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# $\mathbb{M}$-Projective Curvature Tensor and Ricci-Bourguignon Soliton on Lorentzian $\beta$-Kenmotsu Manifold

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**Abstract:** The prime object of the present article is to explore the characteristics of a Lorentzian $\beta$-$\mathscr{K}$enmotsu manifold ($\mathscr{M}_\beta$) admitting an $\mathbb{M}$-projective curvature tensor. We derived characteristic theorems for $\mathscr{M}_\beta$ satisfying $\mathbb{M}$-projectively $\varphi$-recurrent and pseudo-symmetric conditions. $\varphi-\mathbb{M}$-projectively flat $\mathscr{M}_\beta$ is the subject of our next analysis. Additionally, we deal with the $\mathscr{M}_\beta$ satisfying $\mathbb{M}$-projectively Ricci pseudo-symmetric condition. Next, we describe the $\mathscr{M}_\beta$ satisfying $R(\zeta,\mathscr{B}_3)\cdot\mathbb{M}=0$ condition. Further, we discuss cyclic parallel Ricci tensor, $\mathscr{V}^\flat$-parallel Ricci tensor and Ricci-Bourguignon soliton in $\mathscr{M}_\beta$. Finally, we have presented an example of an $\mathscr{M}_\beta$.

**Keywords:** Lorentzian $\beta$-Kenmotsu manifold ($\mathscr{M}_\beta$), $\mathbb{M}$-projective curvature tensor, generalized $\mathscr{V}^\flat$-Einstein manifold, Ricci pseudo-symmetric, and Ricci-Bourguignon soliton (*RBS*).

**2010 Mathematics Subject Classification**. 53C05; 53C15; 53C20; 53C25; 53D15; 53D10.

## 1 Introduction

If a smooth manifold possesses a Lorentzian metric $g$ and a symmetric, $(0,2)$ tensor field with an index of 1, it is referred to as a Lorentzian manifold. Dutch physicist Hendrik Lorentz introduced this concept, which was independently explored by Rosca and Mihai [16]. Since then, numerous geometers have been used to investigate Lorentzian manifolds [26], uncovering several important properties. Our current study focuses on a specific type of manifold known as Lorentzian $\beta$-$\mathscr{K}$enmotsu manifold ($\mathscr{M}_\beta$). First, we present an overview of the development of this manifold.

Within the field of differential geometry, the $\mathbb{M}$-projective curvature tensor stands as a significant tensor, comparable in importance to the conformal curvature tensor. This particular tensor acts as a bridge, connecting the $\mathscr{H}$-projective curvature tensor with three other tensors: the conharmonic, conformal, and concircular curvature tensors.

**Definition 1.1** *Consider $(\mathscr{M}^n,g)$, a smooth manifold of dimension n with $\mathbb{C}^\infty$ classification, endowed with a metric tensor g and a Riemannian connection $\nabla$. In 1971, Pokhariyal and Mishra [21] proposed the $\mathbb{M}$-projective curvature tensor for the Riemannian manifold $(\mathscr{M}^n,g)$, which is expressed as follows:*

$$\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3=R(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3-\frac{1}{2(n-1)}[S(\mathscr{B}_2,\mathscr{B}_3)\mathscr{B}_1-S(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_2+g(\mathscr{B}_2,\mathscr{B}_3)Q\mathscr{B}_1-g(\mathscr{B}_1,\mathscr{B}_3)Q\mathscr{B}_2], \tag{1.1}$$

*for all smooth vector fields $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3\in\chi(\mathscr{M}^n)$, here R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator defined by $S(\mathscr{B}_1,\mathscr{B}_2)=g(Q\mathscr{B}_1,\mathscr{B}_2)$.*

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In 1981, Jean-Pierre Bourguignon [3] proposed the Ricci-Bourguignon (*RB*) flow, building upon unpublished insights from Lichnerowicz and a paper by Aubin [1]. The *RB* flow can be formally defined as [10]

**Definition 1.2** *On a Riemannian manifold $(\mathscr{M}^n, g)$ of dimension n, a set of metrics $g(t)$ is considered to develop according to the RB flow if $g(t)$ conforms to the following developmental equation:*

$$\frac{\partial}{\partial t} g = -2(S - \psi r g), \tag{1.2}$$

*where S represents the Ricci tensor of type $(0,2)$, r denotes the scalar curvature, and $\psi \in \mathbb{R}$ is a constant.*

The previous definition shows that when $\psi = 0$ in (1.2), the flow transforms into Ricci flow. According to [10], various tensors emerge for different values of $\psi$: the Einstein tensor at $\psi = \frac{1}{2}$, the traceless Ricci tensor at $\psi = \frac{1}{n}$, the Schouten tensor at $\psi = \frac{1}{2(n-1)}$, and the Ricci tensor at $\psi = 0$.

In recent studies, Ho [12] explored this flow on locally homogeneous 3-manifolds. Additional research on this subject has been conducted in relation to the product of an Anti de Sitter space with a sphere [2]. Shubham Dwivedi [10] introduced a new extended form of soliton, termed the Ricci-Bourguignon soliton (*RBS* for short), which functions as interpolating soliton connecting Ricci and Yamabe solitons. Moreover, numerous researchers, including Dey [9] and Dey et al. [8], have investigated the Ricci-Bourguignon soliton.

**Definition 1.3** *A Riemannian manifold $(\mathscr{M}^n, g)$ is considered a RB soliton when there exists a vector field $\mathscr{B}_2$ on $\mathscr{M}$ that fulfills the following equation:*

$$\pounds_{\mathscr{B}_2} g + 2S = 2\Lambda g + 2\psi r g, \tag{1.3}$$

*where $\pounds_{\mathscr{B}_2} g$ represents the Lie derivative along the vector field $\mathscr{B}_2$, and $\Lambda \in \mathbb{R}$ is a real constant.*

The classification of the system depends on the value of $\Lambda$: it is considered shrinking when $\Lambda > 0$, steady when $\Lambda = 0$, and expanding when $\Lambda < 0$. In cases where $\mathscr{B}_2 = \nabla f$, with $f$ being a smooth function from $\mathscr{M}$ to $\mathbb{R}$, the system $(\mathscr{M}, g)$ is referred to as a gradient *RBS* (*GRBS* for short). Furthermore, if $\Lambda$ in (1.3) is a smooth function from $\mathscr{M}$ to $\mathbb{R}$, then $(\mathscr{M}, g)$ is termed an almost *RBS* (*ARBS* for short). An *ARBS* becomes a gradient almost *RBS* (*GARBS* for short) when $\mathscr{B}_2 = \nabla f$ for a smooth function $f$ on $\mathscr{M}$. The *GARBS* is categorized as shrinking, steady, or expanding based on whether $\Lambda > 0$, $\Lambda = 0$, or $\Lambda < 0$, respectively. It's worth noting that if $\mathscr{B}_2$ is a Killing vector field, an *ARBS* reduces to a *RBS*, as this condition necessitates $\Lambda$ to be constant.

In [21], the authors explored the relativistic implications of the $\mathbb{M}$-projective curvature tensor. Several authors have explored the $\mathbb{M}$-projective curvature tensor, with notable contributions from Chaubey and Ojha [4], De and Mallick [5], as well as Singh [29]. In recent studies, the authors (see,[17], [24], [19], [18], [25], [30], [32], [33], [20]) have conducted comprehensive investigations into the characteristics of the $\mathbb{M}$-projective curvature tensor across various types of manifolds. Motivated from all these studies, we explore some distinctive attributes of $\mathbb{M}$-projective curvature tensor in $\mathscr{M}_\beta$.

**The layout of the article is as follows:** Section 2 provides a concise introduction to the $\mathscr{M}_\beta$. In section 3, we show that an $\mathbb{M}$-projective $\varphi$-recurrent $\mathscr{M}_\beta$ with invariant scalar curvature $r$ is equivalent to an $\mathbb{M}$-projectively $\varphi$-symmetric manifold. The $\mathbb{M}$-projectively pseudo-symmetric $\mathscr{M}_\beta$ is examined in section 4. Section 5 explores the $\varphi - \mathbb{M}$-projectively flat $\mathscr{M}_\beta$. Subsequently, section 6 analyzes the $\mathscr{M}_\beta$ that satisfies the $\mathbb{M}$-projectively Ricci pseudo-symmetric condition. Section 7 investigates the $\mathscr{M}_\beta$ that fulfills the $R(\zeta, \mathscr{B}_3) \cdot \mathbb{M} = 0$ condition. The focus of section 8 is the cyclic parallel Ricci tensor. The next two sections examine the $\mathscr{V}^\flat$-parallel Ricci tensor and Ricci-Bourguignon soliton. To validate our findings, the concluding portion showcases an illustration of a five-dimensional $\mathscr{M}_\beta$.

## 2 Preliminaries

A smooth manifold of dimension $(2n+1)$ is characterized as Lorentzian $\beta$-Kenmotsu manifold ($\mathscr{M}_\beta$) if it possesses a $(1,1)$-tensor field $\varphi$, a contravariant vector field $\zeta$, a covariant vector field $\mathscr{V}^\flat$ and a Lorentzian metric $g$ which satisfies

$$\varphi^2 \mathscr{B}_1 = \mathscr{B}_1 + \mathscr{V}^\flat(\mathscr{B}_1)\zeta, \qquad g(\mathscr{B}_1, \zeta) = \mathscr{V}^\flat(\mathscr{B}_1), \tag{2.1}$$

$$\mathscr{V}^\flat(\zeta) = -1, \qquad \varphi\zeta = 0, \qquad \mathscr{V}^\flat(\varphi\mathscr{B}_1) = 0, \tag{2.2}$$

$$g(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2) = g(\mathscr{B}_1, \mathscr{B}_2) + \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2), \tag{2.3}$$

$$g(\varphi\mathscr{B}_1, \mathscr{B}_2) = g(\mathscr{B}_1, \varphi\mathscr{B}_2), \tag{2.4}$$


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for all $\mathscr{B}_1, \mathscr{B}_2 \in \chi(\mathscr{M}_\beta)$.
A $\mathscr{M}_\beta$ also fulfills

$$\nabla_{\mathscr{B}_1} \zeta = \beta[\mathscr{B}_1 - \mathscr{V}^\flat(\mathscr{B}_1)\zeta], \tag{2.5}$$

$$(\nabla_{\mathscr{B}_1} \mathscr{V}^\flat)(\mathscr{B}_2) = \beta[g(\mathscr{B}_1, \mathscr{B}_2) - \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)], \tag{2.6}$$

where $\nabla$ represents the covariant differentiation operator.
Moreover, on an $\mathscr{M}_\beta$, the following relations hold [23]:

$$\mathscr{V}^\flat(R(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_3) = \beta^2[g(\mathscr{B}_1, \mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_2) - g(\mathscr{B}_2, \mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_1)], \tag{2.7}$$

$$R(\zeta, \mathscr{B}_1)\mathscr{B}_2 = \beta^2(\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{B}_1 - g(\mathscr{B}_1, \mathscr{B}_2)\zeta), \tag{2.8}$$

$$R(\mathscr{B}_1, \mathscr{B}_2)\zeta = \beta^2(\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{B}_2 - \mathscr{V}^\flat(\mathscr{B}_2)\mathscr{B}_1), \tag{2.9}$$

$$S(\mathscr{B}_1, \zeta) = -(n-1)\beta^2\mathscr{V}^\flat(\mathscr{B}_1), \tag{2.10}$$

$$Q\zeta = -(n-1)\beta^2\zeta, \tag{2.11}$$

$$S(\zeta, \zeta) = (n-1)\beta^2, \tag{2.12}$$

for any $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3 \in \chi(\mathscr{M}_\beta)$, where $R$ and $S$ denote the curvature tensor and Ricci tensor on $\mathscr{M}_\beta$ respectively.

**Definition 2.1** *An $\mathscr{M}_\beta$ is named a generalized $\mathscr{V}^\flat$-Einstein manifold if [31]*

$$S(\mathscr{B}_1, \mathscr{B}_2) = \gamma_1 g(\mathscr{B}_1, \mathscr{B}_2) + \gamma_2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2) + \gamma_3\Phi(\mathscr{B}_1, \mathscr{B}_2), \tag{2.13}$$

*here $S$ represents the Ricci tensor of type $(0,2)$; $\gamma_1, \gamma_2, \gamma_3$ are smooth functions and $\Phi(\mathscr{B}_1, \mathscr{B}_2) = g(\varphi\mathscr{B}_1, \mathscr{B}_2)$, is the fundamental 2-form of $\mathscr{M}_\beta$. If $\gamma_3 = 0$ (resp.,$\gamma_2 = \gamma_3 = 0$), then an $\mathscr{M}_\beta$ reduces to a $\mathscr{V}^\flat$-Einstein (resp., Einstein) manifold.*

In particular, if $\gamma_3 = 0$, equation (2.13) reduces to

$$S(\mathscr{B}_1, \mathscr{B}_2) = \gamma_1 g(\mathscr{B}_1, \mathscr{B}_2) + \gamma_2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2). \tag{2.14}$$

In view of (2.1) and (2.14), we infer

$$Q\mathscr{B}_1 = \gamma_1\mathscr{B}_1 + \gamma_2\mathscr{V}^\flat(\mathscr{B}_1)\zeta. \tag{2.15}$$

Substituting $\mathscr{B}_1 = \mathscr{B}_2 = e_i$ in (2.14), and summing for $1 \le i \le n$, we achieve

$$r = n\gamma_1 - \gamma_2. \tag{2.16}$$

Now, putting $\mathscr{B}_1 = \mathscr{B}_2 = \zeta$ in (2.14) and using (2.1), (2.2) and (2.12), we obtain

$$\gamma_2 - \gamma_1 = (n-1)\beta^2. \tag{2.17}$$

From the condition (2.16) and (2.17), we have

$$\gamma_1 = \frac{r}{(n-1)} + \beta^2 \quad \text{and} \quad \gamma_2 = \frac{r}{(n-1)} + n\beta^2, \tag{2.18}$$

where $r$ is the scalar curvature.
Thus, (2.14) and (2.18) together gives

$$S(\mathscr{B}_1, \mathscr{B}_2) = \left[\frac{r}{(n-1)} + \beta^2\right] g(\mathscr{B}_1, \mathscr{B}_2) + \left[\frac{r}{(n-1)} + n\beta^2\right] \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2). \tag{2.19}$$

## 3 $\mathbb{M}$-Projectively $\varphi$-recurrent $\mathscr{M}_\beta$

**Definition 3.1** *An $\mathscr{M}_\beta$ is considered $\mathbb{M}$-projectively $\varphi$-recurrent if for any non-zero one form $\mathscr{A}$, it fulfills the following condition [28]:*

$$\varphi^2((\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_3) = \mathscr{A}(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_3, \tag{3.1}$$

*where $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4 \in \chi(\mathscr{M}_\beta)$.*

In view of (2.1) in (3.1),

$$(\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3+\mathscr{V}^\flat((\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3)\zeta=\mathscr{A}(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3, \tag{3.2}$$

which yields

$$g((\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3,\mathscr{Z})+\mathscr{V}^\flat((\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{Z})=\mathscr{A}(\mathscr{B}_4)g(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3,\mathscr{Z}). \tag{3.3}$$

Incorporating equation (1.1) into the aforementioned expression and subsequently performing contraction with respect to $\mathscr{B}_1$ and $\mathscr{Z}$ yields

$$\begin{aligned}(n+1)(\nabla_{\mathscr{B}_4}S)(\mathscr{B}_2,\mathscr{B}_3)-g(\mathscr{B}_2,\mathscr{B}_3)dr(\mathscr{B}_4)+2(n-1)\mathscr{V}^\flat((\nabla_{\mathscr{B}_4}R)(\zeta,\mathscr{B}_2)\mathscr{B}_3)+\mathscr{V}^\flat(\mathscr{B}_2)(\nabla_{\mathscr{B}_4}S)(\zeta,\mathscr{B}_3)\\ -g(\mathscr{B}_2,\mathscr{B}_3)(\nabla_{\mathscr{B}_4}S)(\zeta,\zeta)+\mathscr{V}^\flat(\mathscr{B}_3)(\nabla_{\mathscr{B}_4}S)(\mathscr{B}_2,\zeta)=\mathscr{A}(\mathscr{B}_4)[nS(\mathscr{B}_2,\mathscr{B}_3)-rg(\mathscr{B}_2,\mathscr{B}_3)].\end{aligned} \tag{3.4}$$

In light of the equations (2.10), (2.12), (2.5) and (2.6) in (3.4), we achieve

$$\begin{aligned}(n+1)(\nabla_{\mathscr{B}_4}S)(\mathscr{B}_2,\mathscr{B}_3)-g(\mathscr{B}_2,\mathscr{B}_3)dr(\mathscr{B}_4)+2(n-1)\beta^3[g(\mathscr{B}_4,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_2)-\mathscr{V}^\flat(\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_2)\\ -4g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_4)]+\beta\left\{(n-1)\beta^2\mathscr{V}^\flat(\nabla_{\mathscr{B}_4}\mathscr{B}_3)-S(\mathscr{B}_4,\mathscr{B}_3)\right\}\mathscr{V}^\flat(\mathscr{B}_2)\\ +\beta\left\{(n-1)\beta^2\mathscr{V}^\flat(\nabla_{\mathscr{B}_4}\mathscr{B}_2)-S(\mathscr{B}_4,\mathscr{B}_2)\right\}\mathscr{V}^\flat(\mathscr{B}_3)=\mathscr{A}(\mathscr{B}_4)[nS(\mathscr{B}_2,\mathscr{B}_3)-rg(\mathscr{B}_2,\mathscr{B}_3)].\end{aligned} \tag{3.5}$$

Putting $\mathscr{B}_2=\mathscr{B}_3=\zeta$ in (3.5) and using (2.1), we obtain

$$\mathscr{A}(\mathscr{B}_4)=\frac{dr(\mathscr{B}_4)+2(n-1)(2n+1)\beta^3\mathscr{V}^\flat(\mathscr{B}_4)}{r+n(n-1)\beta^2}. \tag{3.6}$$

Thus, we arrive at the following conclusion:

**Theorem 3.1** *In an $n$-dimensional $\mathbb{M}$-projectively $\varphi$-recurrent $\mathscr{M}_\beta$, the non-zero 1-form $\mathscr{A}$ is given by (3.6), provided that $r\neq -n(n-1)\beta^2$.*

Additionally, assuming the scalar curvature of an $\mathbb{M}$-projectively $\varphi$-recurrent $\mathscr{M}_\beta$ remains constant, we can conclude that $dr(\mathscr{B}_4)=0$. Consequently, the equation (3.6) leads to

$$\mathscr{A}(\mathscr{B}_4)=\frac{2(n-1)(2n+1)\beta^3}{r+n(n-1)\beta^2}\mathscr{V}^\flat(\mathscr{B}_4). \tag{3.7}$$

Making use of (3.7) in (3.1), we obtain

$$\varphi^2((\nabla_{\mathscr{B}_4}\mathbb{M})(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3)\neq 0. \tag{3.8}$$

Hence, we can state the following result.

**Theorem 3.2** *The $\mathbb{M}$-projectively $\varphi$-recurrent $\mathscr{M}_\beta$ possessing a constant scalar curvature $r$ never reduces to an $\mathbb{M}$-projectively $\varphi$-symmetric manifold [28].*

## 4 $\mathbb{M}$-Projectively Pseudo-symmetric $\mathscr{M}_\beta$

**Definition 4.1** *An $\mathscr{M}_\beta$ is termed as $\mathbb{M}$-projectively pseudo-symmetric if its curvature tensor satisfies [28]*

$$(R(\mathscr{B}_1,\mathscr{B}_2)\cdot\mathbb{M})(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4=\mathscr{L}_{\mathbb{M}}[((\mathscr{B}_1\wedge\mathscr{B}_2)\cdot\mathbb{M})(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4], \tag{4.1}$$

*on $\mathscr{B}_{1\mathbb{M}}=\left\{x\in\mathscr{M}_\beta:\mathbb{M}\neq 0 \text{ at } x\right\}$, where $\mathscr{L}_{\mathbb{M}}$ is any function on $\mathscr{B}_{1\mathbb{M}}$.*

In view of equation (4.1), we can express it as follows:

$$\begin{aligned}(R(\mathscr{B}_1,\zeta)\cdot\mathbb{M})(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4=\mathscr{L}_{\mathbb{M}}[((\mathscr{B}_1\wedge\zeta)(\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4))-\mathbb{M}((\mathscr{B}_1\wedge\zeta)\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4\\ -\mathbb{M}(\mathscr{B}_3,(\mathscr{B}_1\wedge\zeta)\mathscr{Z})\mathscr{B}_4-\mathbb{M}(\mathscr{B}_3,\mathscr{Z})(\mathscr{B}_1\wedge\zeta)\mathscr{B}_4].\end{aligned} \tag{4.2}$$


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From the left-hand side of (4.2), we can readily obtain

$$\beta^2[\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\mathscr{B}_1 - g(\mathscr{B}_1,\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\zeta - \mathscr{V}^\flat(\mathscr{B}_3)\mathbb{M}(\mathscr{B}_1,\mathscr{Z})\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{B}_3)\mathbb{M}(\zeta,\mathscr{Z})\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\mathscr{B}_1)\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_1 + g(\mathscr{B}_1,\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\zeta]. \tag{4.3}$$

In a similar manner, the right-hand side of (4.2) produces

$$\mathscr{L}_{\mathbb{M}}[\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\mathscr{B}_1 - g(\mathscr{B}_1,\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\zeta - \mathscr{V}^\flat(\mathscr{B}_3)\mathbb{M}(\mathscr{B}_1,\mathscr{Z})\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{B}_3)\mathbb{M}(\zeta,\mathscr{Z})\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\mathscr{B}_1)\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_1 + g(\mathscr{B}_1,\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\zeta]. \tag{4.4}$$

Now, the preceding equation (4.2) can take the form

$$(\mathscr{L}_{\mathbb{M}} - \beta^2)[\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\mathscr{B}_1 - g(\mathscr{B}_1,\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_4)\zeta - \mathscr{V}^\flat(\mathscr{B}_3)\mathbb{M}(\mathscr{B}_1,\mathscr{Z})\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{B}_3)\mathbb{M}(\zeta,\mathscr{Z})\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\mathscr{B}_1)\mathscr{B}_4 + g(\mathscr{B}_1,\mathscr{Z})\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\mathscr{B}_1 + g(\mathscr{B}_1,\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\mathscr{Z})\zeta] = 0. \tag{4.5}$$

Substituting $\mathscr{Z} = \zeta$ into equation (4.5) and utilizing equation (1.1), we find that either $\mathscr{L}_{\mathbb{M}} = \beta^2$ or

$$\mathbb{M}(\mathscr{B}_3,\mathscr{B}_1)\mathscr{B}_4 = -\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4)\mathscr{B}_1 + g(\mathscr{B}_1,\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4)\zeta + \mathscr{V}^\flat(\mathscr{B}_3)\mathbb{M}(\mathscr{B}_1,\zeta)\mathscr{B}_4 - \mathscr{V}^\flat(\mathscr{B}_1)\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_4 + \mathscr{V}^\flat(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\zeta)\mathscr{B}_1 - g(\mathscr{B}_1,\mathscr{B}_4)\mathbb{M}(\mathscr{B}_3,\zeta)\zeta. \tag{4.6}$$

By utilizing equations (1.1), (2.2), (2.3), (2.7), and (2.10), equation (4.6) can be rewritten as

$$\mathbb{M}(\mathscr{B}_3,\mathscr{B}_1)\mathscr{B}_4 = \frac{\beta^2}{2}\Big[g(\mathscr{B}_3,\mathscr{B}_4)\mathscr{B}_1 + \mathscr{V}^\flat(\mathscr{B}_3)g(\mathscr{B}_1,\mathscr{B}_4)\zeta + \mathscr{V}^\flat(\mathscr{B}_4)g(\mathscr{B}_3,\mathscr{B}_1)\zeta\Big] + \frac{1}{2(n-1)}\Big[S(\mathscr{B}_3,\mathscr{B}_4)\mathscr{B}_1 + \mathscr{V}^\flat(\mathscr{B}_3)S(\mathscr{B}_1,\mathscr{B}_4)\zeta + \mathscr{V}^\flat(\mathscr{B}_4)S(\mathscr{B}_3,\mathscr{B}_1)\zeta\Big]. \tag{4.7}$$

Contracting (4.7) with respect to $\mathscr{B}_3$ yields

$$S(\mathscr{B}_1,\mathscr{B}_4) = -\left[\frac{3(n^2-1)\beta^2 + r}{(2n-3)}\right]g(\mathscr{B}_1,\mathscr{B}_4). \tag{4.8}$$

Upon contracting (4.8) once more, we infer

$$r = -n(n+1)\beta^2. \tag{4.9}$$

Hence, we state an important observation:

**Theorem 4.1** *In an n-dimensional $\mathbb{M}$-projectively pseudo-symmetric $\mathscr{M}_\beta$, either $\mathscr{L}_{\mathbb{M}} = \beta^2$ or the manifold reduces to an Einstein manifold having constant scalar curvature $-n(n+1)\beta^2$.*

## 5 $\varphi - \mathbb{M}$-Projectively flat $\mathscr{M}_\beta$

**Definition 5.1** *An $\mathscr{M}_\beta$ is said to be $\varphi - \mathbb{M}$-projectively flat if [25]*

$$\varphi^2(\mathbb{M}(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)\varphi\mathscr{B}_3) = 0, \tag{5.1}$$

*for all $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3 \in \chi(\mathscr{M}_\beta)$.*

It's apparent that $\varphi^2(\mathbb{M}(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)\varphi\mathscr{B}_3) = 0$ holds if

$$g(\mathbb{M}(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)\varphi\mathscr{B}_3,\varphi\mathscr{B}_4) = 0, \tag{5.2}$$

for all $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4 \in \chi(\mathscr{M}_\beta)$.
Making use of (1.1), we have

$$g(R(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)\varphi\mathscr{B}_3,\varphi\mathscr{B}_4) = \frac{1}{2(n-1)}[S(\varphi\mathscr{B}_2,\varphi\mathscr{B}_3)g(\varphi\mathscr{B}_1,\varphi\mathscr{B}_4) - S(\varphi\mathscr{B}_1,\varphi\mathscr{B}_3)g(\varphi\mathscr{B}_2,\varphi\mathscr{B}_4) + g(\varphi\mathscr{B}_2,\varphi\mathscr{B}_3)S(\varphi\mathscr{B}_1,\varphi\mathscr{B}_4) - g(\varphi\mathscr{B}_1,\varphi\mathscr{B}_3)S(\varphi\mathscr{B}_2,\varphi\mathscr{B}_4)]. \tag{5.3}$$

Consider a local orthonormal basis of vector fields on $\mathscr{M}_\beta$, represented as $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \zeta\}$ at any given point. By setting $\mathscr{B}_1 = \mathscr{B}_4 = e_i$ in (5.3) and summing over $i$, where $1 \leq i \leq (n-1)$, we obtain

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi\mathscr{B}_2)\varphi\mathscr{B}_3, \varphi e_i) = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}[S(\varphi\mathscr{B}_2, \varphi\mathscr{B}_3)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi\mathscr{B}_3)g(\varphi\mathscr{B}_2, \varphi e_i) + g(\varphi\mathscr{B}_2, \varphi\mathscr{B}_3)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi\mathscr{B}_3)S(\varphi\mathscr{B}_2, \varphi e_i)], \tag{5.4}$$

which reduces to

$$S(\varphi\mathscr{B}_2, \varphi\mathscr{B}_3) = \frac{r}{(n+1)}g(\varphi\mathscr{B}_2, \varphi\mathscr{B}_3). \tag{5.5}$$

In the view of (2.3) in (5.5), we have

$$S(\mathscr{B}_2, \mathscr{B}_3) = \frac{r}{(n+1)}g(\mathscr{B}_2, \mathscr{B}_3) + \frac{[r+(n^2-1)\beta^2]}{(n+1)}\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3). \tag{5.6}$$

Taking frame field over $\mathscr{B}_2$ and $\mathscr{B}_3$, we acquire

$$r = -\frac{(n^2-1)}{2}\beta^2. \tag{5.7}$$

As a result, we reach the following determination:

**Theorem 5.1** *An $n$-dimensional $\varphi - \mathbb{M}$-projectively flat $\mathscr{M}_\beta$ is $\mathscr{V}^\flat$-Einstein manifold with constant scalar curvature $-\frac{(n^2-1)}{2}\beta^2$.*

## 6 $\mathscr{M}_\beta$ Satisfying $\mathbb{M}$-Projectively Ricci Pseudo-symmetric Condition

Deszcz proposed the concept of *Ricci pseudo symmetric* manifold ([6], [7]). A geometric interpretation for these manifolds, specifically in the Riemannian case, is presented in [14].

**Definition 6.1** *A Riemannian manifold $(\mathscr{M}^n, g)$ is termed as Ricci pseudo-symmetric ([6],[7],[13],[27]) if the Tachibana tensor $\widetilde{Q}(g,S)$ and the tensor $R\cdot S$ are linearly dependent, where*

$$(R(\mathscr{B}_1, \mathscr{B}_2)\cdot S)(\mathscr{B}_3, \mathscr{B}_4) = -S(R(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_3, \mathscr{B}_4) - S(\mathscr{B}_3, R(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_4), \tag{6.1}$$

$$\mathscr{F}_S\widetilde{Q}(g,S)(\mathscr{B}_3, \mathscr{B}_4; \mathscr{B}_1, \mathscr{B}_2) = -S((\mathscr{B}_1 \wedge_g \mathscr{B}_2)\mathscr{B}_3, \mathscr{B}_4) - S(\mathscr{B}_3, (\mathscr{B}_1 \wedge_g \mathscr{B}_2)\mathscr{B}_4), \tag{6.2}$$

*and*

$$(\mathscr{B}_1 \wedge_g \mathscr{B}_2)\mathscr{B}_3 = g(\mathscr{B}_2, \mathscr{B}_3)\mathscr{B}_1 - g(\mathscr{B}_1, \mathscr{B}_3)\mathscr{B}_2, \tag{6.3}$$

*holds for $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4 \in \chi(\mathscr{M}^n)$, and $R$ denotes the curvature tensor of $\mathscr{M}^n$.*

**Definition 6.2** *An $\mathscr{M}_\beta$ is termed as $\mathbb{M}$-projectively Ricci pseudo-symmetric if it's curvature tensor satisfies [22]*

$$(\mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\cdot S)(\mathscr{B}_3, \mathscr{B}_4) = \mathscr{F}_S\widetilde{Q}(g,S)(\mathscr{B}_3, \mathscr{B}_4; \mathscr{B}_1, \mathscr{B}_2), \tag{6.4}$$

*on $\mathscr{B}_{1S} = \left\{x \in \mathscr{M}_\beta : S \neq \frac{r}{n}g \text{ at } x\right\}$, where $\mathscr{F}_S$ is any function on $\mathscr{B}_{1S}$.*

In view of (6.4), we have

$$S(\mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_3, \mathscr{B}_4) + S(\mathscr{B}_3, \mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_4) = \mathscr{F}_S[g(\mathscr{B}_2, \mathscr{B}_3)S(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \mathscr{B}_3)S(\mathscr{B}_2, \mathscr{B}_4) + g(\mathscr{B}_2, \mathscr{B}_4)S(\mathscr{B}_1, \mathscr{B}_3) - g(\mathscr{B}_1, \mathscr{B}_4)S(\mathscr{B}_2, \mathscr{B}_3)]. \tag{6.5}$$

Plugging $\mathscr{B}_3 = \zeta$ in (6.5), we have

$$S(\mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\zeta, \mathscr{B}_4) + S(\zeta, \mathbb{M}(\mathscr{B}_1, \mathscr{B}_2)\mathscr{B}_4) = \mathscr{F}_S[g(\mathscr{B}_2, \zeta)S(\mathscr{B}_1, \mathscr{B}_4) - g(\mathscr{B}_1, \zeta)S(\mathscr{B}_2, \mathscr{B}_4) + g(\mathscr{B}_2, \mathscr{B}_4)S(\mathscr{B}_1, \zeta) - g(\mathscr{B}_1, \mathscr{B}_4)S(\mathscr{B}_2, \zeta)]. \tag{6.6}$$


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By using (2.10) and (1.1) in (6.6) and on simplification, we obtain

$$\beta^2\Big[\Big\{\mathscr{V}^\flat(\mathscr{B}_1)S(\mathscr{B}_2,\mathscr{B}_4)-\mathscr{V}^\flat(\mathscr{B}_2)S(\mathscr{B}_1,\mathscr{B}_4)\Big\}-(n-1)\beta^2\Big\{g(\mathscr{B}_1,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_2)-g(\mathscr{B}_2,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_1)\Big\}\Big]$$
$$-\frac{\beta^2}{2}\Big[2\mathscr{V}^\flat(\mathscr{B}_1)S(\mathscr{B}_2,\mathscr{B}_4)-2\mathscr{V}^\flat(\mathscr{B}_2)S(\mathscr{B}_1,\mathscr{B}_4)-S(\mathscr{B}_2,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_1)+S(\mathscr{B}_1,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_2)$$
$$+(n-1)\beta^2\Big\{g(\mathscr{B}_2,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_1)-g(\mathscr{B}_1,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_2)\Big\}\Big]=\mathscr{F}_S\Big[\mathscr{V}^\flat(\mathscr{B}_2)S(\mathscr{B}_1,\mathscr{B}_4)-\mathscr{V}^\flat(\mathscr{B}_1)S(\mathscr{B}_2,\mathscr{B}_4)$$
$$-(n-1)\beta^2\Big\{g(\mathscr{B}_2,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_1)-g(\mathscr{B}_1,\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_2)\Big\}\Big]. \tag{6.7}$$

Putting $\mathscr{B}_2=\zeta$ in (6.7) and making use of (2.10) and (2.1), it yields

$$(2\mathscr{F}_S+\beta^2)S(\mathscr{B}_1,\mathscr{B}_4)+(n-1)\beta^4(2\mathscr{F}_S+1)g(\mathscr{B}_1,\mathscr{B}_4)=0, \tag{6.8}$$

$$or,\quad S(\mathscr{B}_1,\mathscr{B}_4)=\frac{-(n-1)\beta^4(2\mathscr{F}_S+1)}{(2\mathscr{F}_S+\beta^2)}g(\mathscr{B}_1,\mathscr{B}_4). \tag{6.9}$$

Taking frame field over $\mathscr{B}_1$ and $\mathscr{B}_4$, we infer

$$r=\frac{-n(n-1)\beta^4(2\mathscr{F}_S+1)}{(2\mathscr{F}_S+\beta^2)}. \tag{6.10}$$

Therefore, we propose the following theorems.

**Theorem 6.1** *If an $\mathscr{M}_\beta$ is satisfying $\mathbb{M}$-projectively Ricci pseudo-symmetric condition then the manifold is an Einstein manifold having constant scalar curvature* $\dfrac{-n(n-1)\beta^4(2\mathscr{F}_S+1)}{(2\mathscr{F}_S+\beta^2)}$.

## 7 $\mathscr{M}_\beta$ Satisfying $R(\zeta,\mathscr{B}_3)\cdot\mathbb{M}=0$ Condition

In the present section, we study $\mathscr{M}_\beta$ fulfilling $R(\zeta,\mathscr{B}_3)\cdot\mathbb{M}=0$. Then, we have

$$R(\zeta,\mathscr{B}_3)\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4-\mathbb{M}(R(\zeta,\mathscr{B}_3)\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4-\mathbb{M}(\mathscr{B}_1,R(\zeta,\mathscr{B}_3)\mathscr{B}_2)\mathscr{B}_4-\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)R(\zeta,\mathscr{B}_3)\mathscr{B}_4=0. \tag{7.1}$$

We fetch the equation (2.8) into (7.1) to achieve

$$\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4)\mathscr{B}_3-g(\mathscr{B}_3,\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4)\zeta-\mathscr{V}^\flat(\mathscr{B}_1)\mathbb{M}(\mathscr{B}_3,\mathscr{B}_2)\mathscr{B}_4+g(\mathscr{B}_3,\mathscr{B}_1)\mathbb{M}(\zeta,\mathscr{B}_2)\mathscr{B}_4$$
$$-\mathscr{V}^\flat(\mathscr{B}_2)\mathbb{M}(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_4+g(\mathscr{B}_3,\mathscr{B}_2)\mathbb{M}(\mathscr{B}_1,\zeta)\mathscr{B}_4-\mathscr{V}^\flat(\mathscr{B}_4)\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3+g(\mathscr{B}_3,\mathscr{B}_4)\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\zeta=0. \tag{7.2}$$

By taking inner product with $\zeta$ in (7.2), we acquire

$$\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_3)+g(\mathscr{B}_3,\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4)-\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_3,\mathscr{B}_2)\mathscr{B}_4)+g(\mathscr{B}_3,\mathscr{B}_1)\mathscr{V}^\flat(\mathbb{M}(\zeta,\mathscr{B}_2)\mathscr{B}_4)$$
$$-\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_3)\mathscr{B}_4)+g(\mathscr{B}_3,\mathscr{B}_2)\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\zeta)\mathscr{B}_4)-\mathscr{V}^\flat(\mathscr{B}_4)\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_3)$$
$$+g(\mathscr{B}_3,\mathscr{B}_4)\mathscr{V}^\flat(\mathbb{M}(\mathscr{B}_1,\mathscr{B}_2)\zeta)=0. \tag{7.3}$$

Utilizing equation (1.1) and simplifying, we arrive at

$$g(\mathscr{B}_3,R(\mathscr{B}_1,\mathscr{B}_2)\mathscr{B}_4)+\frac{\beta^2}{2}[g(\mathscr{B}_3,\mathscr{B}_1)g(\mathscr{B}_2,\mathscr{B}_4)-g(\mathscr{B}_1,\mathscr{B}_4)g(\mathscr{B}_3,\mathscr{B}_2)]-\frac{1}{2(n-1)}[-S(\mathscr{B}_2,\mathscr{B}_4)g(\mathscr{B}_3,\mathscr{B}_1)$$
$$+(n-1)\beta^2g(\mathscr{B}_3,\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_2)-(n-1)\beta^2g(\mathscr{B}_3,\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_4)\mathscr{V}^\flat(\mathscr{B}_1)+g(\mathscr{B}_3,\mathscr{B}_2)S(\mathscr{B}_1,\mathscr{B}_4)$$
$$-S(\mathscr{B}_2,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_4)+S(\mathscr{B}_1,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_4)]=0. \tag{7.4}$$

Consider an orthonormal frame field $\{e_i\}_{i=1}^n$ defined at any point on $\mathscr{M}_\beta$. If we set $\mathscr{B}_1=\mathscr{B}_3=e_i$ in (7.4) and summing over $i$, $1\le i\le n$, we have

$$S(\mathscr{B}_2,\mathscr{B}_4)=-\frac{(n-1)\beta^2}{3}g(\mathscr{B}_2,\mathscr{B}_4)+\frac{[n(n-1)\beta^2+r]}{3(n-1)}\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_4). \tag{7.5}$$

As a result, we reach at following determination:

**Theorem 7.1** *If a $\mathscr{M}_\beta$ fulfills the condition $R(\zeta,\mathscr{B}_3)\cdot\mathbb{M}=0$, then the manifold is an $\mathscr{V}^\flat$-Einstein manifold.*

## 8 Cyclic Parallel Ricci Tensor

Assuming that $\mathscr{M}_\beta$ has a *cyclic parallel* Ricci tensor [11], the Ricci tensor $S$ satisfies

$$(\nabla_{\mathscr{B}_1}S)(\mathscr{B}_2,\mathscr{B}_3)+(\nabla_{\mathscr{B}_2}S)(\mathscr{B}_3,\mathscr{B}_1)+(\nabla_{\mathscr{B}_3}S)(\mathscr{B}_1,\mathscr{B}_2)=0. \tag{8.1}$$

By applying the covariant derivative to (2.19) w.r.t. any vector field $\mathscr{B}_1$ and utilizing (2.6), we obtain

$$\begin{aligned}(\nabla_{\mathscr{B}_1}S)(\mathscr{B}_2,\mathscr{B}_3)=&\frac{dr(\mathscr{B}_1)}{n-1}\left\{g(\mathscr{B}_2,\mathscr{B}_3)+\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)\right\}\\&+\beta\left(\frac{r}{n-1}+n\beta^2\right)[g(\mathscr{B}_1,\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)+g(\mathscr{B}_1,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_2)-2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)].\end{aligned} \tag{8.2}$$

Likewise, we can find

$$\begin{aligned}(\nabla_{\mathscr{B}_2}S)(\mathscr{B}_3,\mathscr{B}_1)=&\frac{dr(\mathscr{B}_2)}{n-1}\left\{g(\mathscr{B}_3,\mathscr{B}_1)+\mathscr{V}^\flat(\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_1)\right\}\\&+\beta\left(\frac{r}{n-1}+n\beta^2\right)\left[g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_1)+g(\mathscr{B}_2,\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_3)-2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)\right],\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}(\nabla_{\mathscr{B}_3}S)(\mathscr{B}_1,\mathscr{B}_2)=&\frac{dr(\mathscr{B}_3)}{n-1}\left\{g(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\right\}\\&+\beta\left(\frac{r}{n-1}+n\beta^2\right)[g(\mathscr{B}_3,\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)+g(\mathscr{B}_3,\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_1)-2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)].\end{aligned} \tag{8.4}$$

*Cartan hypersurfaces* are recognized as manifolds with a non-parallel Ricci tensor that satisfies equation (8.1). From this equation, we can deduce that $r$ is constant. Considering equations (8.2)-(8.4) in relation to (8.1), we obtain

$$\beta\left(\frac{r}{n-1}+n\beta^2\right)[g(\mathscr{B}_1,\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)+g(\mathscr{B}_2,\mathscr{B}_3)\mathscr{V}^\flat(\mathscr{B}_1)+g(\mathscr{B}_3,\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)-3\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)]=0. \tag{8.5}$$

Replacing $\mathscr{B}_1$ by $\varphi\mathscr{B}_1$ in (8.5) and applying (2.2), we achieve

$$\beta\left(\frac{r}{n-1}+n\beta^2\right)[g(\varphi\mathscr{B}_1,\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)+g(\mathscr{B}_3,\varphi\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)]=0. \tag{8.6}$$

Plugging $\mathscr{B}_3=\zeta$ in (8.6) and using (2.2), it gives

$$\beta\left(\frac{r}{n-1}+n\beta^2\right)g(\varphi\mathscr{B}_1,\mathscr{B}_2)=0. \tag{8.7}$$

Thus, we get

$$r=-n(n-1)\beta^2.$$

This results in the following:

**Theorem 8.1** *If an $\mathscr{M}_\beta$ possesses a cyclic parallel Ricci tensor, the scalar curvature of the manifold remains constant at $-n(n-1)\beta^2$.*

## 9 $\mathscr{V}^\flat$-Parallel Ricci Tensor

Kon [15] introduced the concept of *$\mathscr{V}^\flat$-parallel Ricci tensor* for an $\mathscr{M}_\beta$.

**Definition 9.1** *An $\mathscr{M}_\beta$ is called $\mathscr{V}^\flat$-parallel if it's Ricci tensor $S$ satisfies*

$$(\nabla_{\mathscr{B}_3}S)(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)=0, \tag{9.1}$$

*holds for $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3\in\chi(\mathscr{M}_\beta)$.*

Let us assume that the Ricci tensor of a $\mathscr{M}_\beta$ is $\mathscr{V}^\flat$-parallel, then (9.1) holds.
From (2.19), we can easily find that

$$S(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2) = \left(\frac{r}{n-1} + \beta^2\right) g(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2). \tag{9.2}$$

By utilizing (2.3) and applying the covariant derivative to (9.2) w.r.t. any vector field $\mathscr{B}_3$, we derive

$$\begin{aligned}(\nabla_{\mathscr{B}_3} S)(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2) = \frac{dr(\mathscr{B}_3)}{n-1} &\left\{g(\mathscr{B}_1, \mathscr{B}_2) + \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\right\} \\ &+ \beta\left(\frac{r}{n-1} + \beta^2\right)[(\nabla_{\mathscr{B}_3}\mathscr{V}^\flat)(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2) + (\nabla_{\mathscr{B}_3}\mathscr{V}^\flat)(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_1)].\end{aligned} \tag{9.3}$$

In view of (2.6) in (9.3), we have

$$\begin{aligned}(\nabla_{\mathscr{B}_3} S)(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2) = \frac{dr(\mathscr{B}_3)}{n-1} &\left\{g(\mathscr{B}_1, \mathscr{B}_2) + \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\right\} \\ &+ \beta\left(\frac{r}{n-1} + \beta^2\right)[g(\mathscr{B}_3, \mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2) + g(\mathscr{B}_3, \mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_1) - 2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)].\end{aligned} \tag{9.4}$$

Thus, (9.1) and (9.4) together give

$$\begin{aligned}\frac{dr(\mathscr{B}_3)}{n-1} \left\{g(\mathscr{B}_1, \mathscr{B}_2) + \mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\right\} + \beta\left(\frac{r}{n-1} + \beta^2\right)&[g(\mathscr{B}_3, \mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2) + g(\mathscr{B}_3, \mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_1) \\ &- 2\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)] = 0.\end{aligned} \tag{9.5}$$

Consider an orthonormal frame field $\{e_i\}_{i=1}^n$ defined at any point of $\mathscr{M}_\beta$. By substituting $\mathscr{B}_1 = \mathscr{B}_2 = e_i$ in (9.5) and performing contraction, we obtain

$$dr(\mathscr{B}_3) = -4\beta\left(\frac{r}{n-1} + \beta^2\right)\mathscr{V}^\flat(\mathscr{B}_3). \tag{9.6}$$

Using this in (9.5), we obtain

$$\beta\left(\frac{r}{n-1} + \beta^2\right)\left[g(\mathscr{B}_3, \mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2) + g(\mathscr{B}_3, \mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_1) - 4g(\mathscr{B}_1, \mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3) - 6\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3)\right] = 0. \tag{9.7}$$

Replacing $\mathscr{B}_1$ by $\zeta$ in the above equation, it yields

$$\beta\left(\frac{r}{n-1} + \beta^2\right)[3\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3) - g(\mathscr{B}_2, \mathscr{B}_3)] = 0. \tag{9.8}$$

Since, $3\mathscr{V}^\flat(\mathscr{B}_2)\mathscr{V}^\flat(\mathscr{B}_3) - g(\mathscr{B}_2, \mathscr{B}_3) \neq 0$, therefore

$$r = -(n-1)\beta^2. \tag{9.9}$$

Conversely, if $r = -(n-1)\beta^2$, one can easily find from (9.4) that

$$(\nabla_{\mathscr{B}_3} S)(\varphi\mathscr{B}_1, \varphi\mathscr{B}_2) = 0, \tag{9.10}$$

for all $\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3 \in \chi(\mathscr{M}_\beta)$.
Hence, we arrive at the following conclusion:

**Theorem 9.1** *An n-dimensional $\mathscr{M}_\beta$ is $\mathscr{V}^\flat$-parallel Ricci tensor iff $\mathscr{M}_\beta$ has constant scalar curvature $-(n-1)\beta^2$.*

## 10 Ricci-Bourguignon Soliton

Assume that an $\mathscr{M}_\beta$ possesses an *RB* soliton with the potential vector field being the Reeb vector field $\zeta$. From equation (1.3), we then obtain

$$\pounds_\zeta g(\mathscr{B}_1,\mathscr{B}_2)+2S(\mathscr{B}_1,\mathscr{B}_2)=2\Lambda g(\mathscr{B}_1,\mathscr{B}_2)+2\psi rg(\mathscr{B}_1,\mathscr{B}_2), \tag{10.1}$$

which gives

$$g(\nabla_{\mathscr{B}_1}\zeta,\mathscr{B}_2)+g(\nabla_{\mathscr{B}_2}\zeta,\mathscr{B}_1)+2S(\mathscr{B}_1,\mathscr{B}_2)=2\Lambda g(\mathscr{B}_1,\mathscr{B}_2)+2\psi rg(\mathscr{B}_1,\mathscr{B}_2). \tag{10.2}$$

Taking into account of (2.5), the above expression takes the form

$$S(\mathscr{B}_1,\mathscr{B}_2)=[\Lambda+\psi r-\beta]g(\mathscr{B}_1,\mathscr{B}_2)+\beta\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2). \tag{10.3}$$

Plugging $\mathscr{B}_2=\zeta$ in (10.3) and making use of (2.1) and (2.10), we achieve

$$[2\beta-(n-1)\beta^2-\Lambda-\psi r]\mathscr{V}^\flat(\mathscr{B}_1)=0. \tag{10.4}$$

Since $\mathscr{V}^\flat(\mathscr{B}_1)\neq 0$, then it follows that

$$\Lambda=2\beta-(n-1)\beta^2-\psi r. \tag{10.5}$$

Hence, we reach to the following:

**Theorem 10.1** *For an n-dimensional $\mathscr{M}_\beta$ with metric $g$ that fulfills the RB soliton $(g,\zeta,\Lambda,\psi)$, where $\zeta$ represents the reeb vector field, then the soliton's nature is shrinking, steady, expanding accordingly* $2\beta-(n-1)\beta^2-\psi r \gtreqless 0$.

We incorporate the identity from (10.5) into (10.3) to acquire

$$S(\mathscr{B}_1,\mathscr{B}_2)=[\beta-(n-1)\beta^2]g(\mathscr{B}_1,\mathscr{B}_2)+\beta\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2). \tag{10.6}$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame field at any point of $\mathscr{M}_\beta$. If we set $\mathscr{B}_1=\mathscr{B}_2=e_i$ in (10.6) and sum over $i$, $1\leq i\leq n$, we have

$$r=-(n-1)(n\beta-1)\beta. \tag{10.7}$$

Hence, we reach at the following conclusion:

**Theorem 10.2** *If an n-dimensional $\mathscr{M}_\beta$ admits an RB soliton $(g,\zeta,\Lambda,\psi)$ whose potential vector field is the Reeb vector field $\zeta$, then the manifold becomes an $\mathscr{V}^\flat$-Einstein manifold and the scalar curvature is of the form* $r=-(n-1)(n\beta-1)\beta$.

## 11 Example of a five-dimensional $\mathscr{M}_\beta$

Consider a five dimensional manifold $\mathscr{M}^5=\{(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)\in\mathbb{R}^5:\vartheta_5\neq 0\}$, where $(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)$ are the standard coordinates in $\mathbb{R}^5$. We choose

$$\rho_1=\cos^2 v_5\frac{\partial}{\partial\vartheta_1},\quad \rho_2=\cos^2 v_5\frac{\partial}{\partial\vartheta_2},\quad \rho_3=\cos^2 v_5\frac{\partial}{\partial\vartheta_3},\quad \rho_4=\cos^2 v_5\frac{\partial}{\partial\vartheta_4},$$
$$\rho_5=\vartheta_1\frac{\partial}{\partial\vartheta_1}+\vartheta_2\frac{\partial}{\partial\vartheta_2}+\vartheta_3\frac{\partial}{\partial\vartheta_3}+\vartheta_4\frac{\partial}{\partial\vartheta_4}+\frac{\partial}{\partial\vartheta_5},$$

which are *linearly independent* at each point of the manifold $\mathscr{M}^5$.
Let the Lorentzian metric $g$ be defined by

$$g(\rho_m,\rho_n)=\begin{cases}1, & if \quad m=n \quad and \quad m,n\in\{1,2,3,4\},\\ -1, & if \quad m=n=5,\\ 0, & otherwise.\end{cases}$$

Let the 1-form $\mathscr{V}^\flat$ satisfy the relation

$$\mathscr{V}^\flat(\rho_5)=-1.$$

Let a $(1,1)$ tensor field $\varphi$ be given by $\varphi(\rho_1)=-\rho_2$, $\varphi(\rho_3)=-\rho_4$, $\varphi(\rho_5)=0$. Then, we have

$$\varphi^2(\mathscr{B}_1)=\mathscr{B}_1+\mathscr{V}^\flat(\mathscr{B}_1)\zeta,$$
$$g(\varphi\mathscr{B}_1,\varphi\mathscr{B}_2)=g(\mathscr{B}_1,\mathscr{B}_2)+\mathscr{V}^\flat(\mathscr{B}_1)\mathscr{V}^\flat(\mathscr{B}_2),$$

for any $\mathscr{B}_1, \mathscr{B}_2 \in \chi(\mathscr{M}^5)$. Thus for $\rho_5 = \zeta$, $\mathscr{M}^5(\varphi, \zeta, \mathscr{V}^\flat, g)$ defines an *almost contact metric structure* on $\mathscr{M}^5$. Now, we have

$$[\rho_1,\rho_2]=0, \quad [\rho_1,\rho_3]=0, \quad [\rho_1,\rho_4]=0, \quad [\rho_1,\rho_5]=(1-2\tan\vartheta_5)\rho_1,$$
$$[\rho_2,\rho_3]=0, \quad [\rho_2,\rho_4]=0, \quad [\rho_2,\rho_5]=(1-2\tan\vartheta_5)\rho_2, \quad [\rho_3,\rho_4]=0,$$
$$[\rho_3,\rho_5]=(1-2\tan\vartheta_5)\rho_3, \quad [\rho_4,\rho_5]=(1-2\tan\vartheta_5)\rho_4.$$

Setting $\rho_5 = \zeta$ and using well known Koszul's formula, we arrive at the following:

$$\begin{cases} \nabla_{\rho_1}\rho_1 = -(1-2\tan\vartheta_5)\rho_1, \quad \nabla_{\rho_2}\rho_1 = 0, \quad \nabla_{\rho_3}\rho_1 = 0, \quad \nabla_{\rho_4}\rho_1 = 0, \quad \nabla_{\rho_5}\rho_1 = 0, \\ \nabla_{\rho_1}\rho_2 = 0, \quad \nabla_{\rho_2}\rho_2 = -(1-2\tan\vartheta_5)\rho_2, \quad \nabla_{\rho_3}\rho_2 = 0, \quad \nabla_{\rho_4}\rho_2 = 0, \quad \nabla_{\rho_5}\rho_2 = 0, \\ \nabla_{\rho_1}\rho_3 = \rho_1, \quad \nabla_{\rho_2}\rho_3 = \rho_2, \quad \nabla_{\rho_3}\rho_3 = -(1-2\tan\vartheta_5)\rho_3, \quad \nabla_{\rho_4}\rho_3 = 0, \quad \nabla_{\rho_5}\rho_3 = 0, \\ \nabla_{\rho_1}\rho_4 = \rho_1, \quad \nabla_{\rho_2}\rho_4 = \rho_2, \quad \nabla_{\rho_3}\rho_4 = 0, \quad \nabla_{\rho_4}\rho_4 = -(1-2\tan\vartheta_5)\rho_4, \quad \nabla_{\rho_5}\rho_4 = 0, \\ \nabla_{\rho_1}\rho_5 = (1-2\tan\vartheta_5)\rho_1, \ \nabla_{\rho_2}\rho_5 = (1-2\tan\vartheta_5)\rho_2, \ \nabla_{\rho_3}\rho_5 = (1-2\tan\vartheta_5)\rho_3, \\ \nabla_{\rho_4}\rho_5 = (1-2\tan\vartheta_5)\rho_4, \ \nabla_{\rho_5}\rho_5 = 0. \end{cases}$$

Hence we can see that for $\beta = (1-2\tan\vartheta_5)$, $\mathscr{M}^5(\varphi, \zeta, \mathscr{V}^\flat, g)$ is an $\mathscr{M}_\beta$ of dimension 5.
The components of the curvature tensor with respect to $\nabla$ are as follows

$$\begin{cases} R(\rho_1,\rho_2)\rho_2 = -\beta^2\rho_1, \ R(\rho_1,\rho_5)\rho_5 = \beta^2\rho_1, \ R(\rho_1,\rho_4)\rho_1 = \beta^2\rho_4, \ R(\rho_2,\rho_4)\rho_2 = \beta^2\rho_4, \\ R(\rho_2,\rho_4)\rho_4 = -\beta^2\rho_2, \ R(\rho_3,\rho_5)\rho_3 = \beta^2\rho_5, \ R(\rho_1,\rho_3)\rho_3 = -\beta^2\rho_1, \ R(\rho_1,\rho_2)\rho_1 = \beta^2\rho_2, \\ R(\rho_1,\rho_5)\rho_1 = \beta^2\rho_5, \ R(\rho_2,\rho_5)\rho_2 = \beta^2\rho_5, \ R(\rho_2,\rho_5)\rho_5 = \beta^2\rho_2, \ R(\rho_3,\rho_4)\rho_4 = -\beta^2\rho_3, \\ R(\rho_4,\rho_5)\rho_5 = \beta^2\rho_4, \ R(\rho_1,\rho_4)\rho_4 = -\beta^2\rho_1, \ R(\rho_1,\rho_3)\rho_1 = \beta^2\rho_3, \ R(\rho_2,\rho_3)\rho_2 = \beta^2\rho_3, \\ R(\rho_2,\rho_3)\rho_3 = -\beta^2\rho_2, \ R(\rho_3,\rho_4)\rho_3 = \beta^2\rho_4, \ R(\rho_4,\rho_5)\rho_4 = \beta^2\rho_5, \ R(\rho_3,\rho_5)\rho_5 = \beta^2\rho_3. \end{cases}$$

From above, we can obtain

$$S(\rho_1,\rho_1) = -4\beta^2, \quad S(\rho_2,\rho_2) = -4\beta^2, \quad S(\rho_3,\rho_3) = -4\beta^2, \quad S(\rho_4,\rho_4) = -4\beta^2, \quad S(\rho_5,\rho_5) = -4\beta^2.$$

$$\Rightarrow \quad r = \sum_{i=1}^{5} S(\rho_i,\rho_i) = -20\beta^2.$$

## Declarations

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