Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/19.2.4

New Exact Solutions for a New Family of $(3+1)$-Dimensional Rosenau's Equations

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Received: Nov. 11, 2024 Accepted :Feb. 5, 2026

Abstract: In this paper, we present a new family of $(3+1)$-dimensional Rosenau's equations. We use a new $expQ(\xi)$-expansion method for solve our new equations. We determine a variety of exact solutions for each equation and expressed in terms of hyperbolic functions, trigonometric functions and rational functions.

Keywords: $\exp(Q(\xi))$-Expansion Method; soliton solution; Rosenau's equation.

2010 Mathematics Subject Classification. 35B09, 35C08, 35C09.

1 Introduction

The study of nonlinear $(3+1)$-dimensional equations is important in the fields of science and technology wave propagation, and engineering areas.

In his 1986 and 1988 papers, Rosenau [1,2] developed a formalism to treat the dynamics of discrete dense systems that can deal with wave-wave and wave-wall interactions that cannot be treated with the Korteweg-de Vries (KdV) equation. Such a formalism leads to a quasi-continuum which is endowed with the leading effects due to discreteness. Rosenau's original equation may be written as

$$v_t \pm (v+v^2)_x + v_{xxxxt} = 0. \tag{1}$$

In the many researches, have been to solve the Rosenau's equation uased various methods [3]-[8]. Motivated by the fact that the $(3+1)$-dimensional equations possess a rich development of scientific phenomena, it is then normal to propose two new sets of $(3+1)$-dimensional nonlinear modified equations. The first set mainly from the development made by Hereman [9,10] where he proposed a new $(3+1)$-dimensional modified KdV given in the form

$$w_t + 6w^2 w_x + w_{xyz} = 0, \tag{2}$$

Using the same sense used for Eq. (2), we introduce two more $(3+1)$-dimensional modified KdV equations given by

$$w_t + 6w^2 w_y + w_{xyz} = 0, \tag{3}$$

and

$$w_t + 6w^2 w_z + w_{xyz} = 0. \tag{4}$$

Following the same way of formulation, we introduce a second set of $(3+1)$-dimensional equations, that possess the modified Rosenau's equation and include the following forms

$$v_t + v_x + v^2 v_y + v_{xzxt} = 0, \tag{5}$$

$$v_t + v_y + v^2 v_z + v_{xxxt} = 0, \tag{6}$$

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and

$$v_t + v_z + v^2 v_x + v_{xyxxt} = 0, \tag{7}$$

which will be called (3+1)-dimensional modified Rosenau's equations.

Our aim is to study the family proposed above by using a new $\exp(Q(\xi))$-expansion method. The computer symbolic system Maple will be used to perform the computational work.

2 Description of a new $\exp(Q(\xi))$-expansion method

Consider the general nonlinear partial differential equations (NLPDEs), say, in two variables,

$$\varphi(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, ...) = 0, \tag{8}$$

where $v = v(x,t)$ is an unknown function, φ is a polynomial in $v(x,t)$ and the subscripts stand for the partial derivatives.

We suppose that the combination of real variables x and t by a complex variable ξ

$$v(x,t) = v(\xi),\ \xi = ax - ct, \tag{9}$$

where a is the wave number and c is the speed of the traveling wave. The traveling wave transformation Eq. (9) converts Eq. (8) into an ordinary differential equation (ODE) for $v = v(\xi)$:

$$F(v, -cv', av'', c^2v'', -cav'', a^2v'', ...) = 0,\ ' \equiv \frac{d}{d\xi}, \tag{10}$$

Suppose the traveling wave solution of Eq. (10) can be expressed by polynomial in $\exp(Q(\xi))$ as:

$$v(\xi) = \sum_{i=0}^{N} a(\exp(Q(\xi)))^i, \tag{11}$$

where the coefficients a_i $(0 \le i \le N), a\ and\ c$ are constants to be determined, such that $a_N \neq 0$ and $Q = Q(\xi)$ satisfies the following ordinary differential equation:

$$Q'(\xi) = \sqrt{\mu \exp(Q(\xi)) + \lambda}, \tag{12}$$

Eq. (12) gives the following solutions:

Case 1. where $\lambda > 0$ $\mu \neq 0$,

$$Q(\xi) = \ln\Big(\frac{-\lambda}{\mu} sech^2\Big(\frac{\sqrt{\lambda}}{2}(\xi + k)\Big)\Big),$$

where k is a constant of integration.

Case 2. where $\lambda < 0$ $\mu \neq 0$,

$$Q(\xi) = \ln\Big(\frac{-\lambda}{\mu} sec^2\Big(\frac{\sqrt{-\lambda}}{2}(\xi + k)\Big)\Big),$$

Case 3. where $\lambda = 0$ $\mu \neq 0$,

$$Q(\xi) = \ln\Big(\frac{4}{\mu(\xi + k)^2}\Big),$$

Case 4. where $\lambda \neq 0$ $\mu = 0$,

$$Q(\xi) = \sqrt{\lambda}(\xi + k),$$

where $A, a_i(i = 0, 1, 2, ..., N), a, c, \mu$ and λ are constants. The positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (10).

Substituting Eq. (11) into Eq. (10), using Eq. (12) repeatedly, and setting the coefficients of the each order of $(\exp(Q(\xi)))^i$ to zero, we obtain a set of nonlinear algebraic equations for $a_i(i = 0, 1, 2, ..., N), a, c, \mu$ and λ. With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i(i = 0, 1, 2, ..., N), a, c$. Substituting the values of $a_i(i = 0, 1, 2, ..., N), a, c$ into Eq. (11) along with general solutions of Eq. (12) completes the determination of the solution of Eq. (8).

3 Application of the method

In this section, we apply the $\exp(Q(\xi))$-expansion method to construct the exact solutions for a new family of $(3+1)$-dimensional Rosenau's equations.

3.1 The first 3D Rosenau's equation

The 3D Rosenau's equation that reads:

$$v_t + v_x + v^2 v_y + v_{xzxxt} = 0, \tag{13}$$

by using

$$v(x,y,z,t) = v(\xi), \xi = ax + by + cz - dt. \tag{14}$$

Eq. (13) reducing to

$$(a-d)v' + bv^2 v' - a^3 c d v^{(5)} = 0. \tag{15}$$

Integrating once Eq. (15), we have

$$(a-d)v + \frac{b}{3}v^3 - a^3 c d v^{(4)} + H = 0, \tag{16}$$

where H is an integrating constant.
Balancing of Eq. (16) give

$$3N = N + 4 \implies N = 2.$$

By the use of Eq. (11), we present the solution of Eq. (16) as:

$$v(\xi) = a_0 + a_1 \exp(Q(\xi)) + a_2(\exp(Q(\xi)))^2. \tag{17}$$

Substituting Eq. (17) in Eq. (16) and using Eq. (12), collecting the coefficients of each power of $\exp(Q(\xi))^i$, $0 \le i \le 6$, setting each coefficients to zero, and solving the resuting system by Maple we have,

$$a_0 = \pm a^2 \lambda \sqrt{\frac{-5c}{3bca^3\lambda^2 - 2b}}, a_1 = \pm 3a^2 \mu \sqrt{\frac{-5c}{3bca^3\lambda^2 - 2b}},$$

$$a_2 = 0, d = \frac{-2a}{3ca^3\lambda^2 - 2}, H = \mp \frac{3}{4} \frac{ca^6\lambda^3 \sqrt{\frac{-5c}{3bca^3\lambda^2 - 2b}}}{3ca^3\lambda^2 - 2},$$

now put is giving by a_0, a_1, a_2 *and* d into Eq. (17) into solution of Eq. (12), give
case 1. where $\lambda > 0$ and $\mu \neq 0$,

$$V_{1,2}(\xi) = a^2 \lambda \sqrt{\frac{-5c}{3bca^3\lambda^2 - 2b}} \Big(\pm 1 \mp 3 sech^2 \Big(\frac{\sqrt{\lambda}}{2}(\xi + k) \Big) \Big),$$

case 2. where $\lambda < 0$ and $\mu \neq 0$,

$$V_{3,4}(\xi) = a^2 \lambda \sqrt{\frac{-5c}{3bca^3\lambda^2 - 2b}} \Big(\pm 1 \mp 3 sec^2 \Big(\frac{\sqrt{-\lambda}}{2}(\xi + k) \Big) \Big),$$

where

$$\xi = ax + by + cz + \frac{2at}{3ca^3\lambda^2 - 2}$$

case 3. where $\lambda = 0$ and $\mu \neq 0$,

$$V_{5,6}(\xi) = \frac{\pm 6a^2 \sqrt{\frac{10c}{b}}}{(ax + by + cz - at + k)^2}.$$

case 4. where $\lambda \neq 0$ and $\mu = 0$, v_7 is constant.

3.2 The second 3D Rosenau's equation

The 3D Rosenau's equation as:

$$v_t + v_y + v^2 v_z + v_{xxxxt} = 0, \tag{18}$$

proceeding with the above method, we develop the following solutions:

$$a_0 = \pm a^2\lambda\sqrt{\frac{-5b}{3bca^4\lambda^2 - 2c}}, a_1 = \pm 3a^2\mu\sqrt{\frac{-5b}{3bca^4\lambda^2 - 2c}},$$

$$a_2 = 0, d = \frac{-2b}{3a^4\lambda^2 - 2}, H = \mp\frac{4}{3}\frac{ba^6\lambda^3\sqrt{\frac{-5b}{3ca^4\lambda^2 - 2c}}}{3a^4\lambda^2 - 2},$$

case 1. where $\lambda > 0$ and $\mu \neq 0$,

$$V_{1,2}(\xi) = a^2\lambda\sqrt{\frac{-5b}{3ca^4\lambda^2 - 2c}}\Big(\pm 1 \mp 3sech^2\Big(\frac{\sqrt{\lambda}}{2}(\xi + k)\Big)\Big),$$

case 2. where $\lambda < 0$ and $\mu \neq 0$,

$$V_{3,4}(\xi) = a^2\lambda\sqrt{\frac{-5b}{3ca^4\lambda^2 - 2c}}\Big(\pm 1 \mp 3sec^2\Big(\frac{\sqrt{-\lambda}}{2}(\xi + k)\Big)\Big),$$

where

$$\xi = ax + by + cz + \frac{2bt}{3a^4\lambda^2 - 2}$$

case 3. where $\lambda = 0$ and $\mu \neq 0$,

$$V_{5,6}(\xi) = \frac{\pm 6a^2\sqrt{\frac{10b}{c}}}{(ax + by + cz - bt + k)^2}.$$

conditions hold

case 4. where $\lambda \neq 0$ and $\mu = 0$, v_7 is constant.

3.3 The third 3D Rosenau's equation

The 3D Rosenau's equation as:

$$v_t + v_z + v^2 v_x + v_{xyxxt} = 0, \tag{19}$$

proceeding with the above method, we develop the following solutions:

$$a_0 = \pm\frac{5acb\lambda}{(3a^3b\lambda^2 - 2)\sqrt{\frac{-5cb}{3a^3b\lambda^2 - 2}}}, a_1 = \mp 3a\mu\sqrt{\frac{-5cb}{3a^3b\lambda^2 - 2}},$$

$$a_2 = 0, d = \frac{-2c}{3a^3b\lambda^2 - 2}, H = \pm\frac{20a^4c^2b^2\lambda^3}{3(3a^3b\lambda^2 - 2)^2\sqrt{\frac{-5cb}{3a^3b\lambda^2 - 2}}}$$

case 1. where $\lambda > 0$ and $\mu \neq 0$,

$$v_{1,2}(\xi) = \pm\frac{5acb\lambda}{(3a^3b\lambda^2 - 2)\sqrt{\frac{-5cb}{3a^3b\lambda^2 - 2}}} \pm a\lambda\sqrt{\frac{-5cb}{3a^3b\lambda^2 - 2}}sech^2\Big(\frac{\sqrt{\lambda}}{2}(\xi + k)\Big),$$

case 2. where $\lambda < 0$ and $\mu \neq 0$,

$$v_{3,4}(\xi) = \pm \frac{5acb\lambda}{(3a^3b\lambda^2-2)\sqrt{\frac{-5cb}{3a^3b\lambda^2-2}}} \pm a\lambda\sqrt{\frac{-5cb}{3a^3b\lambda^2-2}}sec^2\Big(\frac{\sqrt{-\lambda}}{2}(\xi+k)\Big),$$

where

$$\xi = ax+by+cz+\frac{2ct}{3a^3b\lambda^2-2},$$

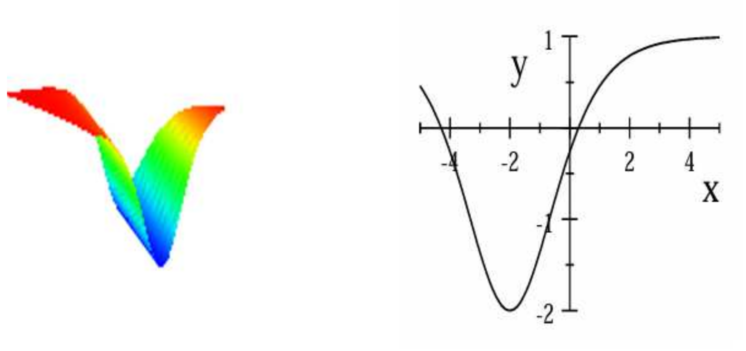
case 3. where $\lambda = 0$ and $\mu \neq 0$,

$$v_{3,4}(\xi) = \frac{\pm 6a\sqrt{10bc}}{(ax+by+cz-ct+k)^2}.$$

case 4. where $\lambda \neq 0$ and $\mu = 0$, v_7 is constant.

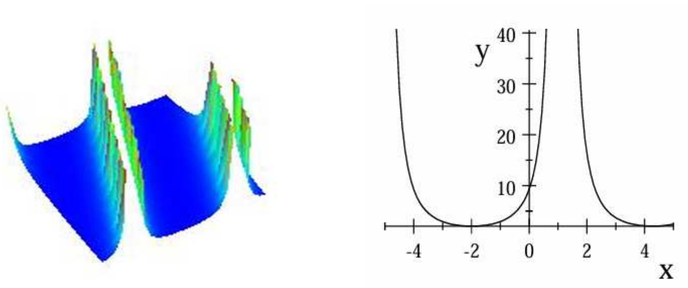
Conclusion

In this paper, we presented a new family of (3 + 1)-dimensional modified Rosenau's equations. A new $\exp(Q(\xi))$-expansion method has been successfully implemented to find new traveling wave solutions for our equations. our equations. We expressed to some our new solutions graphical.



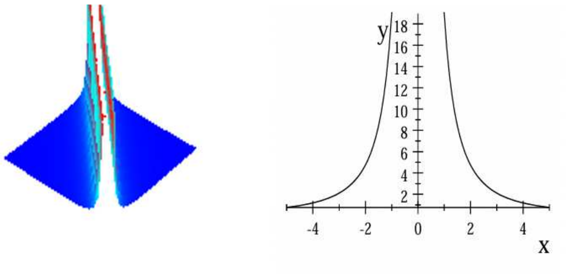
(a) (b)

Fig.1 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_1 with $\lambda > 0$,$c+b+-1$, $a=1$ and $k=2$.



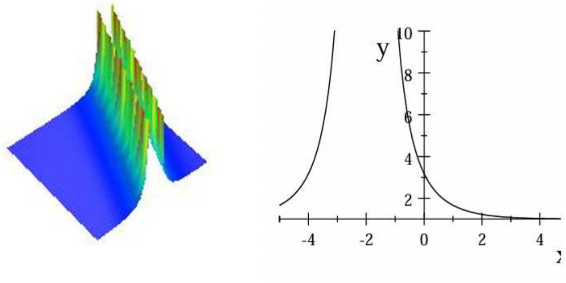
(a) (b)

Fig.2 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_2 with $\lambda < 0$, $c=b=-1$, $a=1$ and $k=2$.



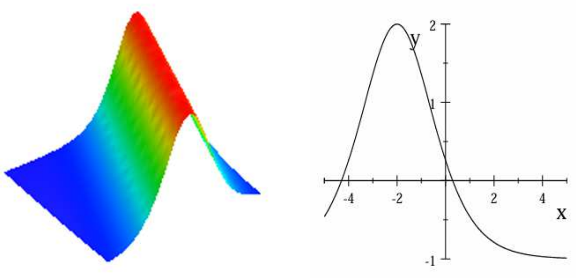
(a) (b)

Fig.3 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_3 with $\lambda = 0$, $c = b = -1$, $a = 1$ and $k = 2$.



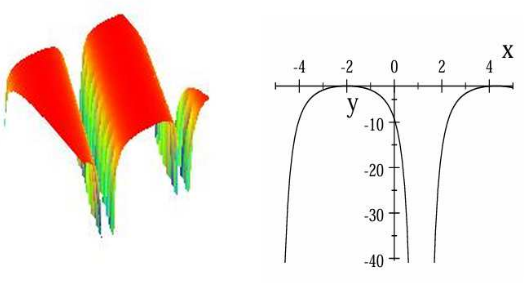
(a) (b)

Fig.4 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_4 with $\lambda = A^2$, $c = b = -1$, $A = 1$ and $k = 2$.



(a) (b)

Fig.5 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_1 with $\lambda > 0$, $c = b = -1$, $a = 1$ and $k = 2$.



(a) (b)

Fig.6 Plot3D from give (a) and Plot2D from give (b) of the solution of first application v_2 with $\lambda < 0$, $c = b = -1$, $a = 1$ and $k = 2$.

Declarations

Competing interests: No
Authors' contributions: Prof.Dr. Mohammed Al-Amry and M.P. Eman Al-Abdali formulated the problem, and M.P. Eman Al-Abdali prepared the manuscript.
Funding: No
Acknowledgments: Authors are thankful to the learned referee for their valuable suggestions regarding improvement of the paper.

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