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Kählerian spacetime and η-Einstein solitons

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Abstract: This paper is dedicated to the study of the geometric composition of a Kählerian spacetime with a η-Einstein soliton. Here, we have delineated the nature such soliton on a Kählerian spacetime to be expanding, steady, or shrinking under various physical conditions depend on isotropic pressure, cosmological constant, energy density, and gravitational constant relations. We have also discussed η-Ricci soliton on some special types of perfect fluid spacetime such as dust fluid, viscous fluid, dark fluid and radiation era.

Keywords: K*ä*hler spacetime; Quasi-conformal curvature tensor; Weakly symmetric; Weakly Ricci symmetric; Perfect fluid; Dust fluid and viscous fluid.

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1 Background and Motivations

In general relativity, a 4-dimensional pseudo-Riemannian manifold equipped with g is regarded as a perfect fluid spacetime. In cosmology, perfect fluids are employed to simulate idealized matter distributions such as the interior of a star or an isotropic pressure. A perfect fluid has a mass density ρ and an isotropic pressure p. If ρ=3p, then radiation fluid is the ideal space fluid. In [18] addressed the use of semi-Riemannian geometry in the theory of relativity. Kaigorodov investigates the curvature structure of spacetime in [13]. The curvature structure and other aspects of spacetime were studied by other geometers ([22], [4], [8], [29], [16], [28], [24], [30], [31], [32], [33], [34]).

According to [27], a quasi-conformal curvature tensor is defined by :

$$\begin{aligned}\mathscr{C}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3 = \alpha^{\dagger}\mathscr{R}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3 + \beta^{\dagger}[\mathscr{S}^{\dagger}(\mathscr{B}_2,\mathscr{B}_3)\mathscr{B}_1 - \mathscr{S}^{\dagger}(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_2 \\ + g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{Q}\mathscr{B}_1 - g(\mathscr{B}_1,\mathscr{B}_3)\mathscr{Q}\mathscr{B}_2] \\ - \frac{\kappa}{n}\left[\frac{\alpha^{\dagger}}{n-1} + 2\beta^{\dagger}\right][g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{B}_1 - g(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_2],\end{aligned} \tag{1}$$

where $\alpha^{\dagger}$ and $\beta^{\dagger}$ are constants. If $\alpha^{\dagger}$=1 and $\beta^{\dagger}$=-$\frac{1}{n-2}$, then (1) reduces to the conformal curvature tensor (or, Weyl tensor) $\mathscr{C}$ of type $(1,3)$ is given by.

$$\begin{aligned}\mathscr{C}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3 &= \mathscr{R}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3 - \frac{1}{n-2}[\mathscr{S}^{\dagger}(\mathscr{B}_2,\mathscr{B}_3)\mathscr{B}_1 - \mathscr{S}^{\dagger}(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_2 \\ &+ g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{Q}\mathscr{B}_1 - g_1(\mathscr{B}_1,\mathscr{B}_3)\mathscr{Q}\mathscr{B}_2] \\ &+ \frac{\kappa}{(n-1)(n-2)}[g_1(\mathscr{B}_2,\mathscr{B}_3)\mathscr{B}_1 - g_1(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_2],\end{aligned} \tag{2}$$

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Also we know that

$$(div\mathscr{C})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3 = \frac{n-3}{n-2}[\{(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2)\mathscr{B}_3 - (\nabla_{\mathscr{B}_2}\mathscr{S}^\dagger)(\mathscr{B}_1)\mathscr{B}_3\} - \frac{1}{n-2}\{(\mathscr{B}_1\kappa)g_1(\mathscr{B}_2,\mathscr{B}_3) - (\mathscr{B}_3\kappa)g_1(\mathscr{B}_1,\mathscr{B}_3)\}]. \tag{3}$$

where κ is the scalar curvature.

A semi-Riemannian manifold is called quasi-conformally at if $\mathscr{C}^\dagger = 0$ for $n > 3$. The quasi-conformal curvature tensor have been studied by several authors in various ways such as ([39],[40],[41],[42]) and many others. The quasi-conformal curvature tensor has significant applications in general relativity, providing valuable insights into the geometric structure and physical properties of space-time. A quasi-conformally flat perfect fluid spacetime is both a de Sitter spacetime and a Robertson–Walker spacetime. This spacetime is studied as a solution within $f(\mathscr{R},\mathscr{G})$ gravity theory, leading to a relationship among the snap, jerk, and deceleration parameters using the flat Friedmann–Robertson–Walker metric. For three different models of $f(\mathscr{R},\mathscr{G})$, the weak energy condition, null energy condition, and dominant energy condition are satisfied, while the strong energy condition is violated, aligning well with recent observational studies indicating that the Universe is in an accelerating phase. Perfect fluid spacetimes with a vanishing quasi-conformal curvature tensor are considered [43]. Also, the divergence of the quasi-conformal curvature tensor is studied in the context of perfect fluids, resulting in the derivation of numerous physical outcomes.

A Riemannian metric g on (Ω^n, g) is called Ricci soliton if it satisfies[12]:

$$(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2) + 2\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) + 2\theta_1 g(\mathscr{B}_1,\mathscr{B}_2) = 0. \tag{4}$$

Moreover, we recall the notion of an η-Ricci soliton [5]:

$$(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2) + 2\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) + 2\theta_1 g(\mathscr{B}_1,\mathscr{B}_2) + 2\theta_2\eta(\mathscr{B}_1)\eta(\mathscr{B}_1) = 0, \tag{5}$$

where θ_1, θ_2 are real constants. Thus an η-Ricci soliton in (Ω^n, g) are structure $(g,\xi,\theta_1,\theta_2)$ that satisfy the equation (5). If θ_2=0, the structure (g,ξ,θ_1) is called Ricci soliton defined by the equation (4) and said to be shrinking, steady, or expanding according to $\theta_1 < 0$, θ_1=0 or $\theta_1 > 0$, respectively [7]:

Since $(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2)$, is given by

$$(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2) = g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2) + g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi), \tag{6}$$

which implies in view of (6) and (5) that

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = -\theta_1 g(\mathscr{B}_1,\mathscr{B}_2) - \theta_2\eta(\mathscr{B}_1)\eta(\mathscr{B}_2) - \frac{1}{2}[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2) + g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)], \tag{7}$$

for any $\mathscr{B}_1$, $\mathscr{B}_2$ in $\chi(\Omega)$.

Now, we consider an η-Einstein soliton on perfect fluid spacetime [3]:

$$(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2) + 2\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) + (2\theta_1 - \kappa)g(\mathscr{B}_1,\mathscr{B}_2) + 2\theta_2\eta(\mathscr{B}_1)\eta(\mathscr{B}_2) = 0, \tag{8}$$

In particular, if θ_2=0, (g,ξ,θ_1) is an Einstein soliton [6]. After that, there are so many researches around solitons equation, for more details ([35],[36],[37],[38]).

Also, from (6) and (8), we get

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = -(\theta_1 - \frac{\kappa}{2})g(\mathscr{B}_1,\mathscr{B}_2) - \theta_2\eta(\mathscr{B}_1)\eta(\mathscr{B}_2) - \frac{1}{2}[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2) + g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)]. \tag{9}$$

A Riemannian manifold (Ω^n, g) is said to be (i) Weakly symmetric (WS)$_n$, if its curvature tensor $\mathscr{R}^\ddagger$ of type $(0,4)$ satisfies

$$\begin{aligned}(\nabla_{\mathscr{B}_1}\mathscr{R}^\ddagger)(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) = {} & \alpha^\star(\mathscr{B}_1)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) + \beta^\star(\mathscr{B}_2)\mathscr{R}^\ddagger(\mathscr{B}_1,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) \\ & + \gamma(\mathscr{B}_3)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_1,\mathscr{B}_4,\mathscr{B}_5) + \delta(\mathscr{B}_4)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_1,\mathscr{B}_5) \\ & + \sigma(\mathscr{B}_5)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_1),\end{aligned} \tag{10}$$

(ii) Weakly Ricci symmetric (WRS)$_4$, if its Ricci tensor $\mathscr{S}^\dagger$ of type $(0,2)$ holds

$$(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2,\mathscr{B}_3) = \alpha^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3) + \beta^\star(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3) + \gamma(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1), \tag{11}$$

where $\alpha^\star$, $\beta^\star$, γ, δ and σ are simultaneous non-vanishing 1-form.

In [21] we notice that if the (Ω^n, g) is (WS)$_n$, then $\beta^\star$=γ=δ=σ. But in our study we take $\beta^\star$=γ=δ=σ=ψ, then (10) and (11) reduces to as

$$(\nabla_{\mathscr{B}_1}\mathscr{R}^\ddagger)(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) = \alpha^\star(\mathscr{B}_1)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) + \psi(\mathscr{B}_2)\mathscr{R}^\ddagger(\mathscr{B}_1,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) + \psi(\mathscr{B}_3)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_1,\mathscr{B}_4,\mathscr{B}_5) + \psi(\mathscr{B}_4)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_1,\mathscr{B}_5) + \psi(\mathscr{B}_5)\mathscr{R}^\ddagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_1), \tag{12}$$

and

$$(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2,\mathscr{B}_3) = \alpha^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3) + \psi(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3) + \psi(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1), \tag{13}$$

where $g(\mathscr{B}_1,\upsilon)$=$\psi(\mathscr{B}_1)$ and $g(\mathscr{B}_1,\vartheta)$=$\alpha^\star(\mathscr{B}_1)$, respectively.

(iii) Special weakly Ricci symmetric (SWRS)$_n$, if its Ricci tensor $\mathscr{S}^\dagger$ of type $(0,2)$ satisfies

$$(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2,\mathscr{B}_3) = 2\mu^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3) + \mu^\star(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3) + \mu^\star(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1) \tag{14}$$

where $\mu^\star$ is a 1-form associated with ω such that $\mu^\star(\mathscr{B}_1)$=$g(\mathscr{B}_1,\omega)$ [26].

In 1988, Chaki [53] defined a non-flat Riemannian manifold as a pseudo-Ricci-symmetric manifold. Generalizing this notion, Tamássy and Binh [54], in 1993, developed the concept of n-dimensional weakly Ricci-symmetric manifolds. If an $n(\geq 3)$-dimensional Riemannian or semi-Riemannian manifold has a non-zero $\mathscr{S}^\dagger$, satisfies the condition defined by (11). According to Chaki, the manifold is referred to as a generalized pseudo-Ricci-symmetric manifold [53] if the vector $\alpha^\star$ in (11) is replaced with $2\alpha^\star$. Hence, the defining condition of a weakly Ricci-symmetric manifold is a little weaker than that of a generalized pseudo-Ricci-symmetric manifold.

The inquiry of weakly Ricci-symmetric space-times is important because this kind of space-time describes how the universe has evolved. In general relativity, the matter tensor of a space-time is the most important component. It is known that the universe's material content behaves as a perfect fluid space-time in typical cosmological models [52]. After that so many authors investigated such spacetime in view of different context ([46],[47],[48],[49],[50],[51]).

We categorize the paper as: After background and motivations, in section 2 basic results of (KST)$_4$ and discuss the soliton on (KST)$_4$ in conjunction with perfect fluid (briefly, PF). In section 3 we investigate a quasi-conformally flat on (KST)$_4$ and deduce some interesting results. Beside this, a quasi-conformally flat on (KST)$_4$ with PF are discuss in section 4. The section 5 are concern with (KST)$_4$ with perfect fluid with divergence free. Also in section 6, we consider dust fluid (KST)$_4$ and viscous fluid (KST)$_4$ and analysis the results. Finally, we summarize our work by discussing some symmetric proprieties on such spacetime in section 7.

The following abbreviations are used throughout the study: (KST)$_4$: Kähler spacetime, EMT: Energy momentum tensor, and GRS: General relativistic spacetime.

2 Basic Properties of Kählerian spacetime

A 4-dimensional manifold has general relativistic perfect fluid spacetime such type of manifold is said to be a Kählerian spacetime (briefly, (KST)$_4$) if its metric satisfies

$$\phi^2\mathscr{B}_1 = -\mathscr{B}_1, \tag{15}$$

$$g(\overline{\mathscr{B}}_1,\overline{\mathscr{B}}_2) = g(\mathscr{B}_1,\mathscr{B}_2), \tag{16}$$

$$(\nabla_{\mathscr{B}_1}\phi)\mathscr{B}_2 = 0, \tag{17}$$

$$\mathscr{S}^\dagger(\overline{\mathscr{B}}_1,\overline{\mathscr{B}}_2) = \mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2), \tag{18}$$

where ϕ is $(1,1)$-tensor field and g is Riemannian metric. We denote $\phi(\mathscr{B}_1)$ by $\overline{\mathscr{B}}_1$.

The EFE with cosmological constant on a (KST)$_4$ [18]:

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) + [\mu^\flat - \frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2) = \tau\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2), \tag{19}$$

$\forall\ \mathscr{B}_1,\mathscr{B}_2$, where $\mu^\flat$ is cosmological constant, τ is the gravitation constant and $\mathscr{T}^\dagger$ is the EMT of type $(0,2)$ defined as for PF [18]

$$\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = (\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2) + pg(\mathscr{B}_1,\mathscr{B}_2), \tag{20}$$

where ρ is the energy density, p is the isotropic pressure of the fluid and $\mathscr{A}^\natural$ is a non-zero 1-form such that $g(\mathscr{B}_1,\mathscr{V})$=$\mathscr{A}^\natural(\mathscr{B}_1)$ for all $\mathscr{B}_1$; $\mathscr{V}$ being the flow vector field of the fluid and satisfies $g(\mathscr{V},\mathscr{V})$=-1.

In view of (19) and (20), we have

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)+[\mu^\flat-p\tau-\frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2)=\tau(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2). \tag{21}$$

Replacing $\mathscr{B}_1$ by $\overline{\mathscr{B}}_1$ and $\mathscr{B}_2$ by $\overline{\mathscr{B}}_2$ in (21), using (16) and (18), we get

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)+[\mu^\flat-p\tau-\frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2)=\tau(\rho+p)\mathscr{A}^\natural(\overline{\mathscr{B}}_1)\mathscr{A}^\natural(\overline{\mathscr{B}}_2). \tag{22}$$

Subtracting (22) from (21) and taking $\mathscr{B}_2$=$\mathscr{V}$, it yield

$$\tau(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)=0. \tag{23}$$

Since, in Kähler spacetime $\tau\neq 0$ and $\mathscr{A}^\natural(\mathscr{B}_1)\neq 0$, we get

$$\rho=-p. \tag{24}$$

Using (24) in (21) , we have

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=\Big(\frac{\kappa}{2}-\mu^\flat+p\tau\Big)g(\mathscr{B}_1,\mathscr{B}_2). \tag{25}$$

Taking $\mathscr{B}_1$=$\mathscr{B}_2$=e_i, $1\leq i\leq 4$, in (25), we obtain

$$\mu^\flat=\frac{1}{4}[\kappa+4p\tau] \tag{26}$$

In view of (26), equation (25) reduces to

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=\frac{\kappa}{4}g(\mathscr{B}_1,\mathscr{B}_2). \tag{27}$$

Also from (7) and (21), we have

$$\Big[\theta_1-\mu^\flat+p\tau+\frac{\kappa}{2}\Big]g(\mathscr{B}_1,\mathscr{B}_2)+[\tau(p+\rho)+\theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)=-\frac{1}{2}\left[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2)+g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)\right]. \tag{28}$$

On contracting (28) over $\mathscr{B}_1$ and $\mathscr{B}_2$, we get

$$4\theta_1-\theta_2=4\mu^\flat-2\kappa-(3p-\rho)\tau-\mathrm{div}\xi. \tag{29}$$

Again putting $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (28), it yields

$$-2\theta_1+2\theta_2=-2\mu^\flat+\kappa-2\rho\tau. \tag{30}$$

In view of (29) and (30), we have

$$\theta_1=\mu^\flat-\frac{\kappa}{2}-p\tau-\frac{1}{3}div\xi \;\text{ and }\; \theta_2=-\tau(p+\rho)-\frac{1}{3}div\xi. \tag{31}$$

If θ_2=0, then we get the Ricci soliton with θ_1=$\frac{1}{2}[\mu^\flat-\kappa+2\tau\rho]$, which is consistent if ρ=$\frac{1}{2\tau}[\kappa-\mu^\flat]$, growing if $\rho>\frac{1}{2\tau}[\kappa-\mu^\flat]$ and reducing if $\rho<\frac{1}{2\tau}[\kappa-\mu^\flat]$.
Thus we state the results.

Proposition 1. *If a perfect fluid (KST)$_4$ satisfies the EFE then, the spacetime represents Einstein.*

Since an Einstein spacetime gives $div\mathscr{C}$=0, where $'div'$ denotes the divergence, hence from a result of Mantica et al. [17] it infers that the spacetime represent a generalized Robertson–Walker spacetime (GRW). Thus we get the result.

Corollary 1. *A perfect fluid (KST)$_4$ satisfying EFE is a generalized Robertson–Walker spacetime.*

Theorem 1. *An η-Ricci soliton $(g,\xi,\theta_1,\theta_2)$ on perfect fluid (KST)$_4$ satisfies EFE with cosmological constant, then the soliton is steady, expanding, or shrinking according as ρ=$\frac{1}{2\tau}[\kappa-\mu^\flat]$, $\rho>\frac{1}{2\tau}[\kappa-\mu^\flat]$, or $\rho<\frac{1}{2\tau}[\kappa-\mu^\flat]$.*

Also, for radiation fluid $\rho = 3p$ then θ_1=$\mu^\flat - \frac{\kappa}{2} - p\tau - \frac{1}{3}div\xi$ and θ_2=-4$p\tau - \frac{div\xi}{3}$. Again from (9) and (21), we obtain

$$\left[\theta_1 - \mu^\flat + p\tau\right] g(\mathscr{B}_1, \mathscr{B}_2) + [\tau(p+\rho) + \theta_2] \mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2) = -\frac{1}{2}\left[g(\nabla_{\mathscr{B}_1}\xi, \mathscr{B}_2) + g(\mathscr{B}_1, \nabla_{\mathscr{B}_2}\xi)\right]. \tag{32}$$

On contracting (32) over $\mathscr{B}_1$ and $\mathscr{B}_2$, we obtain

$$4\theta_1 - \theta_2 = 4\mu^\flat - 3p\tau + \tau\rho - \mathrm{div}\xi \tag{33}$$

Putting $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (32), we yields

$$\theta_1 - \theta_2 = \mu^\flat + \rho\tau. \tag{34}$$

From (33) and (34), we have

$$\theta_1 = \mu^\flat - p\tau - \frac{1}{3}div\xi \;\text{ and } \theta_2 = -\tau(p+\rho) - \frac{1}{3}div\xi. \tag{35}$$

If θ_2=0 then we get the Einstein soliton with θ_1=$\mu^\flat + \tau\rho$, which is consistent if ρ=$-\frac{\mu^\flat}{\tau}$, growing if $\rho > -\frac{\mu^\flat}{\tau}$ and reducing if $\rho < -\frac{\mu^\flat}{\tau}$.

So we state

Theorem 2. *An η-Einstein soliton $(g, \xi, \theta_1, \theta_2)$ on perfect fluid (KST)$_4$ satisfying EFE with cosmological constant then the soliton is steady, if ρ=$-\frac{\mu^\flat}{\tau}$, expanding if $\rho > -\frac{\mu^\flat}{\tau}$, or shrinking if $\rho < -\frac{\mu^\flat}{\tau}$.*

Now, for the radiation fluid ρ=3p then θ_1=$\frac{1}{3}[3\mu^\flat - 3p\tau + div\xi]$ and θ_2=-$\frac{1}{3}[12p\tau + div\xi]$.

3 Quasi-conformally flat on Kählerian spacetime

Let $\overline{\Omega}_4$ be a general relativistic spacetime. Then from (1), we have

$$\begin{aligned}\mathscr{C}^\dagger(\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4) &= \alpha^\dagger \mathscr{R}^\dagger(\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4)\\ &\quad + \beta^\dagger\Big[\mathscr{S}^\dagger(\mathscr{B}_2, \mathscr{B}_3)g(\mathscr{B}_1, \mathscr{B}_4) - \mathscr{S}^\dagger(\mathscr{B}_1, \mathscr{B}_3)g(\mathscr{B}_2, \mathscr{B}_4)\\ &\qquad + g(\mathscr{B}_2, \mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2, \mathscr{B}_4)\Big]\\ &\quad - \frac{\kappa}{n}\left[\frac{\alpha^\dagger}{n-1} + 2\beta^\dagger\right][g(\mathscr{B}_2, \mathscr{B}_3)g(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \mathscr{B}_3)g(\mathscr{B}_2, \mathscr{B}_4)].\end{aligned} \tag{36}$$

If $\mathscr{C}^\dagger(\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4)$=0, then equation (36) exhibits the form

$$\begin{aligned}\alpha^\dagger \mathscr{R}^\dagger(\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4) &= -\beta^\dagger\Big[\mathscr{S}^\dagger(\mathscr{B}_2, \mathscr{B}_3)g(\mathscr{B}_1, \mathscr{B}_4) - \mathscr{S}^\dagger(\mathscr{B}_1, \mathscr{B}_3)g(\mathscr{B}_2, \mathscr{B}_4)\\ &\qquad + g(\mathscr{B}_2, \mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2, \mathscr{B}_4)\Big]\\ &\quad + \frac{\kappa}{n}\left[\frac{\alpha^\dagger}{n-1} + 2\beta^\dagger\right][g(\mathscr{B}_2, \mathscr{B}_3)g(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \mathscr{B}_3)g(\mathscr{B}_2, \mathscr{B}_4)].\end{aligned} \tag{37}$$

Taking a frame field over $\mathscr{B}_1$ and $\mathscr{B}_4$, we yield

$$(\alpha^\dagger + 2\beta^\dagger)\mathscr{S}^\dagger(\mathscr{B}_2, \mathscr{B}_3) = (\alpha^\dagger + 2\beta^\dagger)\frac{\kappa}{4}g(\mathscr{B}_2, \mathscr{B}_3). \tag{38}$$

Replacing $\mathscr{B}_2$ by $\overline{\mathscr{B}_2}$ and $\mathscr{B}_3$ by $\overline{\mathscr{B}_3}$ in (38), using ((16) and (18), then (38) remain unchanged, which means that quasi-conformally flat Kähler spacetime is an Einstein manifold. Consequently, we state that

Theorem 3. *A quasi-conformally flat (KST)$_4$ is an Einstein spacetime provided $\alpha^\dagger$+2$\beta^\dagger \neq 0$.*

Corollary 2. *A quasi-conformally flat (KST)$_4$ is a perfect fluid spacetime provided $\alpha^\dagger$+2$\beta^\dagger \neq 0$.*

Again, from (37) and (38), it yield

$$\mathscr{R}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4)=\frac{\kappa}{12}\left\{g(\mathscr{B}_2,\mathscr{B}_3)g(\mathscr{B}_1,\mathscr{B}_4)-g(\mathscr{B}_1,\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_4)\right\}. \tag{39}$$

This encourages the following:

Theorem 4. *A quasi-conformally flat (KST)$_4$ is of constant curvature provided $\alpha^{\dagger}+2\beta^{\dagger}\neq 0$.*

As per Stephani [25] we state our finding.

Corollary 3. *A quasi-conformally flat (KST)$_4$ is the di Sitter spacetime as well as Robertson–Walker spacetime provided $\alpha^{\dagger}+2\beta^{\dagger}\neq 0$.*

A Riemannian manifold (Ω^n,g), $(n>3)$ is conformally flat if its curvature is constant. As a result, we obtain:

Corollary 4. *A quasi-conformally flat (KST)$_4$ is conformally flat provided $\alpha^{\dagger}+2\beta^{\dagger}\neq 0$.*

According to Petrov [20], we obtain the next outcome:

Theorem 5. *A quasi-conformally flat (KST)$_4$ is of Petrov type O provided $\alpha^{\dagger}+2\beta^{\dagger}\neq 0$.*

Also in view of (19) and (38), we have

$$\mathscr{T}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2)=\frac{1}{\tau}[\mu^{\flat}-\frac{\kappa}{4}]g(\mathscr{B}_1,\mathscr{B}_2), \tag{40}$$

which implies that

$$(\nabla_{\mathscr{B}_3}\mathscr{T}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2)=-\frac{1}{4\tau}d\kappa(\mathscr{B}_3)g(\mathscr{B}_1,\mathscr{B}_2). \tag{41}$$

Since, κ is constant then from (41) we get $(\nabla_{\mathscr{B}_3}\mathscr{T}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2)$=0. Thus we have

Theorem 6. *A quasi-conformally flat (KST)$_4$ satisfying EFE with cosmological constant, the energy momentum tensor is covariant constant provided $\alpha^{\dagger}+2\beta^{\dagger}\neq 0$.*

Now, we recall the following relation from [15] that:

$$\mathfrak{L}_{\xi}\mathscr{D}^{\dagger}-2\omega\mathscr{D}^{\dagger}=0, \tag{42}$$

where $\mathscr{D}^{\dagger}$ denote a geometrical (or physical) quantity and ω is a scalar. We use the metric inheritance symmetry for $\mathscr{D}^{\dagger}$=g in (42) so ξ is the KVF if ω=0. Then

$$(\mathfrak{L}_{\xi}g)(\mathscr{B}_1,\mathscr{B}_2)=2\omega g(\mathscr{B}_1,\mathscr{B}_2). \tag{43}$$

Now for curvature collineation (CC) we have from ([9],[10]):

$$(\mathfrak{L}_{\xi}\mathscr{R}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3)=0. \tag{44}$$

A perfect fluid (KST)$_4$ with quasi-conformal curvature tensor coupled with a Killing vector filed ξ is a curvature collineation. Then we have

$$(\mathfrak{L}_{\xi}g)(\mathscr{B}_1,\mathscr{B}_2)=0. \tag{45}$$

After, taking Lie-derivative of (36), using (44) and (45), we get $(\mathfrak{L}_{\xi}\mathscr{C}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3)$=0. Thus we state that

Theorem 7. *If a perfect fluid (KST)$_4$ admits a quasi-conformal curvature tensor, where ξ is CC then the Lie derivative of curvature tensor vanishes along ξ.*

Again, if we consider the matter collineation then the EMT $\mathscr{T}^{\dagger}$, satisfies

$$(\mathfrak{L}_{\xi}\mathscr{T}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2)=0. \tag{46}$$

Also, from (19) and (38), we have

$$\left(\mu^{\flat}-\frac{\kappa}{4}\right)g(\mathscr{B}_1,\mathscr{B}_2)=\tau\mathscr{T}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2). \tag{47}$$

Taking the Lie derivatives of (47) along ξ and using (45), we get

$$(\mathfrak{L}_{\xi}\mathscr{T}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2)=0, \tag{48}$$

which implies that perfect fluid (KST)$_4$ admit matter collineation. So we assert result.

Theorem 8. *A quasi-conformally flat perfect fluid (KST)$_4$ satisfies the EFE with a cosmological constant admit matter collineation if and only if ξ is a Killing vector field.*

Especially, if ξ is a conformal Killing vector field (briefly, CKVF). Next, we have

$$(\mathfrak{L}_\xi g)(\mathscr{B}_1,\mathscr{B}_2) = 2\omega g(\mathscr{B}_1,\mathscr{B}_2). \tag{49}$$

After, taking the Lie derivatives of (47) along ξ and using (49) we acquire

$$\left(\mu^\flat - \frac{\kappa}{4}\right) 2\omega g(\mathscr{B}_1,\mathscr{B}_2) = \tau(\mathfrak{L}_\xi \mathscr{T}^\dagger)(\mathscr{B}_1,\mathscr{B}_2). \tag{50}$$

In view of (47), Eq.(50) reduces to

$$(\mathfrak{L}_\xi \mathscr{T}^\dagger)(\mathscr{B}_1,\mathscr{B}_2) = 2\omega \mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2). \tag{51}$$

As per above consequence, the EMT possesses the Lie inheritance ability along ξ. We require the outcome.

Theorem 9. *If a quasi-conformally flat perfect fluid (KST)$_4$ satisfies the EFE with a cosmological term admits a conformal Killing vector field if and only if the EMT has the symmetric inheritance property.*

4 Perfect fluid (KST)$_4$ WITH $\mathscr{C}^\dagger = 0$

In this part, we consider perfect fluid (KST)$_4$ which satisfies $\mathscr{C}^\dagger = 0$ and EFE without the cosmological constant, that is, $\mu^\flat$=0. Then from (21) and (38), we have

$$-\left(\frac{\kappa}{4} + p\tau\right) g(\mathscr{B}_1,\mathscr{B}_2) = \tau(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2). \tag{52}$$

Putting $\mathscr{B}_1$ by $\overline{\mathscr{B}}_1$ and $\mathscr{B}_2$ by $\overline{\mathscr{B}}_2$ in (52) and using (16), we get

$$-\left(\frac{\kappa}{4} + p\tau\right) g(\mathscr{B}_1,\mathscr{B}_2) = \tau(\rho+p)\mathscr{A}^\natural(\overline{\mathscr{B}}_1)\mathscr{A}^\natural(\overline{\mathscr{B}}_2). \tag{53}$$

Subtracting (52) from (53), it yield

$$(\rho+p) = 0,\ \tau \neq 0,\ \mathscr{A}^\natural(\mathscr{B}_1) \neq 0. \tag{54}$$

which means that the fluid exhibits cosmological constant behavior and from (54), we get ρ=-p, that is, spacetime is rapidly expanding in cosmology which is called inflation. Since quasi-conformally flat perfect fluid spacetime is an Einstein manifold, that is, the manifold of constant scalar curvature κ. Then from (52) implies that the pressure p is constant and hence from (54), ρ is also constant.

Now, the energy and force equation for a PF [18]:

$$\mathscr{V}\rho = -(\rho+p)div\mathscr{V}, \tag{55}$$

$$(\rho+p)\nabla_{\mathscr{V}}\mathscr{V} = -grad(p) - (\mathscr{V}p)\mathscr{V}, \tag{56}$$

respectively. Since both the parameter ρ and p are constant. Then from (55) and (56) we get $div\mathscr{V}$=0 and $\nabla_{\mathscr{V}}\mathscr{V}$=0. This means that the expansion scalar and acceleration vector are both vanishes. As a result, our finding:

Theorem 10. *A quasi-conformally flat perfect fluid (KST)$_4$ obeying EFE without cosmological term, then the acceleration vector and expansion scalar are vanishes.*

Further, from (47) we have

$$\kappa = -\tau\psi, \tag{57}$$

where $\psi^\dagger$=trace $\mathscr{T}^\dagger$. In view of (57), Eq.(19) reduces

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = \tau\left\{\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2) - \frac{\psi^\dagger}{2} g(\mathscr{B}_1,\mathscr{B}_2)\right\}. \tag{58}$$

According to [1], the EFE for a purely electromagnetic distribution in (KST)$_4$ is as follows:

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = \tau\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2). \tag{59}$$

With the help of (58) and (59) we get $\psi^\dagger$=0. So from (57) we get κ=0. Thus, from (39) we obtain $\mathscr{R}^\ddagger(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4)$=0, it confirms the flatness of spacetime. As a consequence, we get

Theorem 11. *If a quasi-conformally flat perfect fluid (KST)$_4$ admits EFE without cosmological term, then for a purely electromagnetic distribution spacetime is an Euclidean space.*

Also, taking $\mathscr{B}_1$=$\mathscr{B}_2$=e_i in (52), we get

$$\kappa = \tau(\rho - 3p) \tag{60}$$

In view of (60), Eq.(38) take the form

$$\mathscr{S}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2) = \frac{\tau(\rho-3p)}{4} g(\mathscr{B}_1,\mathscr{B}_2). \tag{61}$$

We define $\mathscr{Q}$ and the $(0,2)$-tensor $\mathscr{S}^{\dagger 2}$ as

$$g(\mathscr{Q}\mathscr{B}_1,\mathscr{B}_2) = \mathscr{S}^{\dagger}(\mathscr{B}_1,\mathscr{B}_2), \tag{62}$$

and

$$\mathscr{S}^{\dagger 2}(\mathscr{B}_1,\mathscr{B}_2) = \mathscr{S}^{\dagger}(\mathscr{Q}\mathscr{B}_1,\mathscr{B}_2). \tag{63}$$

Therefore from (61) , we have

$$\mathscr{S}^{\dagger}(\mathscr{Q}\mathscr{B}_1,\mathscr{B}_2) = \frac{\tau^2(\rho-3p)^2}{16} g(\mathscr{B}_1,\mathscr{B}_2). \tag{64}$$

After taking $\mathscr{B}_1$=$\mathscr{B}_2$=e_i in (64), we yield

$$\|\mathscr{Q}\|^2 = \frac{\tau^2(\rho-3p)^2}{4}. \tag{65}$$

So, we state

Theorem 12. *The square length of the Ricci operator $\mathscr{Q}$ on a quasi-conformally flat perfect fluid (KST)$_4$ is $\frac{\tau^2(\rho-3p)^2}{4}$.*

5 Perfect fluid (KST)$_4$ WITH $(div\mathscr{C}^{\dagger}) = 0$

In this section we consider a perfect fluid (KST)$_4$ obeying EFE without cosmological term admit divergence-free quasi-conformal curvature tensor and deduce some result.

Definition 1. *A perfect fluid (KST)$_4$ is said to be quasi-conformal conservative if*

$$(div\mathscr{C}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3) = 0, \tag{66}$$

where $'div'$ stands for divergence. Now, Eq.(36) can be written as

$$\begin{aligned}(\mathrm{div}\mathscr{C}^{\dagger})(\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3) &= \beta^{\dagger}\left\{(\nabla_{\mathscr{B}_1}\mathscr{S}^{\dagger})(\mathscr{B}_2,\mathscr{B}_3) - (\nabla_{\mathscr{B}_2}\mathscr{S}^{\dagger})(\mathscr{B}_1,\mathscr{B}_3)\right\} \\ &\quad - \frac{\alpha^{\dagger}+6\beta^{\dagger}}{12}\left\{g(\mathscr{B}_2,\mathscr{B}_3)\,d\kappa(\mathscr{B}_1) - g(\mathscr{B}_1,\mathscr{B}_3)\,d\kappa(\mathscr{B}_2)\right\}.\end{aligned} \tag{67}$$

Since, κ is constant, that is $d\kappa$=0, so from (66) and (67) we get

$$(\nabla_{\mathscr{B}_1}\mathscr{S}^{\dagger})(\mathscr{B}_2,\mathscr{B}_3) - (\nabla_{\mathscr{B}_2}\mathscr{S}^{\dagger})(\mathscr{B}_2,\mathscr{B}_3) = 0. \tag{68}$$

With the help of (68), from equation (59) we yield

$$(\nabla_{\mathscr{X}}\mathscr{T}^{\dagger})(\mathscr{B}_2,\mathscr{B}_3) = (\nabla_{\mathscr{Y}}\mathscr{T}^{\dagger})(\mathscr{B}_1,\mathscr{B}_3), \tag{69}$$

which means that the EMT $\mathscr{T}^{\dagger}$ is of Codazzi type. So we have the following result

Theorem 13. *A perfect fluid (KST)$_4$ obeying EFE without cosmological constant satisfies the divergence free quasi-conformal curvature then the EMT, $\mathscr{T}^{\dagger}$ is of Codazzi type.*

According to (see, [14, 11]) and the Theorem 13 we state that

Theorem 14. *A perfect fluid (KST)$_4$ obeying EFE without cosmological term admit divergence free quasi-conformal curvature then its velocity vector field is proportional to the gradient vector field of the energy density.*

As per Barnes [2], we have

Theorem 15. *A perfect fluid (KST)$_4$ obeying EFE without cosmological term admit divergence free quasi-conformal curvature then the spacetime is of type I, D or O.*

According to [45], a spacetime with a Codazzi-type EMT represents a Yang Pure Space (YPS). Guilfoyle and Nolan [44] defined "Yang Pure Space" as a Lorentzian manifold (Ω^4, g), where the metric tensor of the manifold solves Yang's equations defined by (68) and proved that a perfect fluid spacetime with $p+\rho \neq 0$ is a YPS if and only if the spacetime is a Robertson-Walker spacetimes. Thus from (69) it is clear that (PFKST)$_4$ with $p+\rho \neq 0$ satisfies Codazzi type EMT is a Robertson-Walker spacetimes. In this sequel, we reflect that

Theorem 16. *A perfect fluid (KST)$_4$ obeying the EFE coupled with a Codazzi type EMT is a Robertson-Walker spacetime.*

6 Dust fluid (KST)$_4$ admits $\mathscr{C}^\dagger = 0$

We recall the EMT on dust fluid (KST)$_4$ [23]:

$$\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = \rho\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2), \tag{70}$$

From (47) and (70) we obtain

$$[\mu^\flat - \frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2) = \tau\rho\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2), \tag{71}$$

If we replace $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (71) we have

$$\mu^\flat = \frac{\kappa}{4} - \tau\rho \tag{72}$$

Again taking contraction over $\mathscr{B}_1$ and $\mathscr{B}_2$, Eq. (71) leads to

$$\mu^\flat = \frac{\kappa}{4} - \frac{\tau\rho}{4} \tag{73}$$

In view of (72) and (73) we get ρ=0, then from (70), we have $\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2)$=0. As a result, we state the outcome:

Theorem 17. *A quasi-conformally flat dust fluid (KST)$_4$ satisfying EFE with cosmological constant is vacuum.*

Also from (19) and (70), we have

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = [-\mu^\flat + \frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2) + \tau\rho\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2) \tag{74}$$

By virtue of (74), Eq. (7) takes the form

$$[-\mu^\flat + \frac{\kappa}{2} + \theta_1]g(\mathscr{B}_1,\mathscr{B}_2) + [\rho\tau + \theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2) = -\frac{1}{2}[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2) + g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)], \tag{75}$$

After contracting over $\mathscr{B}_1$ and $\mathscr{B}_2$ it leads to

$$4\theta_1 - \theta_2 = 4\mu^\flat - 2\kappa + \tau\rho - \mathrm{div}\,\xi \tag{76}$$

Again, if we put $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (75), we get

$$\theta_1 - \theta_2 = \mu^\flat - \frac{\kappa}{2} + \rho\tau. \tag{77}$$

With the help of (76) and (77) we yield

$$\theta_1 = \mu^\flat - \frac{\kappa}{2} - \frac{div\xi}{3} \quad \text{and} \quad \theta_2 = -(\rho\tau + \frac{div\xi}{3}) \tag{78}$$

It is obvious that if θ_2=0, we get the Ricci soliton with θ_1=-$\mu^\flat - \frac{\kappa}{2} + \rho\tau$, which is consistent if ρ=$\frac{1}{\tau}[\frac{\kappa}{2} - \mu^\flat]$, growing if $\rho > \frac{1}{\tau}[\frac{\kappa}{2} - \mu^\flat]$ and reducing if $\rho < \frac{1}{\tau}[\frac{\kappa}{2} - \mu^\flat]$.
The consequence is as follows:

Theorem 18. *An η-Ricci soliton $(g,\xi,\theta_1,\theta_2)$ on a dust fluid (KST)$_4$ satisfying EFE with cosmological constant, then the soliton is steady, expanding, or shrinking according as $\rho=\frac{1}{\tau}[\frac{\kappa}{2}-\mu^\flat]$, $\rho>\frac{1}{\tau}[\frac{\kappa}{2}-\mu^\flat]$, or $\rho<\frac{1}{\tau}[\frac{\kappa}{2}-\mu^\flat]$.*

Now, if the fluid is of radiation type, that is, ρ=3p then θ_1=$\mu^\flat-\frac{\kappa}{2}-\frac{div\xi}{3}$ and θ_2=-$(3p\tau+\frac{div\xi}{3})$.
Also taking (74) and (9) we have

$$[-\mu^\flat+\theta_1]g(\mathscr{B}_1,\mathscr{B}_2)+[\tau\rho+\theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)=-\frac{1}{2}[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2)+g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)], \tag{79}$$

On contracting over $\mathscr{B}_1$ and $\mathscr{B}_2$, Eq.(79) leads to

$$4\theta_1-\theta_2=\tau\rho+4\mu^\flat-\operatorname{div}\xi \tag{80}$$

Again, if we put $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (79), we get

$$\theta_1-\theta_2=\mu^\flat+\tau\rho. \tag{81}$$

With the help of (80) and (81) we yield

$$\theta_1=\mu^\flat-\frac{div\xi}{3}\quad\text{and}\quad\theta_2=-\tau\rho-\frac{div\xi}{3} \tag{82}$$

Now, if θ_2=0 then we obtain the Einstein soliton with θ_1=$\mu^\flat+\tau\rho$, which is consistent if $\mu^\flat$=-$\tau\rho$, growing if $\mu^\flat>-\tau\rho$ and reducing if $\mu^\flat<-\tau\rho$.
Thus we have:

Theorem 19. *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ on a dust fluid (KST)$_4$ satisfying EFE with cosmological constant, then the soliton is consistent if $\mu^\flat=-\tau\rho$, growing if $\mu^\flat>-\tau\rho$, or reducing if $\mu^\flat<-\tau\rho$.*

Again we consider the EMT $\mathscr{T}^\dagger$ for viscous fluid on (KST)$_4$ ([18],[19]):

$$\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)+pg(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{H}^\dagger(\mathscr{B}_1,\mathscr{B}_2), \tag{83}$$

where $\mathscr{H}^\dagger$ denotes the anisotropic pressure of the fluid.
Using (47) in (83), we get

$$[\mu^\flat-\frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2)=\tau[(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_1)+pg(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{H}^\dagger(\mathscr{B}_1,\mathscr{B}_2)], \tag{84}$$

For fix,$\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (84), it yields

$$\rho=\frac{1}{\tau}[\frac{\kappa}{4}-\mu^\flat-\tau\mathscr{I}^\dagger], \tag{85}$$

where $\mathscr{I}^\dagger$=$\mathscr{H}^\dagger(\mathscr{V},\mathscr{V})$.
Again contracting (84) over $\mathscr{B}_1$ and $\mathscr{B}_2$, we obtain

$$p=\frac{1}{3\tau}[4\mu^\flat-\kappa-\rho\tau-\tau\mathscr{N}^\dagger], \tag{86}$$

where $\mathscr{N}^\dagger$=Trace of $\mathscr{H}^\dagger$. Thus we can state:

Theorem 20. *If a quasi-conformally flat viscous fluid (KST)$_4$ satisfying EFE with cosmological constant, the energy density and the isotropic pressure are given by* (85) *and* (86)*.*

Again from (19) and (83), we have

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=[\frac{\kappa}{2}-\mu^\flat+p\tau]g(\mathscr{B}_1,\mathscr{B}_2)+\tau[(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)+\mathscr{H}^\dagger(\mathscr{B}_1,\mathscr{B}_2)]. \tag{87}$$

Keeping in mind (87), Eq. (7) can be written as

$$\begin{aligned}\left[\frac{\kappa}{2}-\mu+p\tau+\theta_1\right]g(\mathscr{X},\mathscr{Y})+[\tau(\rho+p)+\theta_2]\mathscr{A}^\natural(\mathscr{X})\mathscr{A}^\natural(\mathscr{Y})\\+\tau\mathscr{H}^\dagger(\mathscr{X},\mathscr{Y})=-\frac{1}{2}[g(\nabla_{\mathscr{X}}\xi,\mathscr{Y})+g(\mathscr{X},\nabla_{\mathscr{Y}}\xi)].\end{aligned} \tag{88}$$

After contraction over $\mathscr{B}_1$ and $\mathscr{B}_2$, where $\mathscr{N}^\dagger$=Trace of $\mathscr{H}^\dagger$ leads to

$$4\theta_1 - \theta_2 = 4\mu^\flat - 3p\tau - 2\kappa + \tau\rho - \tau\mathscr{N}^\dagger - \operatorname{div}\xi \tag{89}$$

Again, if we put $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (88), where $\mathscr{I}^\dagger$=$\mathscr{H}^\dagger(\mathscr{V},\mathscr{V})$ have

$$\theta_1 - \theta_2 = \mu^\dagger - \frac{\kappa}{2} + \tau(\rho + \mathscr{I}^\dagger) \tag{90}$$

With the help of (89) and (90) we yield

$$\theta_1 = \mu^\flat - \frac{\tau(\rho-3p)}{2} - \frac{\tau}{3}\left[\mathscr{I}^\dagger + 3p + \mathscr{N}^\dagger\right] - \frac{div\xi}{3} \text{ and } \theta_2 = -\tau\left[p + \rho + \frac{4\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3}\right] - \frac{div\xi}{3} \tag{91}$$

In particular if θ_2=0 then we get the Ricci soliton with θ_1=$\mu^\flat - \frac{\tau(\rho-3p)}{2} + \tau(\rho+\mathscr{I}^\dagger)$ which is consistent if p=$\rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$, growing if $p > \rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$ and reducing if $p < \rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$.
We state the result:

Theorem 21. *An η-Ricci soliton $(g,\xi,\theta_1,\theta_1)$ on a viscous fluid (KST)$_4$ satisfying EFE with cosmological constant. Then the soliton is*
(i) steady if p=$\rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$,
(ii) expanding if $p > \rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$,
(iii) shrinking if $p < \rho + (\frac{2}{3})\mathscr{I}^\dagger - \frac{2}{3}(\frac{\mu^\flat}{\tau})$.

Next, for radiation fluid ρ=3p then θ_1=$\mu^\flat - \frac{\tau}{3}\left[\mathscr{I}^\dagger + 3p + \mathscr{N}^\dagger\right]$ and θ_2=-$\tau(4p + \frac{4\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3})$-$\frac{div\xi}{3}$.
Also, utilizing (87), Eq. (9) take the form

$$\begin{aligned}\left[\mu^\flat - \theta_1 + p\tau\right] g(\mathscr{B}_1,\mathscr{B}_2) - [\tau(\rho+p)+\theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)&\\ - \tau\mathscr{H}^\dagger(\mathscr{B}_1,\mathscr{B}_2) = \frac{1}{2}\left[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2) + g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)\right].&\end{aligned} \tag{92}$$

On contracting over $\mathscr{B}_2$ and $\mathscr{B}_2$, where $\mathscr{N}^\dagger$=Trace of $\mathscr{H}^\dagger$ leads to

$$-4\theta_1 + \theta_2 = -4\mu^\flat + 3p\tau - \tau\rho + \tau\mathscr{N}^\dagger + \operatorname{div}\xi. \tag{93}$$

Again, if we put $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (92), where $\mathscr{I}^\dagger$=$\mathscr{H}^\dagger(\mathscr{V},\mathscr{V})$ reduces to

$$\theta_1 - \theta_2 = \mu^\flat + \tau\rho + \tau\mathscr{I}^\dagger. \tag{94}$$

From (93) and (94) we yield

$$\theta_1 = \mu^\flat - \tau p - \frac{\mathscr{I}^\dagger\tau}{3} - \frac{\mathscr{N}^\dagger\tau}{3} - \frac{div\xi}{3} \quad\text{and}\quad \theta_2 = -\tau\left[p + \rho + \frac{4\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3}\right] - \frac{div\xi}{3}. \tag{95}$$

It is clear that if θ_2=0 then we obtain the Einstein soliton with θ_1=$\mu^\flat - \tau p - \frac{\mathscr{I}^\dagger\tau}{3} - \frac{\mathscr{N}^\dagger\tau}{3} - \frac{div\xi}{3}$ which is steady if p=$\frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$, expanding if $p > \frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$ and shrinking if $p < \frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$. As a consequence, we get:

Theorem 22. *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ on a viscous fluid (KST)$_4$ satisfying EFE with cosmological constant. Then the soliton is*
(i) steady if p=$\frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$,
(ii) expanding if $p > \frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$,
(iii) shrinking if $p < \frac{\mu^\flat}{\tau} - \frac{\mathscr{I}^\dagger}{3} + \frac{\mathscr{N}^\dagger}{3} - \frac{div\xi}{3\tau}$.

Moreover, if there is radiation fluid ρ=3p then θ_1=$\mu^\flat - \tau p - \frac{\mathscr{I}^\dagger \tau}{3}$-$\frac{\mathscr{N}^\dagger \tau}{3}$-$\frac{div\xi}{3}$ and θ_2=-$\tau\left[4p+\frac{4\mathscr{I}^\dagger}{3}+\frac{\mathscr{N}^\dagger}{3}\right]-\frac{div\xi}{3}$.

Finally, we investigate whether or not a viscous fluid on a quasi-conformally flat (KST)-4, may permit heat flux, or not. In order to do this, we take into account the EMT, $\mathscr{T}^\dagger$ ([18],[19]):

$$\mathscr{T}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)+pg(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{F}^\dagger(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_2)\mathscr{F}^\dagger(\mathscr{B}_1), \tag{96}$$

where $\mathscr{F}^\dagger(\mathscr{B}_1)$=$g(\mathscr{B}_1,\xi)$ for all vector fields $\mathscr{B}_1$; ξ being the heat flux vector field and $g(\mathscr{V},\xi)$=0, that is, $\mathscr{F}(\mathscr{V})$=0. Using (47) in (96), we have

$$\left(\mu^\flat-\frac{\kappa}{4}\right)g(\mathscr{B}_1,\mathscr{B}_2)=\tau[(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)+pg(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{F}^\dagger(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_2)\mathscr{F}^\dagger(\mathscr{B}_1)], \tag{97}$$

Putting $\mathscr{B}_2$=$\mathscr{V}$ in (97), it yield

$$\mathscr{F}^\dagger(\mathscr{B}_1)=-\frac{1}{\tau}[\mu^\flat-\frac{\kappa}{4}+\tau p]\mathscr{A}^\natural(\mathscr{B}_1). \tag{98}$$

Therefore, we have:

Theorem 23. *A quasi-conformally flat viscous fluid (KST)$_4$ satisfying EFE with cosmological constant admit heat flux, provided* $\mu^\flat-\frac{\kappa}{4}+\tau p\neq 0$.

Also using (96) in (21), we have

$$\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_2)=[p-\mu^\flat+\frac{\kappa}{2}]g(\mathscr{B}_1,\mathscr{B}_2)+\tau[(\rho+p)\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{F}^\dagger(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_2)\mathscr{F}^\dagger(\mathscr{B}_1)]. \tag{99}$$

By virtue of (99), Eq.(7) can be written as

$$\begin{aligned}\left[p+\frac{\kappa}{2}-\mu^\flat+\theta_1\right]g(\mathscr{B}_1,\mathscr{B}_2)&+[\tau(\rho+p)+\theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)\\ &+\tau\left[\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{F}^\dagger(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_2)\mathscr{F}^\dagger(\mathscr{B}_1)\right]\\ &=-\frac{1}{2}\left[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2)+g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)\right].\end{aligned} \tag{100}$$

Taking contracting over $\mathscr{B}_1$ and $\mathscr{B}_2$, we get

$$4\theta_1-b\theta_2=4\mu^\flat-4p-2\kappa+\tau(p+\rho)-\mathrm{div}\xi \tag{101}$$

Putting $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (100) , yield

$$-2\theta_2+2\theta_2=2p+\kappa-2\mu^\flat-\tau(p+\rho). \tag{102}$$

From (101) and (102), we obtain

$$\theta_1=\mu^\flat-p+\frac{1}{6}\tau(p+\rho)-\frac{\tau(\rho-3p)}{2}-\frac{div\xi}{3}\ \text{ and }\ \theta_2=\frac{\tau(p+\rho)}{3}-\frac{div\xi}{3}. \tag{103}$$

If θ_2=0 then we get the Ricci soliton with θ_1=$-p+\mu^\flat-\frac{\kappa}{2}+\frac{\tau(\rho+p)}{2}$ which is consistent if p=$\frac{\mu^\flat}{1-\tau}$, growing if $p>\frac{\mu^\flat}{1-\tau}$ and reducing if $p<\frac{\mu^\flat}{1-\tau}$. Thus we state

Theorem 24. *An η-Ricci soliton $(g,\xi,\theta_1,\theta_2)$ in a (VFKST)$_4$ satisfying EFE with cosmological constant, then the soliton is steady, expanding, or shrinking according as* $p=\frac{\mu^\flat}{1-\tau},\ \ p>\frac{\mu^\flat}{1-\tau},\ \ p<\frac{\mu^\flat}{1-\tau}$.

Also, for radiation fluid ρ=3p then θ_1=$\mu^\flat-p$+$\frac{2p\tau}{3}$-$\frac{div\xi}{3}$ and θ_2=$\frac{4p\tau}{3}$-$\frac{div\xi}{3}$.

Finally, we use (99) in (9) we obtain

$$\begin{aligned}\left[p-\mu^\flat+\theta_1\right]g(\mathscr{B}_1,\mathscr{B}_2)&+[\tau(\rho+p)+\theta_2]\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{A}^\natural(\mathscr{B}_2)\\ &+\tau\left[\mathscr{A}^\natural(\mathscr{B}_1)\mathscr{F}^\dagger(\mathscr{B}_2)+\mathscr{A}^\natural(\mathscr{B}_2)\mathscr{F}^\dagger(\mathscr{B}_1)\right]\\ &=-\frac{1}{2}\left[g(\nabla_{\mathscr{B}_1}\xi,\mathscr{B}_2)+g(\mathscr{B}_1,\nabla_{\mathscr{B}_2}\xi)\right].\end{aligned} \tag{104}$$

Now, contracting (104) over $\mathscr{B}_1$ and $\mathscr{B}_2$, we have

$$4\theta_1+\theta_2=4\mu^\flat-4p+\tau(p+\rho)-\mathrm{div}\xi \tag{105}$$

Putting $\mathscr{B}_1$=$\mathscr{B}_2$=$\mathscr{V}$ in (104), it yields

$$\theta_1-\theta_2=\mu^\flat-p+\tau(p+\rho). \tag{106}$$

From (105) and (106) , we obtain

$$\theta_1=\mu^\flat-p+\frac{1}{8}[5\tau(p+\rho)-div\xi] \text{ and } \theta_2=\frac{1}{8}[-3\tau(p+\rho)-div\xi]. \tag{107}$$

If θ_2=0 then we get the Eistein soliton with θ_1=$\mu^\flat-p+\tau(p+\rho)$, which is consistent if p=$\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$, growing if $p>\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$ and reducing if $p<\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$. So, we reflect the outcome

Theorem 25. *An η-Einstein soliton $(g,\xi,\theta_1,\theta_2)$ in a (VFKST)$_4$ satisfying EFE with cosmological constant. Then the soliton is steady, expanding, or shrinking according as $p=\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$, $p>\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$, $p<\frac{1}{1-\tau}(\mu^\flat+\tau\rho)$.*

In particular, for radiation fluid ρ=$3p$ then θ_1=$\mu^\flat-p+\frac{1}{8}[20p\tau-div\xi]$ and θ_2=$\frac{1}{8}[-12p\tau-div\xi]$.

7 Symmetric on Kählerian spacetime properties

We consider a type of symmetric properties on a quasi-conformally flat (KST)$_4$ and deduce some notable results. Let (Ω^4,g) be a (KST)$_4$ then we have

$$\mathscr{R}^\dagger(\overline{\mathscr{B}}_2,\overline{\mathscr{B}}_3,\mathscr{B}_4,\mathscr{B}_5)=\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) \tag{108}$$

which implies that,

$$(\nabla_{\mathscr{B}_6}\mathscr{R})(\overline{\mathscr{B}}_2,\overline{\mathscr{B}}_3,\mathscr{B}_4,\mathscr{B}_5)=(\nabla_{\mathscr{B}_6}\mathscr{R})(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5) \tag{109}$$

In view of (10) and (109), we have

$$\begin{aligned}\psi(\overline{\mathscr{B}}_2)\mathscr{R}^\dagger(\mathscr{B}_6,\overline{\mathscr{B}}_3,\mathscr{B}_4,\mathscr{B}_5)+\psi(\overline{\mathscr{B}}_3)\mathscr{R}^\dagger(\overline{\mathscr{B}}_2,\mathscr{B}_6,\mathscr{B}_4,\mathscr{B}_5)\\ =\psi(\mathscr{B}_2)\mathscr{R}^\dagger(\mathscr{B}_6,\mathscr{B}_3,\mathscr{B}_4,\mathscr{B}_5)+\psi(\mathscr{B}_3)\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_6,\mathscr{B}_4,\mathscr{B}_5).\end{aligned} \tag{110}$$

For fix, $\mathscr{B}_3$=$\mathscr{B}_4$=e_i in (110) and taking the summation over i, $1\le i\le 4$, we yield

$$\begin{aligned}\psi(\overline{\mathscr{B}}_2)\mathscr{S}^\dagger(\mathscr{B}_6,\overline{\mathscr{B}}_5)+\mathscr{R}^\dagger(\overline{\mathscr{B}}_2,\mathscr{B}_6,\overline{\mathscr{B}}_5,\overline{\rho})\\ =\psi(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_6,\overline{\mathscr{B}}_5)-\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_6,\overline{\mathscr{B}}_5,\rho).\end{aligned} \tag{111}$$

Using (38) in (111) and summing over $\mathscr{B}_6$=$\mathscr{B}_5$=e_i, $1\le i\le 4$, we obtain

$$\mathscr{S}^\dagger(\mathscr{B}_2,\rho)=\frac{\kappa}{4}g(\mathscr{B}_2,\rho). \tag{112}$$

As per above sequel, we have

Theorem 26. *If a quasi-conformally flat (KST)$_4$ is a weakly symmetric then ρ and $\overline{\rho}$ are the eigen vector of the Ricci tensor $\mathscr{S}$ corresponding to the eigen value $\frac{\kappa}{4}$.*

Again, taking $\mathscr{B}_1$=$\mathscr{B}_5$=e_i, $1\le i\le 4$, in (10), we have

$$\begin{aligned}(div\mathscr{R}^\dagger)(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4)=\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\vartheta)+\psi(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_3,\mathscr{B}_4)\\ +\psi(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_4)+\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\upsilon).\end{aligned} \tag{113}$$

Also using (38) in Bianchi second identity, we get

$$(div\mathscr{R}^\dagger)(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4)=0. \tag{114}$$

In view of (113) and (114), we obtain

$$\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\vartheta)+\psi(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_3,\mathscr{B}_4)+\psi(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{U})+\mathscr{R}^\dagger(\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4,\upsilon)=0. \tag{115}$$

Taking $\mathscr{B}_3$=$\mathscr{B}_4$=e_i, $1\le i\le 4$, in (115) and using (38), it yield

$$\kappa\{g(\vartheta,\upsilon)-4\}=0. \tag{116}$$

This implies that either κ=0, or $g(\vartheta,\upsilon)$=4. Thus as per above discussion, from [49] we can state that:

Theorem 27. *If a quasi-conformally flat (KST)$_4$ is a weakly symmetric then the spacetime is of zero curvature, or the relation $g(\vartheta,\upsilon)$=4 holds on (Ω^4,g).*

For quasi-conformally flat (KST)$_4$ the scalar curvature κ is constant then $(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2,\mathscr{B}_3)$=0. Now from (11), we yield

$$\alpha^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3)+\psi(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3)+\psi(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{117}$$

In view of (38) and (117), we obtain

$$\frac{\kappa}{4}[\alpha^\star(\mathscr{X})g(\mathscr{B}_2,\mathscr{B}_3)+\psi(\mathscr{B}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\psi(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)]=0. \tag{118}$$

which implies that either κ=0, or

$$\alpha^\star(\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_3)+\psi(\mathscr{B}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\psi(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{119}$$

Let $\kappa\neq 0$, then for $\mathscr{B}_2$=$\overline{\mathscr{B}}_2$ and $\mathscr{B}_3$=$\overline{\mathscr{B}}_3$ in (119) we yield

$$\alpha^\star(\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_3)+\psi(\overline{\mathscr{B}}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\psi(\overline{\mathscr{B}}_3)g(\overline{\mathscr{B}}_2,\mathscr{B}_1)=0. \tag{120}$$

With the help of (119) and (120), we assert

$$\psi(\overline{\mathscr{B}}_2)g(\mathscr{B}_1,\overline{\mathscr{B}}_3)-\psi(\mathscr{B}_2)g(\mathscr{B}_1,\overline{\mathscr{B}}_3)+\psi(\overline{\mathscr{B}}_3)g(\overline{\mathscr{B}}_2,\mathscr{B}_1)-\psi(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{121}$$

Taking $\mathscr{B}_1$=$\mathscr{B}_3$=e_i in (121) and summing over i, $1\leq i\leq 4$, we get $\psi(\mathscr{B}_2)$=0,$\forall$ $\mathscr{B}_2$ which it is not possible on Ω^4. Thus we assert the result.

Theorem 28. *There does not exist a weakly Ricci-symmetric quasi-conformally flat (KST)$_4$ with non-zero scalar curvature tensor.*

Finally, Taking cyclic sum of (14), we have

$$\begin{aligned}&(\nabla_{\mathscr{B}_1}\mathscr{S}^\dagger)(\mathscr{B}_2,\mathscr{B}_3)+(\nabla_{\mathscr{B}_2}\mathscr{S}^\dagger)(\mathscr{B}_3,\mathscr{B}_1)+(\nabla_{\mathscr{B}_3}\mathscr{S}^\dagger)(\mathscr{B}_1,\mathscr{B}_2)\\&\quad=4[\mu^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3)+\mu^\star(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3)+\mu^\star(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1)].\end{aligned} \tag{122}$$

In particular, if Ω^4 admits cyclic Ricci tensor then (122) reduces

$$\mu^\star(\mathscr{B}_1)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_3)+\mu^\star(\mathscr{B}_2)\mathscr{S}^\dagger(\mathscr{B}_1,\mathscr{B}_3)+\mu^\star(\mathscr{B}_3)\mathscr{S}^\dagger(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{123}$$

Using (38) in (123), we get

$$\frac{\kappa}{4}[\mu^\star(\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_3)+\mu^\star(\mathscr{B}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\mu^\star(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)]=0, \tag{124}$$

which implies that either κ=0, or

$$\mu^\star(\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_3)+\mu^\star(\mathscr{B}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\mu^\star(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{125}$$

If $\kappa\neq 0$, then (125) holds. Now replacing $\mathscr{B}_2$ and $\mathscr{B}_3$ by $\overline{\mathscr{B}}_2$ and $\overline{\mathscr{B}}_3$ in (125), we obtain

$$\mu^\star(\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_3)+\mu^\star(\overline{\mathscr{B}}_2)g(\mathscr{B}_1,\overline{\mathscr{B}}_3)+\mu^\star(\overline{\mathscr{B}}_3)g(\overline{\mathscr{B}}_2,\mathscr{B}_1)=0. \tag{126}$$

In view of (125) and (126), we have

$$\mu^\star(\overline{\mathscr{B}}_2)g(\mathscr{B}_1,\overline{\mathscr{B}}_3)-\mu^\star(\mathscr{B}_2)g(\mathscr{B}_1,\mathscr{B}_3)+\mu^\star(\overline{\mathscr{B}}_3)g(\overline{\mathscr{B}}_2,\mathscr{B}_1)-\mu^\star(\mathscr{B}_3)g(\mathscr{B}_2,\mathscr{B}_1)=0. \tag{127}$$

For fix, $\mathscr{B}_1$=$\mathscr{B}_3$=e_i in (127) and summing over i, $1\leq i\leq 4$, we get $\mu^\star(\mathscr{B}_2)$=0,$\forall$ $\mathscr{B}_2$. This leads the result.

Theorem 29. *There does not exist a special weakly Ricci-symmetric quasi-conformally flat (KST)$_4$ having non-zero scalar curvature tensor if it admits a cyclic Ricci tensor.*

Corollary 5. *A special weakly Ricci-symmetric quasi-conformally flat (KST)$_4$ can not be an Einstein manifold if the 1-form $\mu^\star\neq 0$.*

8 Example

Let $\Omega = \{(x,y,z,t) \in \Re^4 : t \neq 0\}$, where (x,y,z,t) are the standard coordinates of $\Re^4$. Let e_1, e_2, e_3 and e_4 be a set of linearly independent vector fields of (Ω, g) defined as

$$e_1 = e^{x-at}\frac{\partial}{\partial x}, \quad e_2 = e^{y-at}\frac{\partial}{\partial y}, \quad e_3 = e^{z-at}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t},$$

where $a \neq 0$. We define the Riemannian metric g on Ω^4 by:

$$g^\star_{ij} = g^\star(e_i, e_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ -1 & \text{if} \quad i = j = 4 \\ 1 & \text{otherwise} \end{cases}.$$

Let η be the 1-form coupled to the metric as

$$\eta(\mathscr{B}_1) = g(\mathscr{B}_1, e_4)$$

for any $\mathscr{B}_1 \in \Gamma(\mathscr{T}\Omega)$. Also, let the $(1,1)$-tensor field ϕ is defined by

$$\phi(e_1) = e_1, \qquad \phi(e_2) = e_2, \qquad \phi(e_3) = e_3, \qquad \phi(e_4) = 0,$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, by using the linearity of ϕ and g, we have

$$[e_1, e_4] = ae_1, \;\; [e_2, e_4] = ae_2, \;\; [e_3, e_4] = ae_3.$$

Also for e_4=ξ, the Koszul's formula for the Lorentzian metric g gives

$$\begin{array}{llll} \nabla_{e_1}e_1 = a\,e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = ae_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = a\,e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = ae_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = a\,e_4, & \nabla_{e_3}e_4 = ae_3, \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0. \end{array}$$

Thus we conclude the structure (ϕ, ξ, η, g) is a Lorentzian structure on Ω. Consequently, $\Omega^4(\phi, \xi, \eta, g)$ is a Lorentzian manifold (or four dimensional spacetime.)

Using above relations, the existing components of the curvature tensor:

$$\begin{array}{lll} \mathscr{R}^\dagger(e_1,e_2)e_1 = -a^2e_2, & \mathscr{R}^\dagger(e_1,e_3)e_1 = -a^2e_3, & \mathscr{R}^\dagger(e_1,e_4)e_1 = -a^2e_4, \\ \mathscr{R}^\dagger(e_1,e_2)e_2 = a^2e_1, & \mathscr{R}^\dagger(e_2,e_3)e_2 = -a^2e_3, & \mathscr{R}^\dagger(e_2,e_4)e_2 = -a^2e_4, \\ \mathscr{R}^\dagger(e_1,e_3)e_3 = a^2e_1, & \mathscr{R}^\dagger(e_2,e_3)e_3 = a^2e_2, & \mathscr{R}^\dagger(e_3,e_4)e_3 = -a^2e_4, \\ \mathscr{R}^\dagger(e_1,e_4)e_4 = -a^2e_1, & \mathscr{R}^\dagger(e_2,e_4)e_4 = -a^2e_2, & \mathscr{R}^\dagger(e_3,e_4)e_4 = -a^2e_3. \end{array}$$

Also, the Ricci tensor $\mathscr{S}^\dagger$ of (Ω^4, g) is defined as $\mathscr{R}^\dagger(\mathscr{B}_1, \mathscr{B}_2) = \sum_{i=1}^4 \varepsilon_i\, g(\mathscr{R}^\dagger(e_i, \mathscr{B}_1)\mathscr{B}_2, e_i)$, where $\varepsilon_i = g^\star(e_i, e_i)$, i, j=1, 2, 3, 4. So we obtain

$$\mathscr{S}^\dagger(e_i, e_j) = \begin{bmatrix} 3a^2 & 0 & 0 & 0 \\ 0 & 3a^2 & 0 & 0 \\ 0 & 0 & 3a^2 & 0 \\ 0 & 0 & 0 & -3a^2 \end{bmatrix}.$$

Therefore, $\kappa = 6a^2$. This leads to the spacetime is Ricci semisymmetric.

Let $\{e_1, e_2, e_3\}$ be a basis of the tangent space at any point. For any vector $\mathscr{B}_1, \mathscr{B}_2 \in \chi(\Omega^4)$, we have

$$\mathscr{B}_1 = a_1e_1 + b_1e_2 + c_1e_3 + d_1e_4, \;\; \mathscr{B}_2 = a_2e_1 + b_2e_2 + c_2e_3 + d_2e_4,$$

where $a_i, b_i, c_i, d_i \in \Re\backslash\{0\}$ such that $a_1a_2 + b_1b_2 + c_1c_2 = 0, \;\; d_1d_2 \neq 0$ for all $i = 1,2,3,4$. Thus

$g(\mathscr{B}_1, \mathscr{B}_2)$=$\mathrm{a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2}$, and $\mathrm{Ric}(X,Y) = -2\big(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2\big)$.

So we conclude that $\mathscr{S}^\dagger(\mathscr{B}_1, \mathscr{B}_2)$=$-2g(\mathscr{B}_1, \mathscr{B}_2)$, therefore we notice that Ω is an Einstein manifold.

9 Conclusions

In general theory of relativity, the matter of content of the spacetime is described by choosing the suitable form of energy momentum tensor $\mathscr{T}$. Since the matter content of the universe is considered to working like a perfect fluid such as dust fluid and viscous fluid in the standard cosmological models as a connected 4-dimensional Lorentzian manifold. In this framework Einstein's equation play the fundamental role to construct the cosmological model.

String theory often involves compactification of extra spatial dimensions to describe a four-dimensional universe. η-Einstein solitons can be relevant to the geometry of these compact spaces, particularly Calabi-Yau spaces, which are important for constructing realistic string theory scenarios. In string theory, the vibrational modes of strings in a particular compact space determine the types of particles observed. The presence of η-Einstein solitons might influence the spectrum of these vibrational states, potentially impacting the types of particles (like gauge bosons, quarks, leptons, etc.).

Moreover, η-Einstein solitons can be used to construct solutions to modified gravity theories, where Einstein's field equations are generalized, often to address issues like dark energy or cosmic acceleration. An η-Einstein solitons, which generalize standard Einstein solitons by introducing a potential function [12], we study a type soliton is called an η-Einstein soliton on Kählerian spacetime (briefly, $(KST)_4$) coupled with quasi-conformal curvature tensor. The study of such a new types of solitons is of high interest for researchers from different fields due to its wide applications in general relativity, cosmology, quantum field theory, string theory, thermodynamics, mathematical physics, etc. That is why, we depict some geometrical and physical properties of an η-Einstein soliton on Kählerian spacetimes and deduce that the significance of such soliton depend on energy density, isotropic pressure,the cosmological constant and gravitational constant.

Declarations

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