2} k^n}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{n-m-1} (-1)^{n-m-l-1} \binom{n-m-1}{l} \\ &\times \frac{1}{[n-(i/2)-l-1]} \int_0^\infty u^j \; [-\ln G(u)]^{n-(i/2)-1} [G(u)]^{k-1} g(u) \; du. \end{aligned} \tag{19}$$

Again, set $v = -\ln G(u)$ in (19), to get

$$\begin{aligned} E[(R_m^{(k)})^i (R_n^{(k)})^j] &= \frac{\beta^{(i+j)/2} k^n}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{n-m-1} (-1)^{n-m-l-1} \binom{n-m-1}{l} \\ &\times \frac{1}{[n-(i/2)-l-1]} \int_0^\infty v^{n-(i+j)/2-1} \; e^{-kv} dv, \end{aligned}$$

which gives the result given in (15). Hence, Theorem 3.1 is proved.

Identity 1 *For* $1 \le m \le n-1$

$$\sum_{l=0}^{n-m-1} (-1)^{n-m-l-1} \binom{n-m-1}{l} \frac{1}{(n-l-1)} = \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}. \tag{20}$$

Proof. Setting $i = j = 0$ in (15), we get the required result.

Remark. At $i = 0$ in (15), we have

$$E(R_n^{(k)})^j = \frac{(\beta k)^{j/2}}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{n-m-1} (-1)^{n-m-l-1} \binom{n-m-1}{l} \frac{\Gamma(n-(j/2))}{(n-l-1)}. \tag{21}$$

Now, using (20) in (21), we get

$$E(R_n^{(k)})^j = (\beta k)^{j/2} \frac{\Gamma(n-(j/2))}{\Gamma(n)},$$

which is the exact expression for the single moments of k^{th} lower record values from IRD, as obtained in (7).

Remark. By putting $k = 1$ in (15), we get the explicit expressions for the product moments of lower record values from IRD as

$$\begin{aligned} E[(X_{L(m)})^i (X_{L(n)})^j] &= \frac{\beta^{(i+j)/2}}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{n-m-1} (-1)^{n-m-l-1} \binom{n-m-1}{l} \\ &\times \frac{\Gamma(n-(i+j)/2)}{[n-(i/2)-l-1]}. \end{aligned} \tag{22}$$

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Theorem 4. *For the IRD as given in (1) and the other conditions as stated in Theorem 3.1*

$$E[(R_m^{(k)})^i(R_n^{(k)})^j] = E[(R_m^{(k)})^i(R_{n-1}^{(k)})^j] - \frac{j}{2\beta k}E[(R_m^{(k)})^i(R_n^{(k)})^{j+2}]. \tag{23}$$

Proof. Using the result of Singh and Khan [40], we have

$$E[(R_m^{(k)})^i(R_n^{(k)})^j] - E[(R_m^{(k)})^i(R_{n-1}^{(k)})^j] = -\frac{jk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_\alpha^\beta\int_\alpha^t t^i u^{j-1}$$

$$\times[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}[-\ln G(u)+\ln G(t)]^{n-m-1}[G(u)]^k du dt. \tag{24}$$

Now, applying (3) in (24), we get the result given in (23).

Remark. At $i=0$, Theorem 3.2 reduces to Theorem 2.2.

Remark. At $k=1$ in (23), the recurrence relations for the product moments of lower record values is given by

$$E[(T_{L(m)})^i(T_{L(n)})^j] = E[(T_{L(m)})^i(T_{L(n-1)})^j] - \frac{j}{2\beta}E[(T_{L(m)})^i(T_{L(n)})^{j+2}]. \tag{25}$$

Remark. The product moments of the k^{th} lower record values from the IRD, for the various choices of the parameter β and different values of k, are calculated using (15). The results are presented in Tables 3.1, 3.2, 3.3, and 3.4 below.

Table 3.1 ($\beta = 0.5, k = 1$)

n / m	2	3	4	5	6	7	8	9	10
1	1.0000	0.6667	0.5333	0.4571	0.4063	0.3694	0.3410	0.3183	0.2995
2		0.3333	0.2667	0.2286	0.2032	0.1847	0.1705	0.1591	0.1498
3			0.2000	0.1714	0.1524	0.1385	0.1279	0.1193	0.1123
4				0.1429	0.1270	0.1154	0.1066	0.0995	0.0936
5					0.1111	0.1010	0.0932	0.0870	0.0819
6						0.0909	0.0839	0.0783	0.0737
7							0.0769	0.0718	0.0676
8								0.0667	0.0627
9									0.0588

Table 3.2 ($\beta = 1.5, k = 1$)

n / m	2	3	4	5	6	7	8	9	10
1	3.0000	2.0000	1.6000	1.3714	1.2190	1.1082	1.0230	0.9548	0.8986
2		1.0000	0.8000	0.6857	0.6095	0.5541	0.5115	0.4774	0.4493
3			0.6000	0.5143	0.4571	0.4156	0.3836	0.3580	0.3370
4				0.4286	0.38095	0.3463	0.3197	0.2984	0.2808
5					0.3333	0.3030	0.2797	0.2611	0.2457
6						0.2727	0.2517	0.2350	0.2211
7							0.2308	0.2154	0.2027
8								0.2000	0.1882
9									0.1765

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Table 3.3 ($\beta = 1.5, k = 3$)

m \ n	2	3	4	5	6	7	8	9	10
1	9.0000	6.0000	4.8000	4.1143	3.6571	3.3248	3.0689	2.8643	2.6958
2		3.0000	2.4000	2.0571	1.8286	1.6623	1.5345	1.4322	1.3479
3			1.8000	1.5429	1.3714	1.2468	1.1509	1.0741	1.0109
4				1.2857	1.1429	1.0390	0.9590	0.8951	0.8425
5					1.0000	0.9091	0.8392	0.7832	0.7371
6						0.8182	0.7552	0.7049	0.6634
7							0.6923	0.6462	0.6081
8								0.6000	0.5647
9									0.5294

Table 3.4 ($\beta = 2, k = 3$)

m \ n	2	3	4	5	6	7	8	9	10
1	12.0000	8.0000	6.4000	5.4857	4.8762	4.4329	4.0920	3.8191	3.5945
2		4.0000	3.2000	2.7429	2.4381	2.2165	2.0460	1.9096	1.7972
3			2.4000	2.0571	1.8286	1.6623	1.5345	1.4322	1.3479
4				1.7143	1.5238	1.3853	1.2787	1.1935	1.1233
5					1.3333	1.2121	1.1189	1.0443	0.9829
6						1.0909	1.0070	0.9399	0.8846
7							0.9231	0.8615	0.8109
8								0.8000	0.7529
9									0.7059

In the above Table 3.1 to 3.4, the following points are observed:

- The product moment values are highest when $n = 2$ and decrease as n increases.
- The diagonal elements $(n = m+1)$ decrease progressively.
- For fixed m, as n increases, the moment values decrease.
- Increasing both β and k leads to significantly larger expected lower record values and product moment.

4 Characterization

This section presents characterization theorems for the given distribution, derived from recurrence relations for the single and product moments, as well as for conditional expectation.

Theorem 5. *Fix a positive integer $k \geq 1$ and let j be a non-negative integer. A necessary and sufficient condition for a random variable T to be distributed with PDF given by (1) is that*

$$E(R_n^{(k)})^j = E(R_{n-1}^{(k)})^j - \frac{j}{2\beta k} E(R_n^{(k)})^{j+2}, \ \ n \geq 1. \tag{26}$$

Proof. The necessary part follows from (12). On the other hand if the recurrence relation (26) is satisfied, then on using Bieniek and Syznal [12], we have

$$\frac{jk^{n-1}}{\Gamma(n)} \int_0^\infty t^{j-1}[-\ln G(t)]^{n-1}[G(t)]^k dt = \frac{jk^{n-1}}{2\beta\Gamma(n)} \int_0^\infty t^{j+2}[-\ln G(t)]^{n-1}[G(t)]^{k-1} g(t) dt,$$

which implies

$$\int_0^\infty t^{j-1}[-\ln G(t)]^{n-1}[G(t)]^{k-1} \Big\{ G(t) - \frac{t^3}{2\beta} g(t) \Big\} dt = 0. \tag{27}$$

Now, on application of the Müntz-Szász theorem (see Hwang and Lin [24]) in (27), we get

$$G(t) = \frac{t^3}{2\beta} g(t),$$

which proves that $g(x)$ has the form as given in (1).

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Remark. If $k=1$ in (26), we get the characterizing result based on lower record values as

$$E(T_{L(n)})^j = E(T_{L(n-1)})^j - \frac{j}{2\beta}E(T_{L(n)})^{j+2}. \tag{28}$$

Theorem 6. *For a positive integer k , let i and j be non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with PDF given by (1) is*

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] = E[(R_m^{(k)})^i (R_{n-1}^{(k)})^j] - \frac{j}{2\beta k}E[(R_m^{(k)})^i (R_n^{(k)})^{j+2}],$$
$$\textit{for } 1 \le m \le n-1. \tag{29}$$

Proof. The necessary part follows from (23). On the other hand if the relation in (29) is satisfied, then on using Singh and Khan[40], we have

$$\frac{jk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_0^\infty \int_0^x t^i u^{j-1}[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}[-\ln G(u)+\ln G(t)]^{n-m-1}$$
$$[G(u)]^k dudt = \frac{jk^{n-1}}{2\beta\Gamma(m)\Gamma(n-m)}\int_0^\infty \int_0^t t^i u^{j+2}[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}$$
$$\times[-\ln G(u)+\ln G(t)]^{n-m-1}[G(u)]^{k-1}g(t)dudt,$$

which implies that

$$\int_0^\infty \int_0^t t^i u^{j-1}[-\ln G(t)]^{m-1}\frac{g(t)}{G(t)}[-\ln G(u)+\ln G(t)]^{n-m-1}[G(u)]^{k-1}$$
$$\times\left\{G(u)-\frac{u^3}{2\beta}g(u)\right\}dudt = 0. \tag{30}$$

Applying a generalization of the Müntz-Szász theorem, (see for example Hwang and Lin [24]) to (30), we get

$$G(u) = \frac{u^3}{2\beta}g(u),$$

which proves that $g(u)$ has the form as given in (1).

Remark. At $k=1$ in (29), the characterization result for the IRD based on lower record values is given as:

$$E[(T_{L(m)})^i (T_{L(n)})^j] = E[(T_{L(m)})^i (T_{L(n-1)})^j] - \frac{j}{2\beta}E[(T_{L(m)})^i (T_{L(n)})^{j+2}]. \tag{31}$$

Theorem 7. *Let T be a non-negative random variable having an absolutely continuous DF $G(x)$, with $G(0)=0$ and $0 \le G(t) \le 1$. Then for all $t>0$,*

$$E[(R_n^{(k)})^{(-2)} \mid (R_m^{(k)}) = t] = \frac{1}{t^2} + \frac{(n-m)}{\beta k} \tag{32}$$

if and only if

$$G(t) = \exp\left(-\frac{\beta}{t^2}\right),\ \ t>0,\ \ \beta>0.$$

Proof. From (6), we have

$$E[(R_n^{(k)})^{(-2)} \mid (R_m^{(k)}) = t] = \frac{k^{n-m}}{\Gamma(n-m)}\int_0^t \frac{1}{u^2}[-\ln G(u)+\ln G(t)]^{n-m-1}$$
$$\times\left(\frac{G(u)}{G(t)}\right)^{k-1}\frac{f(u)}{G(t)}du. \tag{33}$$

By setting $w = \frac{G(u)}{G(t)} = \frac{e^{-\beta/u^2}}{e^{-\beta/t^2}}$ in (33), we have

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$$E[(R_n^{(k)})^{(-2)} \mid (R_m^{(k)}) = t] = \frac{k^{n-m}}{\Gamma(n-m)} \int_0^1 \left\{ \frac{1}{t^2} - \frac{1}{\beta} \ln w \right\} w^{k-1} (-\ln w)^{n-m-1} dw. \tag{34}$$

On using (10) in (34), we have the result given in (32).

To prove the sufficiency part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)} \int_0^t \frac{1}{u^2} [-\ln G(u) + \ln G(t)]^{n-m-1} [G(u)]^{k-1} f(u) du = [G(t)]^k g_{n|m}(t), \tag{35}$$

where

$$g_{n|m}(t) = \frac{1}{t^2} + \frac{(n-m)}{\beta k}.$$

Differentiating both the sides of (35) with respect to x , we get

$$-\frac{k^{n-m} g(t)}{G(t)\Gamma(n-m-1)} \int_0^t \frac{1}{u^2} [-\ln G(u) + \ln G(t)]^{n-m-2} [G(u)]^{k-1} g(u) du$$
$$= g'_{n|m}(t)[G(t)]^k + k\, g_{n|m}(t)[G(t)]^{k-1} g(t). \tag{36}$$

Dividing (36) by $[G(t)]^k$, we find that

$$k\, g_{n|m+1}(t)[G(t)]^{k-1} g(t) = g'_{n|m}(t)[G(t)]^k + k\, g_{n|m}(t)[G(t)]^{k-1} g(t).$$

Therefore,

$$\frac{g(t)}{G(t)} = \frac{g'_{n|m}(t)}{k[g_{n|m+1}(t) - g_{n|m}(t)]} = \frac{2\beta}{t^3}, \tag{37}$$

where

$$g'_{n|m}(t) = \tfrac{-2}{t^3} \quad \text{and} \quad g_{n|m+1}(t) - g_{n|m}(t) = \tfrac{-1}{\beta k}$$

which proves that

$$G(t) = \exp\left(-\frac{\beta}{t^2}\right), \;\; t > 0, \;\; \beta > 0.$$

Remark. If $k = 1$ in (32), the characterizing result based on lower record values is given by

$$E[(T_{L(n)})^{-2} \mid (T_{L(m)}) = t] = \frac{1}{t^2} + \frac{(n-m)}{\beta}. \tag{38}$$

Now we will illustrate the application of the characterization result (32) through a simulation study, using $k = 1$ for simplicity. In this study, we simulate lower record values from the distribution for three different values of the parameter β, as shown in Table 4.1 below.

Based on simulated lower record values in Table 4.1, record values were estimated using a characterization result given in (32) under varying truncation conditions as shown in Table 4.2. The accuracy of the estimates improved with increasing n and m. The estimator showed low mean squared error (MSE) particularly for lower values of the shape parameter β (e.g., $\beta = 0.5$). While estimation at higher β (e.g., $\beta = 2$) showed larger deviations, the estimator remained consistent overall. These results support the efficacy of the characterization-based method for estimating lower record values from the IRD. Now, we apply the result (32) on a real data set, for which we take the data of coating weights (gm/m^2) from ALAF industry, Tanzania. ALAF industry deals with improving the quality of steel roofing through several processes, one of them being the coating process. The data set contains 72 observations on coating weights by chemical method on top center side (TCS), which are given below in Table 4.3.

From this data, we have selected lower record values as given below:

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Table 4.1 Simulated lower record values from IRD.

β	Lower record values					
0.5	0.967798	0.782594	0.387303	0.331151	0.291211	0.251646
	0.248894	0.247087	0.239338	0.236909	0.234381	0.231599
	0.219273	0.210116				
1	1.214196	0.919927	0.637123	0.581398	0.485747	0.455052
	0.451544	0.448539	0.403757	0.388883	0.387849	0.38173
	0.367962	0.344776	0.302508	0.298535	0.294736	
2	1.717132	1.300973	0.901028	0.822221	0.686951	0.64354
	0.638579	0.63433	0.570998	0.549964	0.548502	0.539848
	0.520377	0.487587	0.427811	0.422192	0.416819	

Table 4.2 Estimation of record values using characterization result based on simulated dataset

β	n	m	$TruncationPoint(t)$	T	$\hat{T}$	$(T-\hat{T})$	$(T-\hat{T})^2$	MSE
0.5	4	3	0.387303	0.331151	0.339686	-0.008536	0.000073	0.000331
	8			0.247089	0.244950	0.002138	0.000005	
	12			0.231599	0.201348	0.030252	0.000915	
	8	7	0.248894	0.247089	0.234775	0.012314	0.000152	0.000564
	10			0.236909	0.212514	0.024395	0.000595	
	13			0.219273	0.188503	0.030769	0.000945	
	10	9	0.239338	0.236909	0.226704	0.010206	0.000104	0.000369
	12			0.231599	0.206472	0.025128	0.000631	
	14			0.210116	0.190841	0.019275	0.000372	
1	4	3	0.637123	0.581398	0.537331	0.044067	0.001942	0.003841
	7			0.451544	0.393338	0.058206	0.003388	
	11			0.387849	0.309144	0.078705	0.006194	
	8	7	0.451544	0.448539	0.411535	0.037005	0.001369	0.002295
	13			0.367962	0.302828	0.065134	0.004242	
	17			0.294736	0.259024	0.035712	0.001275	
	12	11	0.387849	0.38173	0.361604	0.020126	0.000405	0.000319
	14			0.344776	0.321949	0.022827	0.000521	
	16			0.298535	0.293008	0.005527	0.00003	
2	3	2	1.300973	0.901028	0.957461	-0.056433	0.003185	0.003537
	6			0.64354	0.62127	0.02227	0.000496	
	10			0.549964	0.466718	0.083246	0.00693	
	9	8	0.63433	0.570998	0.578775	-0.007777	0.00006	0.001667
	12			0.539848	0.472179	0.067669	0.004579	
	15			0.427811	0.408751	0.01906	0.000363	
	13	12	0.539848	0.520377	0.50435	0.016027	0.000257	0.000268
	15			0.427811	0.450318	-0.022507	0.000507	
	17			0.416819	0.410606	0.006213	0.000039	

Table 4.3 Coating weights by chemical method on top center side (TCS).

36.8	47.2	35.6	36.7	55.8	58.7	42.3	37.8	55.4	45.2	31.8	48.3
45.3	48.5	52.8	45.4	49.8	48.2	54.5	50.1	48.4	44.2	41.2	47.2
39.1	40.7	40.3	41.2	30.4	42.8	38.9	34.0	33.2	56.8	52.6	40.5
40.6	45.8	58.9	28.7	37.3	36.8	40.2	58.2	59.2	42.8	46.3	61.2
58.4	38.5	34.2	41.3	42.6	43.1	42.3	54.2	44.9	42.8	47.1	38.9
42.8	29.4	32.7	40.1	33.2	31.6	36.2	33.6	32.9	34.5	33.7	39.9

36.8 35.6 31.8 30.4 28.7

Now, by applying the characterization result (32) on these observed records, we have obtained the estimates of record values at different truncation points. The estimated values along with their $MSEs$, for different n, m and β, are tabulated below in Table 4.4.

By applying the characterization result (32) to a real-world dataset of coating weights from ALAF Industries, we evaluated the accuracy of estimated record values under varying shape parameters. The method showed best performance

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Table 4.4 Estimation of record values using characterization result based on data from ALAF industries.

β	n	m	$Truncation Point(t)$	T	$\hat{T}$	$(T-\hat{T})$	$(T-\hat{T})^2$	MSE
2	2	1	36.8	35.6	33.86588	1.73412	3.00716	1.14342
	3	2	35.6	31.8	32.92365	-1.12365	1.26259	
	4	3	31.8	30.4	29.84901	0.55099	0.30359	
	5	4	30.4	28.7	28.68216	0.01783	0.00032	
5	3	1	36.8	31.8	34.39661	-2.59661	6.74239	4.76045
	4	2	35.6	30.4	33.41067	-3.01067	9.06413	
	5	3	31.8	28.7	30.21049	-1.51049	2.28159	
	5	4	30.4	28.7	29.67657	-0.97657	0.95369	
7	2	1	36.8	35.6	35.88489	-0.28489	0.08116	3.75911
	4	2	35.6	30.4	33.99463	-3.59463	12.92139	
	4	3	31.8	30.4	31.20392	-0.80392	0.64629	
	5	4	30.4	28.7	29.87798	-1.17798	1.38763	
10	2	1	36.8	35.6	36.15223	-0.55223	0.30496	3.33949
	3	2	35.6	31.8	35.01256	-3.21256	10.32055	
	4	3	31.8	30.4	31.37921	-0.97921	0.95886	
	5	4	30.4	28.7	30.03175	-1.33175	1.77357	

for $\beta = 2$, with minimal MSE, indicating it may best describe the underlying distribution of coating weights. Estimation accuracy consistently improved with more available record data (increasing m), reinforcing the method's practical utility in industrial quality monitoring where lower thresholds (records) are critical.

5 Conclusion

This study investigates the moment properties of the k^{th} lower record values from the IRD. Explicit expressions for exact moments, along with recurrence relations for both single and product moments, have been derived. Several special cases have also been examined in detail. Furthermore, key statistical measures including the mean, variance, skewness, kurtosis, and product moments have been computed based on the developed formulations. The recurrence relations and conditional expectations of k^{th} lower record values have been utilized to establish a characterization of the distribution. Finally, the practical applicability of the characterization theorem based on conditional expectation has been demonstrated through both simulated and real-world data sets.

Declarations

Competing interests: The authors declare that they have no competing interests.
Authors' contributions: Dr. Haseeb Athar identified and formulated the research problem. Dr. Haseeb Athar and Ms. Saima Zarrin developed the theoretical results jointly. Ms. Saima Zarrin conducted the computational work and data analysis, and Dr. Rafiqullah Khan verified the results. All the authors contributed equally to the writing of the introduction and the overall manuscript, and have read and approved the final version.
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